Introduction

- Reference book for this class: Mechanical Vibration (S. Rao) 5th edition (Prentice Hall)

Mechanical Vibration (Structural Dynamics): A broad field of engineering or applied mechanics

Engineering mechanics: It is one of the oldest disciplines in engineering and it's the field that deal with the action of forces or environmental effect on a body and how that body react to forces.

Main courses in engineering mechanic (solid):

1) Statics

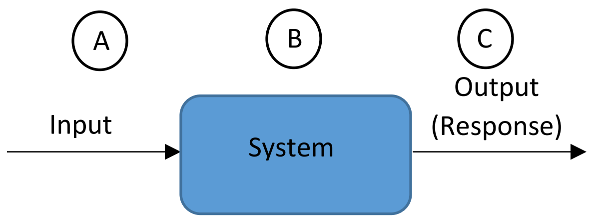
2) Mechanics of material

3) Dynamics

4) Kinematics

5) Mechanical vibrations

Any field in engineering can be represent by following diagram



This is all we do in study engineering mechanics. The only differences is related to different assumptions or the nature of those components.

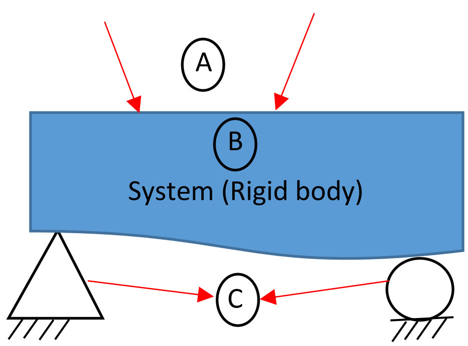
Statics:

A) Forces or load (time independent)

B) Assumed system be a rigid body (particles)

C) Forces

Example:



Because in statics, we have rigid body as the system, it cannot absorb energy and cannot be deformed, it simply transfer the whole input force to supports.

Mechanics of material:

A) Forces or load (time independent)
B) Assumed system be a flexible body (deformable) – (Elasticity)
C) $\underbrace{\text{Forces/ Internal Forces}}_{\text{Stress } (\sigma)}$ (shear and moment inside of a beam)/ $\underbrace{\text{Deformation}}_{\text{Strain } (\varepsilon)}$

Dynamics (rigid bodies):

A) Forces (time dependent)
B) Assumed system be a rigid body
C) Time dependent forces (Also, velocity, acceleration, etc.)

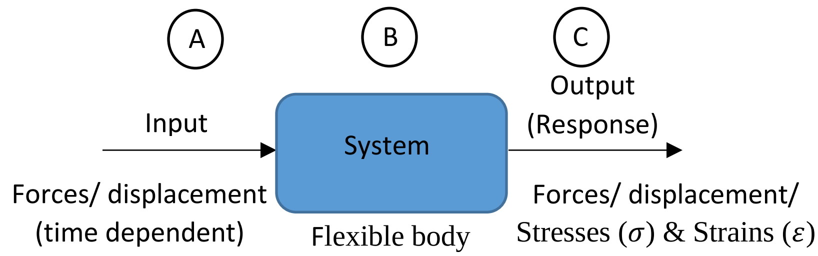
Kinematics (rigid bodies):

A) Input motion
B) Assumed system be a rigid body
C) Output motion (position, displacement, velocity, acceleration, etc.)

Mechanical vibrations:

A) Forces (time dependent) or any other time dependent phenomena can that causes a change in the system (e.g. displacement). The forces can be desirable (like in engines) or undesirable (like earth quick)!
B) Assumed system be a flexible body. That means not whole the force/energy that goes to the system doesn't get out of the system and some part of it absorb by system (damping). A flexible body has inertia, elasticity, and energy absorption (dissipation).
C) Forces/ Displacements/ Stresses(σ) & Strains (ε)

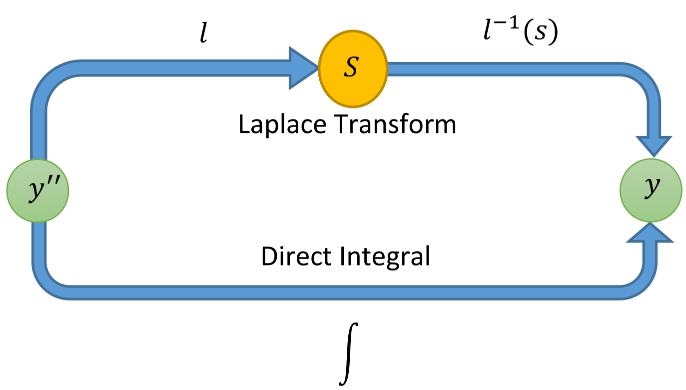
What we need for this course



- The ultimate goal in Mechanical vibration course: is it possible to set up a mathematical model that has all of these elements in it and represents the whole system?

  For formulating mechanical vibration equations, we will use the second order of ordinary differential equation (ODE).

- There are two ways for solving an ODE:
  1) Laplace transform (l)
  2) Direct formulation/ Direct integration

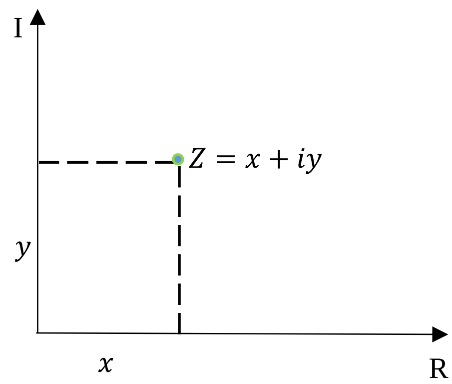


Rather than solving a very complicated integral, in the Laplace transform method, we are first mapped the equation to another domain and then we doing inverse mapping to solve the problem.

In this course, we are dealing with some simple second order differential equations, so we are using direct approach for solving the problems.

- We are working with complex numbers in this course.

  The real numbers have two dimensions: Real dimension & Imaginary dimension



Some number may have the imaginary part or may not (for instance, $5 = 5 + i(0)$)

Complex algebra has its own operation:

1) Summation/ subtraction

If: $Z_1 = x_1 + iy_1$, $Z_2 = x_2 + iy_2$

$Z_t = Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

2) Multiplication

Note: $i = \sqrt{-1}$ so $i^2 = i.i = \sqrt{-1}.\sqrt{-1} = -1$
$Z_t = Z_1.Z_2 = (x_1 + iy_1).(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

3) Division

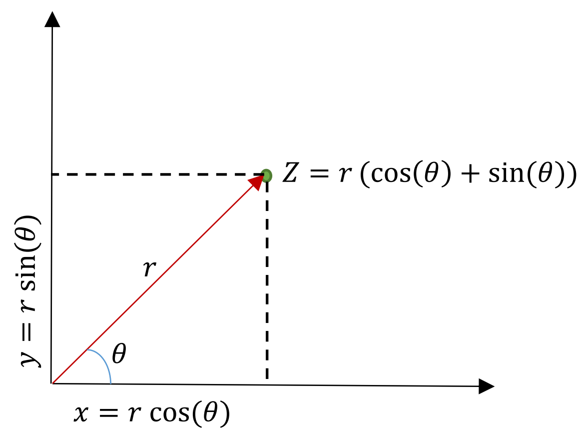
$Z_t = \frac{Z_1}{Z_2} \Rightarrow Z_t.Z_2 = Z_1 \Rightarrow$ From this equality we can find Z_t

$$Z_t = \frac{x_1x_2 + y_1y_2 + i(y_1x_2 + x_1y_2)}{{x_1}^2 + {y_1}^2}$$

Note: If two complex numbers are equal that means real parts of them are equal and imaginary parts of them are equal too.

$Z_1 = Z_2 \Rightarrow x_1 = x_2$ & $y_1 = y_2$

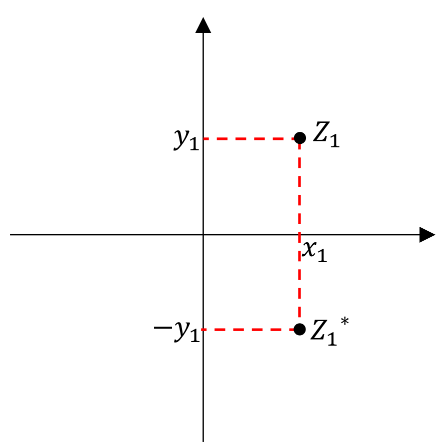
There is another way to look at an analogy of real numbers with using polar coordinates.



Also, “Z” can be written in form of $Z = re^{i\theta}$.

- Complex conjugate: Two numbers which are symmetric with respect to the real axis (they have same real values and opposite imaginary values)

  $Z_1 = x_1 + iy_1$
  $Z_1{}^* = x_1 - iy_1$



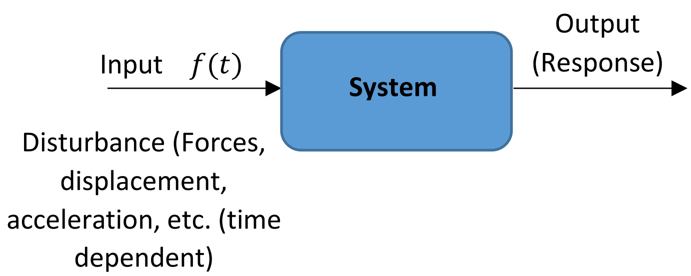
Mechanical Vibration

- What is the objective?

The objective of mechanical vibration is we want to analyze and understand the behavior of a system under the action of a desirable/undesirable motion (to use this vibration in an effective way or eliminate it from system).

In vibration we have focus on the **system** and that is the most important part. Also, we have to understand how model and present the force.

- System can be simple or very complex. For the complex system, as an engineer we have to simplify the system to be more understandable.
- Force is time dependent.



System is that element with all inherent characteristics of the real physical structure. Any physical structure has in general no more than three major inherent properties or characteristics that define basically what any system made of or is capable of doing in order to resist the action of a complain. These properties are: **1) Mass (m) 2) Elasticity (k) 3) Energy Absorption (c).**

Mass for the inertia of the system which as the result of the action of the force moves in the certain direction. The system or that structure has the ability to resist this motion. This resistance can comes in two distinct ways: **Elasticity,** which is the structure resists by going through some deformation, bending, reacting to the action of the force or **Energy Absorption,** that system or structure resist the imposed motion or the force by trying to dissipate the effect of that action or dissipating energy that resists that motion.

The goal of vibrations analysis, is finding the **system** as the most appropriate mathematical model of a real/physical structure or mechanism.

Idealization

- Idealization: 1) Making logical assumptions based on knowing the physics of the structure 2) Defining some things that can help us to develop the proper mathematical model.

1) Disturbance ($f(t)$): This force has certain forms or a behavior
2) Motion: We are dealing with two types of motion: Translational and rotational

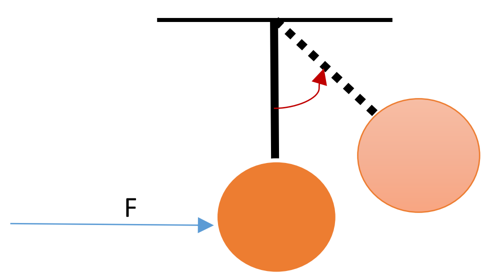
- What is the main difference between translational and rotational motion?

In both of them we are talking about the mass go to some motion. But in translational motion we are look at the motion in terms of what is happening to the center of mass. The distribution of the mass or the way the mass is distributed or the inertia **are not** so important in translational motion while in rotational motion mass distribution is super important.

Translational motion: Dealing with the mass or the center of the mass or how the mass is lumped at one point.
Rotational motion: Dealing with mass moment of inertia

3) Degree of freedom: it describes when the structure start to move how every single mass element in that structure moves. If there is one independent deformation that the rest of structure can be defined according to that, we will have one degree of freedom.

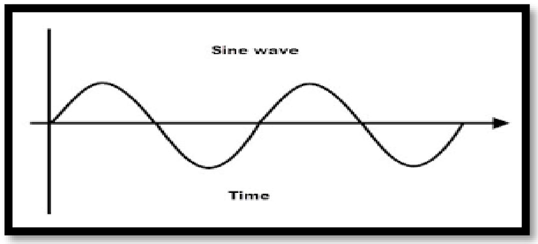
Example: In the following light, if we assume whole wright of structure is just concentrate in the light (chain is weightless), then we will have one degree of freedom and we can describe motion of everything else respect to the blob. In other word, if you get the overall motion of a structure represented by the motion of a single point we call this one degree of freedom.



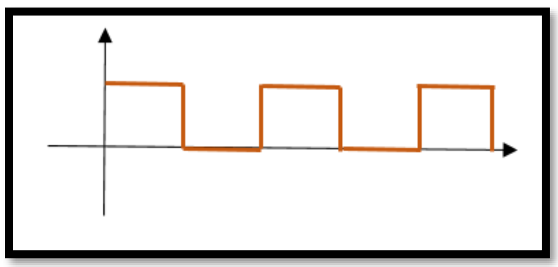
If we can assume the entire mass of a structure or system is lumped in a single point which moving in a certain direction (can be a translational or rotational) then we will have a system with a single degree of freedom.

4) Type and shapes of $f(t)$: This force can be in various shape and various types.

   a) Harmonic force: Sinusoidal cyclic force (like the engine of the car)



   b) Periodic (not harmonic) force:



   c) General force: 1) Deterministic: like a blast load, an impact load etc. 2) Random force: (completely unpredictable and predictability associated with some probability) like earthquake, wind load, etc.

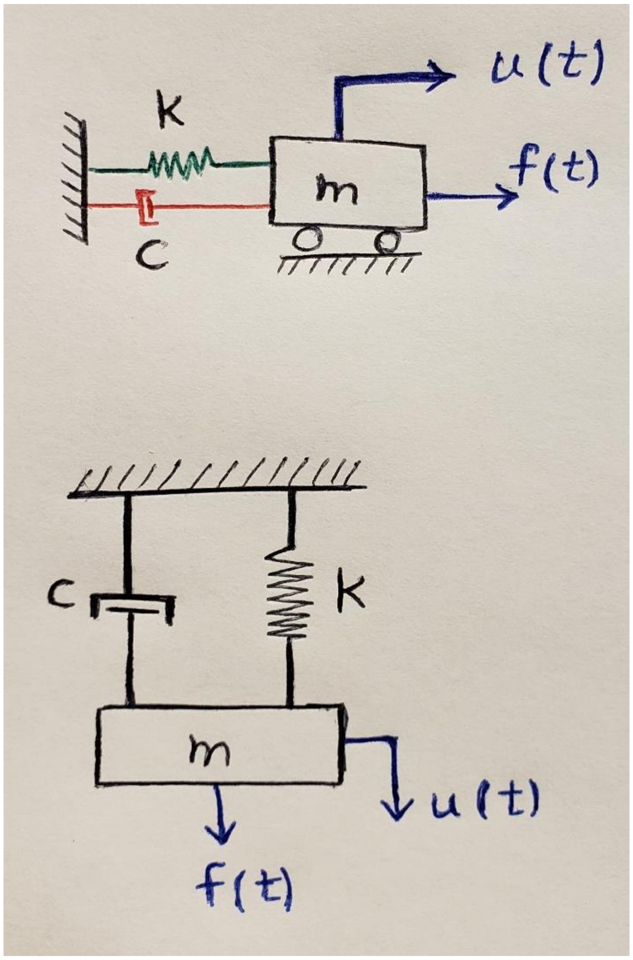
Mechanical vibration can be categorized on two distinct ways.

First, based on type of force that acts on the system. So, the system can vibrating without the force acting on it and it is called **free vibration**. For instance, wind has been blowing and move the trees then wind stops but trees still vibrating and moving. **Or** the system is vibrating due to the action of the force that continuously is acting on it and exist and it is called **forced vibration**.

The following system has one degree of freedom. This system includes a mass (m) which is moving because of a force (f(t)) in one direction (translational). This motion (displacement) is a function of time (u(t)). Also, we have something that is trying to stop the motion of the mass (it is named stiffness or elasticity element (k)). We will assume this system is linear, so that deformation would be a linear function of the displacement. We can represent that by a linear force which can be shown by a spring. We will have another kind of resistance against the motion which is the energy absorption and we call it damping element (c).

Mechanical Vibration

In this course, we use “u” rather than “x” to be able to analyze the multi directional motion and not make any confusion with x-y frame. Also, we assume small values for “u” and that means the deformation of structure and material will be remain in the elastic range.



Input: $\boldsymbol{f(t)}$

System: (m, k, c)

Output/response: u(t)

All structures in the World can be can be described by the motion of a single point where the entire mass of the structure is lumped at that point. So, we can assumed and present most structures in the World by a single degree of freedom (S.D.O.F) model.

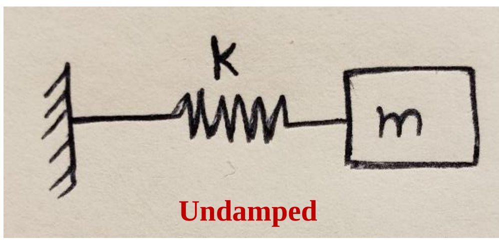
Mechanical vibration has two main Objectives:

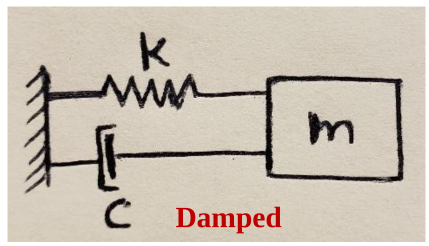
1) How to model a physical system as S.D.O.F. system. If we have a structure, how we can define the stiffness element, damping element, and mass of structure.
2) How to find the response (displacement u(t), velocity $\dot{u}$, acceleration $\ddot{u}$, stress δ and strain ε, etc.)

A physical system in general can be presented in one of the following forms:

1) Undamped System: System with not ability to absorb energy (or negligible)

2) Damped System: System with ability to absorb energy

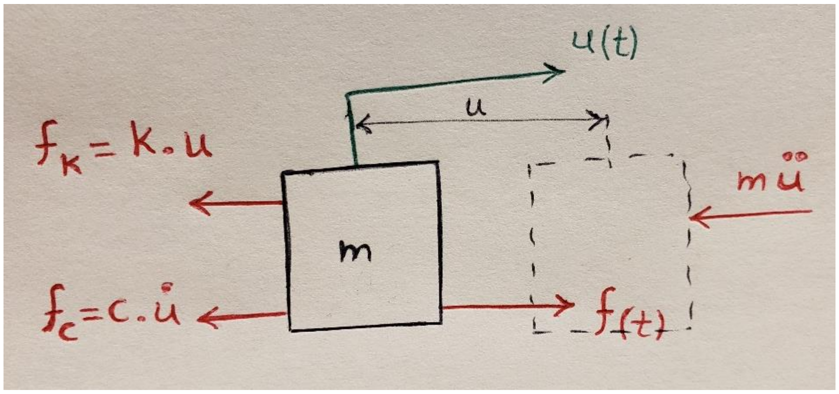




<table>
<tr><th>Type of Vibration</th><th>Type of System</th><th colspan="2">Type of Force</th></tr>
<tr><td rowspan="2">Free Vibration (Force was in the past but No longer any force now)</td><td>Undamped System</td><td rowspan="2" colspan="2"></td></tr>
<tr><td>Damped System</td></tr>
<tr><td rowspan="4">Forced Vibration (Continues force all the time)</td><td rowspan="2">Undamped System</td><td colspan="2">Harmonic</td></tr>
<tr><td colspan="2">Periodic (not harmonic)</td></tr>
<tr><td rowspan="2">Damped System</td><td rowspan="2">General</td><td>Deterministic</td></tr>
<tr><td>Random</td></tr>
</table>

For solving the system we will use the Free Body Diagram. There two ways to set up the free body diagram in dynamics and solving dynamic vibration problems:

1) Newton's low
2) D' Alembert's principle : In this method in order to set up or solve a dynamic problem, you treat the problem as a static case by drawing the free body diagram and place the D'Alembert's force which is basically is an inertia force on it in opposite direction of the motion (mass times acceleration $m\,\ddot{u}$). Then with this free body diagram, we can write the equations of equilibrium which are similar to the static equilibrium.



In this diagram:

f_k: Force of elasticity. This force is a linear function to displacement and can be find from multiplication of displacement to some constant value (stiffness coefficient).

f_c: Damping force. In reality this force is a very complex and hard to calculate. To simplify that, we are making an assumption and define this force as multiplication of velocity to some constant value (damping coefficient).

Equations of equilibrium:

$$\sum F = 0$$

$$\boxed{m\ddot{u} + c\dot{u} + ku = f(t)} \qquad \text{Equation of motion}$$

System **Response** **Input**

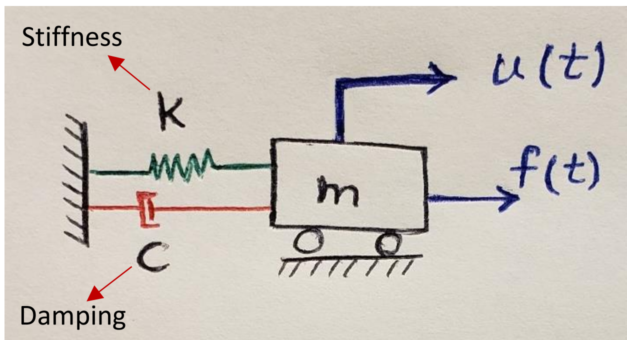
Equation of motion: The mathematical representation form of the entire physical system, input, and response.

Whole mechanical vibration problems will be solved by using "equation of motion" for different condition. Our focus in this course is on how to set up model of a physical system and convert it to the "equation of motion" , then solve the equation to find response and find out what is the meaning of that response and what kind of information that will give us.

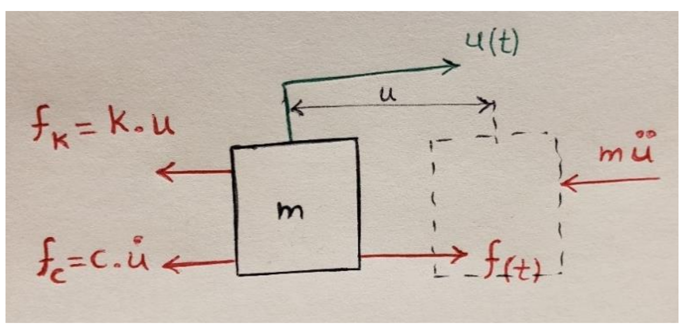
Undamped Free Vibration S.D.O.F System

Summary of last time:

Any physical system that with a single coordinate can describe the motion, it can be modeled as a single degree of freedom system.
Any system can be assumed which is moving in only one direction that would be a single degree of freedom system. This system would be include the moving mass (translation/rotational) and motion of system resisted by two elements: stiffness element and damping element.



Then, we just used D' Alembert's principle which says, you can set up the free body diagram of a dynamic system by including a fictitious force which we call force of inertia. This force for a translational motion would be $m\ddot{u}$ and for rotational motion would be mass moment of inertia multiply to rotational acceleration.



Equations of equilibrium:

$$\sum F = 0$$

$$\boxed{m\ddot{u} + c\dot{u} + ku = f(t)}$$ Equation of motion

System **Response** **Input**

The mechanical vibration consists of two important areas:

1. **We need to know how to set up the S.D.O.F model of a physical system**
   Basically till now, we just accept that any system in the world can be model as a combination of mass, stiffness, and damping elements but if we have a physical system how we can model it with those parameters?
2. **Solving the equation of motion**

Classification of Equation of Motion:

In this class we will study:

Case 1: Undamped system ($C = 0$): We assume the physical system has no energy absorption capability. So, whole the force only causes the deformation (like what you studied in mechanics of material with only difference that here the force is function of time)

1a) Undamped System-Free Vibration ($f(t) = 0$): Source of vibration is not exist anymore.

1b) Undamped System-Forced Vibration ($f(t) \neq 0$): Source of vibration is still exist.

1b1) Undamped System-Forced Vibration with Harmonic force

1b2) Undamped System-Forced Vibration with Periodic force

1b3) Undamped System-Forced Vibration with General force

Case 2: Damped system ($C \neq 0$): We assume the physical system has energy absorption capability.

2a) Damped System-Free Vibration ($f(t) = 0$)

2b) Damped System-Forced Vibration ($f(t) \neq 0$)

2b1) Damped System-Forced Vibration with Harmonic force

2b2) Damped System-Forced Vibration with Periodic force

2b3) Damped System-Forced Vibration with General force

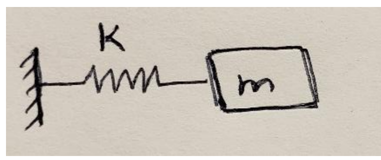
Case 1: Undamped system ($C = 0$):

1a) Undamped System-Free Vibration ($f(t) = 0$)

First of all we need to define some characteristic of parameters which are very important in mechanical vibration and basically they are basis to **set up the S.D.O.F model of a physical system**.

Study an undamped system with free vibration give us some specific information and essential characteristics about our system.

This system include only stiffness and mass as you can see in following schematic model (or can be a rotational system).



$$\boxed{m\ddot{u} + ku = 0}$$

Equation of motion for

Undamped system-free vibration

How to solve this differential equation?

$$mx^2 + k = 0$$

$$\boxed{x_{1,2} = \pm i\sqrt{\frac{k}{m}}}$$

It has two roots, so we will have two solutions

Two solutions:

$$u_1(t) = c_1 e^{i\sqrt{\frac{k}{m}}t}$$

$$u_2(t) = c_2 e^{-i\sqrt{\frac{k}{m}}t}$$

General solution:

$$u(t) = c_1 e^{i\sqrt{\frac{k}{m}}t} + c_2 e^{-i\sqrt{\frac{k}{m}}t}$$

We can write "$u\ (t)$" in the form of *sin* and *cos* (Note: $e^{i\theta} = cos\,\theta + i\,sin\,\theta$):

$$u(t) = c_1\left(\cos\left(\sqrt{\frac{k}{m}}\ t\right) + i\ sin\ \left(\sqrt{\frac{k}{m}}\ t\right)\right) + \ c_2\left(\cos\left(\sqrt{\frac{k}{m}}\ t\right) - i\ sin\ \left(\sqrt{\frac{k}{m}}\ t\right)\right)$$

$$u(t) = (c_1 + c_2)\left(\cos\left(\sqrt{\frac{k}{m}}\ t\right)\right) + \ (c_1 - c_2)\ i\ \left(sin\ \left(\sqrt{\frac{k}{m}}\ t\right)\right)$$

Note: $(c_1 + c_2)$ & $i\ (c_1 - c_2)$ both are constant values. So, we can replace them with some other constant values.

$$\boxed{u(t) = A_1\left(\cos\left(\sqrt{\frac{k}{m}}\ t\right)\right) + \ A_2\ \left(sin\ \left(\sqrt{\frac{k}{m}}\ t\right)\right)}$$

So, the displacement response for an undamped system with free vibration would be a harmonic sinusoidal function.

The frequency of that harmonic oscillation named "Natural Frequency" of the system. Actually, this frequency represents the natural characteristics of the system and each system in the World has its own unique natural frequency (because each system has different mass and stiffness).

Natural Frequency: $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$

General Solution: $\boxed{u(t) = A_1 \cos(\omega_n\ t) + \ A_2\ \ sin\ (\omega_n\ t)}$

Undamped Free Vibration S.D.O.F System

For finding the values of A_1 & A_2, we need two initial conditions:

$t = 0$ $\qquad u(0) = u_0$ $\qquad$ & $\qquad \dot{u}(0) = \dot{u}_0$

$$u(0) = A_1 \cancelto{1}{\cos(0)} + A_2 \cancelto{0}{\sin(0)} = A_1 = u_0$$

$$\dot{u}(t) = -\omega_n A_1 \sin(\omega_n t) + \omega_n A_2 \cos(\omega_n t)$$

$$\dot{u}(0) = -\omega_n A_1 \cancelto{0}{\sin(0)} + \omega_n A_2 \cancelto{1}{\cos(0)} = \omega_n A_2 = \dot{u}_0$$

So:

$$\boxed{A_1 = u_0 \quad \& \quad A_2 = \frac{\dot{u}_0}{\omega_n}}$$

Solution in terms of some initial conditions: $\boxed{u(t) = u_0 \cos(\omega_n t) + \frac{\dot{u}_0}{\omega_n} \sin(\omega_n t)}$

Also, we can simplify this General solution with multiply and dividing this equation by the constant value of $\sqrt{{A_1}^2 + {A_2}^2}$:

$$u(t) = \sqrt{{A_1}^2 + {A_2}^2}\left(\overset{a}{\frac{A_1}{\sqrt{{A_1}^2 + {A_2}^2}}} \cos(\omega_n t) + \overset{b}{\frac{A_2}{\sqrt{{A_1}^2 + {A_2}^2}}} \sin(\omega_n t)\right)$$

$a^2 + b^2 = 1 \quad \Leftrightarrow \quad \sin^2\theta + \cos^2\theta = 1$

(Always there is an angle θ which can satisfy this equation. Also, we can replace $\sqrt{{A_1}^2 + {A_2}^2}$ with some other constant value A which can be find from the initial condition.

$a = sin\theta \quad , \quad b = cos\theta$

From initial condition:

$A = \sqrt{{A_1}^2 + {A_2}^2} = \sqrt{{u_0}^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2}$ = amplitude

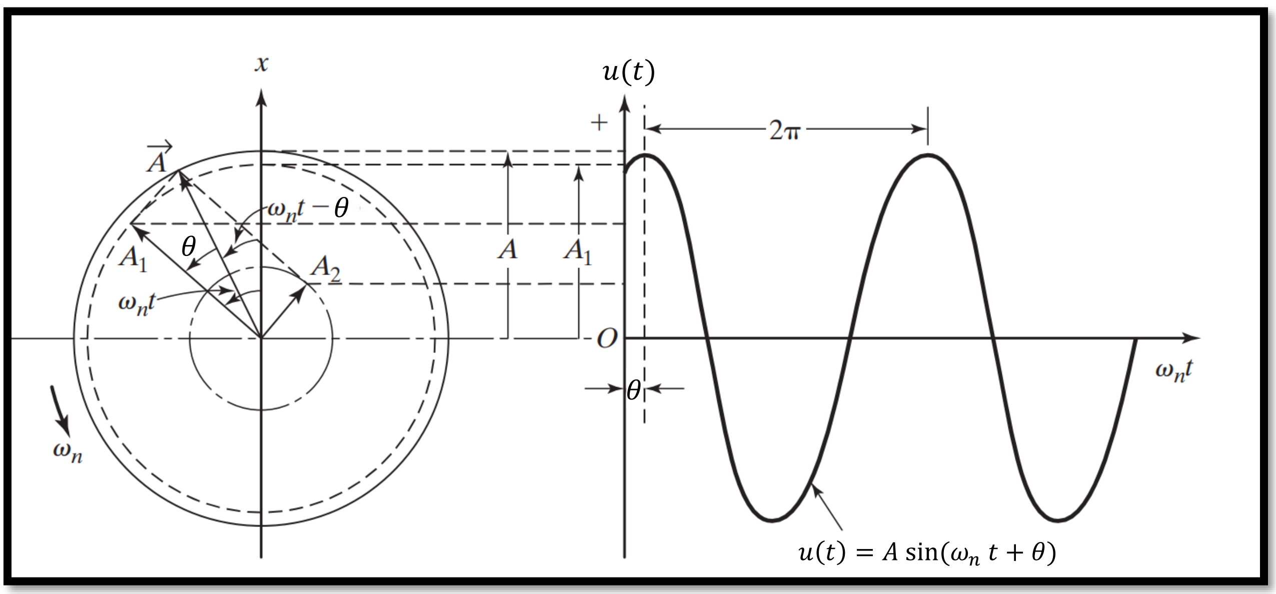
$\theta = tan^{-1}\left(\frac{A_1}{A_2}\right) = tan^{-1}\left(\frac{u_0\omega_n}{\dot{u}_0}\right)$= Phase angle

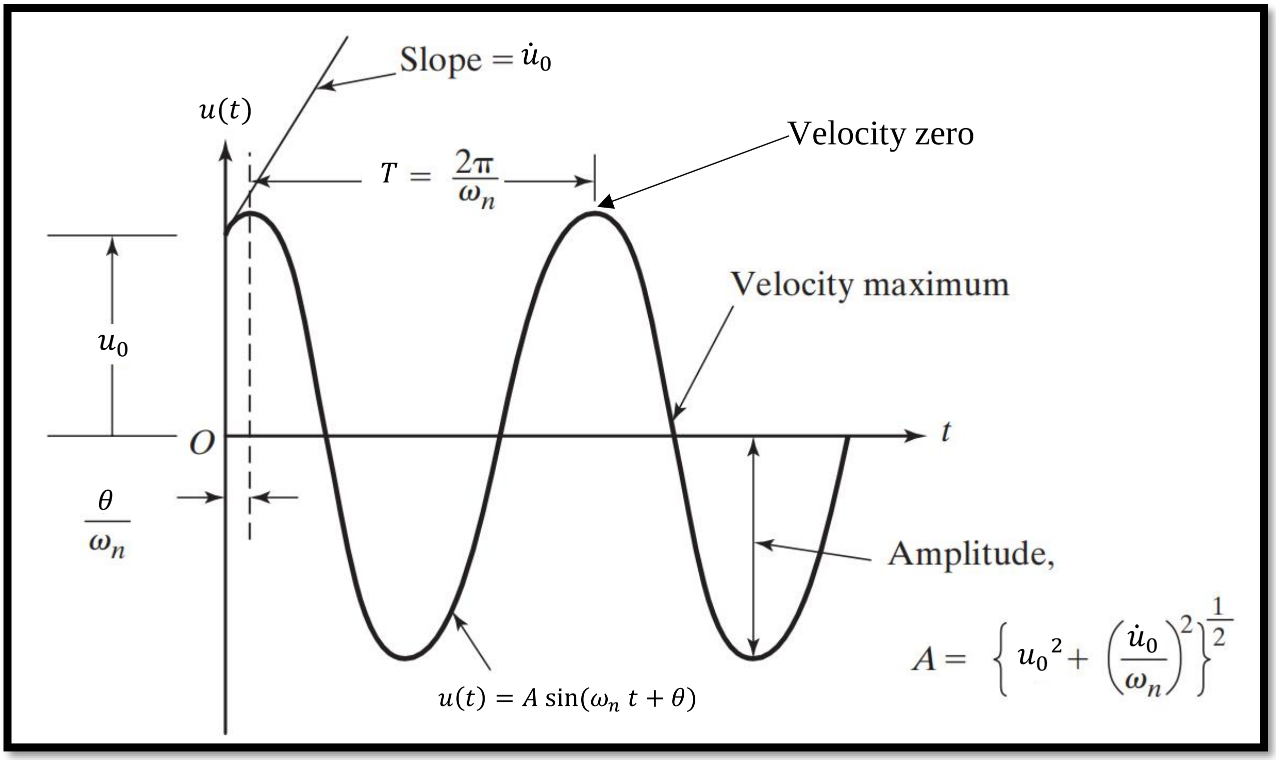
Phase angle shows how the signal lags behind the sinusoidal function.

$$u(t) = A\big(sin\theta \cos(\omega_n\, t) + \; cos\theta \;\; sin\,(\omega_n\, t)\big)$$

General Solution (in the term of single sinusoidal form):

$$\boxed{u(t) = A\,\sin(\omega_n\, t + \theta)}$$





Velocity is the derivative of displacement function and would be in form of sin function. Therefore, whenever displacement is maximum, velocity is zero and for zero displacement, we will have maximum velocity and the change from zero velocity to maximum velocity takes ($t = \frac{\pi}{2\omega_n}$).

- As it discussed before, one of the important concepts for any physical system is natural frequency.
- The motion of S.D.O.F. for an undamped system is harmonic (it repeats itself after each period $T = \frac{2\pi}{\omega_n}$ sec).
- $\omega_n = \sqrt{\frac{k}{m}}$ *(rad/sec)*
- Hertz: We can define the natural frequency in another form (unit: s^{-1}): $f = \frac{\omega_n}{2\pi}$ *(Hz)*
- If we write the mass in term of the weight natural frequency would be equal:

  $\omega_n = \sqrt{\frac{k.g}{W}} \Longrightarrow f = \frac{1}{2\pi}\sqrt{\frac{k.g}{W}}$

  However, "g" is constant value & $\frac{W}{k} = \delta_{st}$ (δ_{st}: static displacement/deformation of the system)

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k.g}{W}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \Longrightarrow T = \frac{1}{f} = 2\pi\sqrt{\frac{\delta_{st}}{g}}}$$

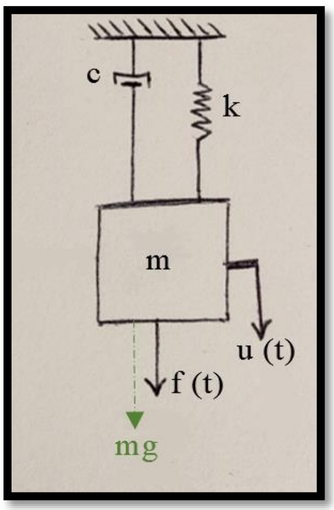
That means the natural frequency of any system can be calculate by static equilibrium. So, if you know the stiffness (k) of the system and weight (W) of the system you can find the natural frequency of that system!

Model a System as S.D.O.F

Summary of last time:

- We talked about a single degree of freedom system.
- The D' Alembert's principle was expressed.
- Based on damped/Undamped and free vibration/forced vibration we classified all possible S.D.O.F. systems.
- We analyzed part of Undamped System-Free Vibration and defined an important characteristic of system, natural frequency (ω_n)

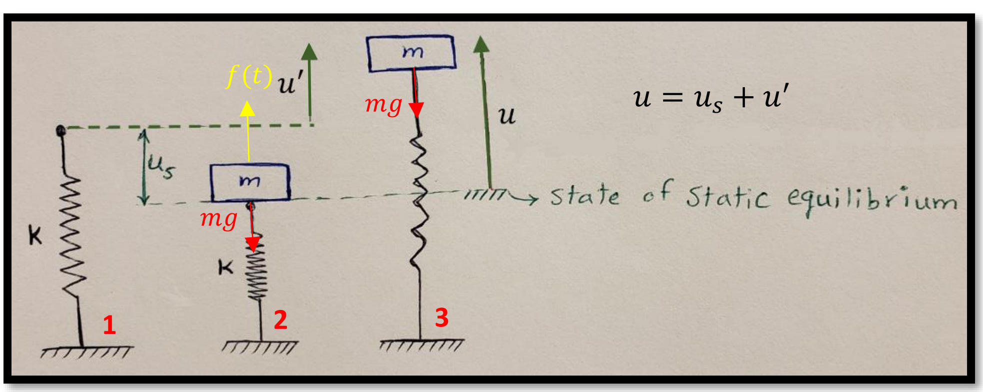
Look at this system:



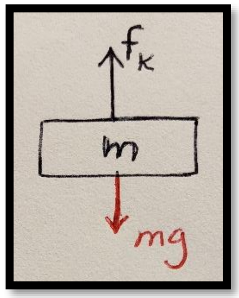
Equation of motion: $m\ddot{u} + c\dot{u} + ku = f(t)$

Where is the gravitation force (mg) in the equation of motion? Why it's not appear in this equation?

The equation of motion is based on the assumption that the dynamic motion starts from the state of static equilibrium.



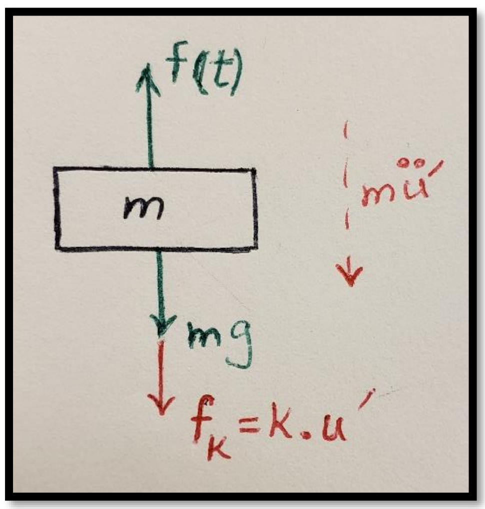
Free Body Diagram 1 to 2:



$$f_k = k.u_s = mg$$

$$\boxed{u_s = \frac{mg}{k}}$$

Free Body Diagram 1 to 3:



$$m\ddot{u}' + ku' = f(t) - mg$$

$$u = u_s + u' \Longrightarrow u' = u - u_s = u - \frac{mg}{k}$$

$$\frac{mg}{k} = constant\ value \Longrightarrow \ddot{u}' = \ddot{u}$$

$$m\ddot{u} + k\left(u - \frac{mg}{k}\right) = m\ddot{u} + ku - \cancel{mg} = f(t) - \cancel{mg}$$

$$\Longrightarrow \boxed{m\ddot{u} + ku = f(t)}$$

So, as you can see mg is not part of the equation of motion when the dynamic motion starts from the state of static equilibrium.

Before continue talking about undamped System-Free Vibration, we want to know **how to set up the S.D.O.F model of a physical system?**

First of all, you need to know some basic concept of stiffness (review this part from mechanics of material)

Based on the complexity of the system, there are **three** main approaches can be used to model a system.

1) Simple cases: Direct Derivation
2) Medium cases: Using Structural analysis approach
3) Complex cases: Energy method:

- Conservation of energy
- Lagrange's approach (for more complex systems)
- Rayleigh's method (for the most complex systems)

1) Direct Derivation

Direct Derivation: Using Free Body Diagram and driving the equation of motion.

What is the stiffness? Ability of the structure to resist about deformation (Or force that generate a unit displacement).

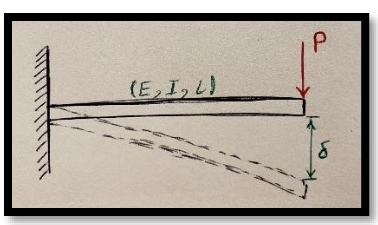
Consider we have a one directional system like a spring:

$$F = k.x$$

In this case for any input force, if displacement become equal 1 ($x = 1$), then stiffness of system (k) would be equal to the applied load (F).

$$F = k.\cancel{x}^{\,1} \Longrightarrow F = k$$

Example 1: We have a beam (system) which is fixed from one end and load is applying at the other end of it.

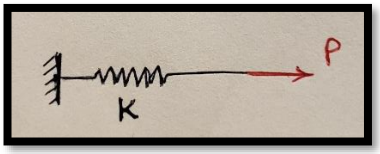


One direction translational deformation case

If we have the properties of this beam (E, I, L) we can write the displacement for the load (P):

$$\delta = \frac{PL^3}{3EI}$$

Can you replace this beam with a single spring? Yes, because if you look at the figure, the beam resisting against the displacement in the same direction of the applied load. So, this system is equivalent to:



What is the stiffness of this beam equivalent?

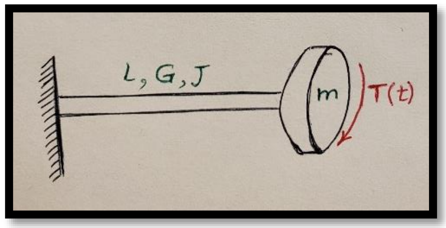
If we put displacement of system equal to one, from previous discussion the force would be equal to the stiffness ($F = k$)

$$\overset{1}{\cancel{\delta}} = \frac{\overset{k}{\cancel{P}}L^3}{3EI} \Longrightarrow \boxed{k = \frac{3EI}{L^3}}$$

So, this is how we can find an equivalent model of a physical system.

Model a System as S.D.O.F

Example 2: We have a shaft with a disk (lumped mass) is subjected to a time dependent torque. This shaft has the geometric properties of length (L), shear module (G), and moment of inertia of the cross section (J). Find a single degree of freedom model that represent the physical system. (We have to find mass and stiffness equivalent to this system)



One direction rotational deformation case

One of the objectives is finding the natural frequency of this system to be able to write the equation of motion for this system.

In this system, we have a torsional force and a shaft which is resisting an angular deformation. The rotational motion of the disk is resisted by the shaft (torsional spring).

$$\theta = \frac{TL}{JG}$$

So, if we put θ (rotational displacement/angular deformation) equal one then the load T would be equivalent to torsional stiffness of the spring (k_t).

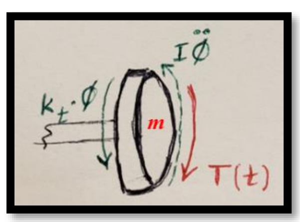
$$\overset{1}{\cancel{\theta}} = \frac{\overset{k_t}{\cancel{T}}L}{JG} \Longrightarrow \boxed{k_t = \frac{JG}{L}}$$

($J = \frac{\pi D^4}{32}$, D: Diameter of the shaft)

Free Body Diagram:

I: Moment of inertia

$\ddot{\phi}$: Angular acceleration



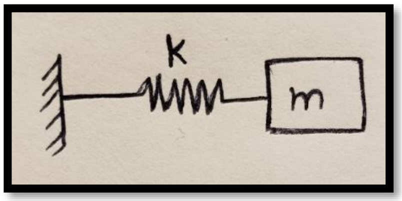
Equation of motion:

$$I\ddot{\emptyset} + k_t\emptyset = T(t)$$

$$I\ddot{\emptyset} + \frac{JG}{L}\emptyset = 0$$

$$m\ddot{u} + ku = f(t) \Rightarrow \boxed{\omega_n = \sqrt{\frac{JG}{LI}}}$$

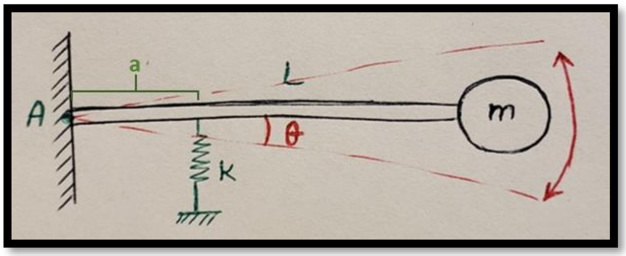
This system is equivalent to:



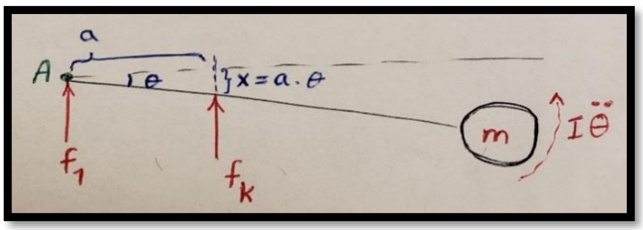
- What is the difference between $\emptyset$ & θ? $\emptyset$ is time-dependent angular deformation when θ is static deformation and stiffness is based on static deformation.

$$\omega_n = \sqrt{\frac{k}{M_{total}}} = \sqrt{\frac{k}{\left(M + \frac{33}{140}m\right)}}$$

Example 3: A rigid beam with length of "L" is pinned to the wall at point A and a mass "m" is connected to the other end of this beam. A spring with constant of "k" is attached somewhere along the beam to the bottom of it. This beam is oscillating up and down with angle of θ. Find the equivalent system?



Free Body Diagram:



Note: If angle θ is small enough, we can assume "x" in the Free Body diagram will be equal to: $(x = a.\theta)$.
Note: Because we have a rotational motion about point "A", we will use $\sum M_A = 0$.

$\sum M_A = 0$

$I\ddot{\theta} + f_k(a) = 0$

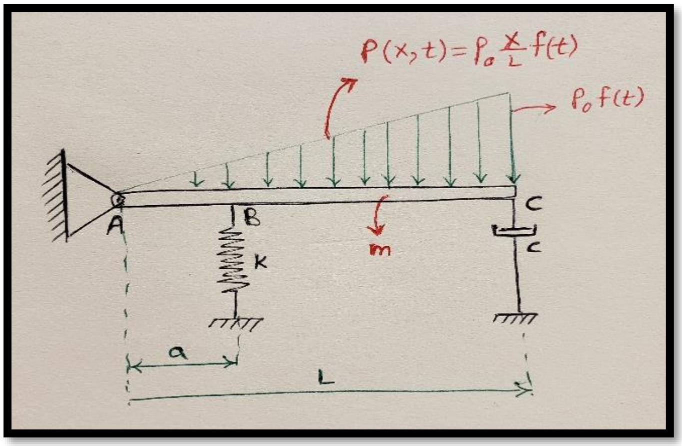
Note: We assume the bar is mass less and the inertia term is just related to the lumped mass (point mass), so $I = mL^2$.

$$mL^2\ddot{\theta} + k(a\theta).a = mL^2\ddot{\theta} + ka^2\theta = 0$$

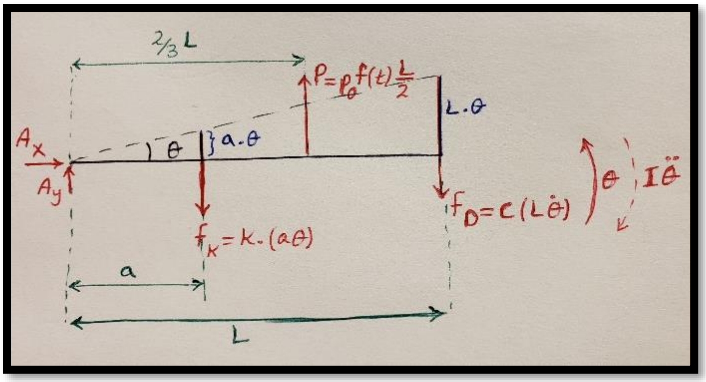
$$m\ddot{u} + ku = f(t)$$

$$m = mL^2 \quad , \quad k = ka^2 \Rightarrow \boxed{\omega_n = \frac{a}{L}\sqrt{\frac{k}{m}}}$$

Example 4: A rigid bar with a mass of "m" and length of "L" is pinned at point A to the wall. This bar is supported by a damper with a coefficient of "c" at point "C" and a spring with stiffness of "k" at point "B". This bar is subjected to a triangular distributed load ($p(x,t) = p_0\frac{x}{L}f(t)$). Find the equation of motion and natural frequency of the system?



Free Body Diagram:



Note: If angle θ is small enough, we can assume displacement at point "B" and "C" in the Free Body diagram will be equal to $a.\theta$ & $L.\theta$ respectively.

Note: Force of spring would be $f_k = k(a.\theta)$ & force of damp would be $f_D = c(L.\dot{\theta})$
Note: The distributed force can be replaced by a single concentrated force at 2/3 of the length of the beam.

$\sum M_A = 0$

$$I\ddot{\theta} + f_D(L) + f_k(a) = \left[p_0 f(t)\frac{L}{2}\right]\left(\frac{2}{3}L\right)$$

$$I = \frac{mL^2}{3}$$

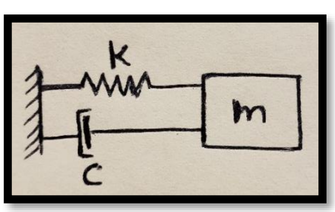
$$\left(\frac{mL^2}{3}\right)\ddot{\theta} + c(L.\dot{\theta})(L) + k(a.\theta)(a) = \left[p_0 f(t)\frac{L}{\cancel{2}}\right]\left(\frac{\cancel{2}}{3}L\right)$$

$$\boxed{\left(\frac{mL^2}{3}\right)\ddot{\theta} + (cL^2)\dot{\theta} + (ka^2)\theta = p_0\frac{L^2}{3}f(t)}$$ Equation of Motion

$$m\ddot{u} + c\dot{u} + ku = f(t)$$

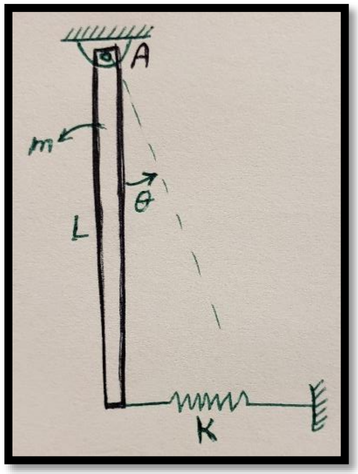
$$m = \frac{mL^2}{3}\ , \quad k = ka^2 \Rightarrow \boxed{\omega_n = \frac{a}{L}\sqrt{\frac{3k}{m}}}$$

This system is equivalent to:



Model a System as S.D.O.F

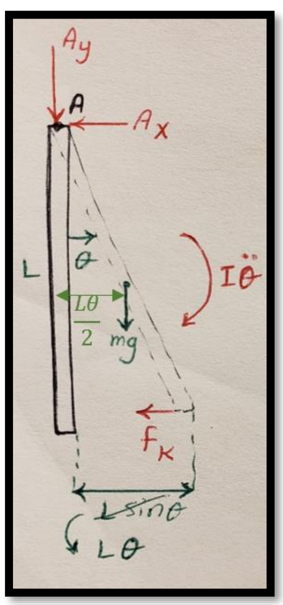
Example 5: Look at following simple structure. That includes a vertical rigid bar with mass of "m" and length of "L" which is pinned at top and attached to a horizontal spring with a coefficient (k) from bottom. This bar can rotate back and forth about the pinned point. Find the equivalent mass, equivalent stiffness, and equivalent natural frequency.



Free Body Diagram:

We will have 5 forces on this system:

- Two forces in x & y directions to hold the bar
- If we assume θ be small then the reaction force from the spring would be $f_k = \text{k}.(L\theta)$
- Force from the weight of bar (mg) which applied to the center of mass
- Force of inertia which is equal to $I\ddot{\theta}$



$\sum M_A = 0$

$I\ddot{\theta} + \left(mg\frac{L\theta}{2}\right) + f_k(L) = 0$

I for the rigid bar about point "A" would be $I = \frac{mL^2}{3}$ & $f_k = k.L\theta$.

$$\left(\frac{mL^2}{3}\right)\ddot{\theta} + \frac{mgL}{2}\theta + k.L\theta(L) = \left(\frac{mL^2}{3}\right)\ddot{\theta} + \frac{mgL}{2}\theta + kL^2\theta = \left(\frac{mL^2}{3}\right)\ddot{\theta} + \left(\frac{mgL}{2} + kL^2\right)\theta = 0$$

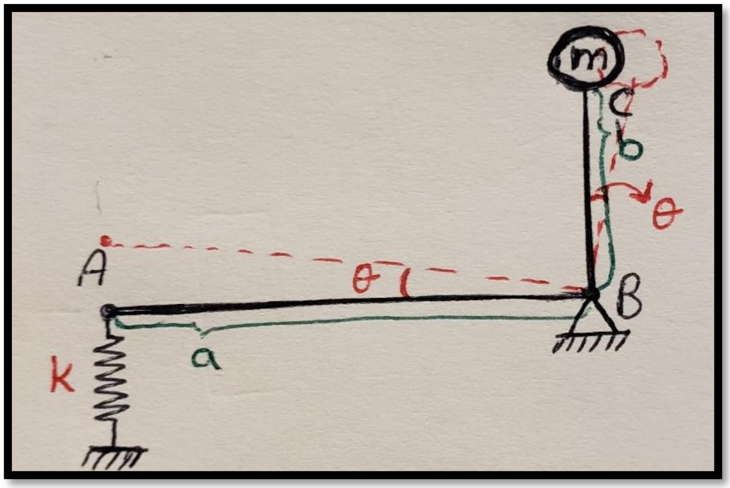
$$m\ddot{u} + ku = f(t)$$

$$\boxed{m_{equivalent} = \frac{mL^2}{3}}$$

$$\boxed{k_{equivalent} = \frac{mgL}{2} + kL^2}$$

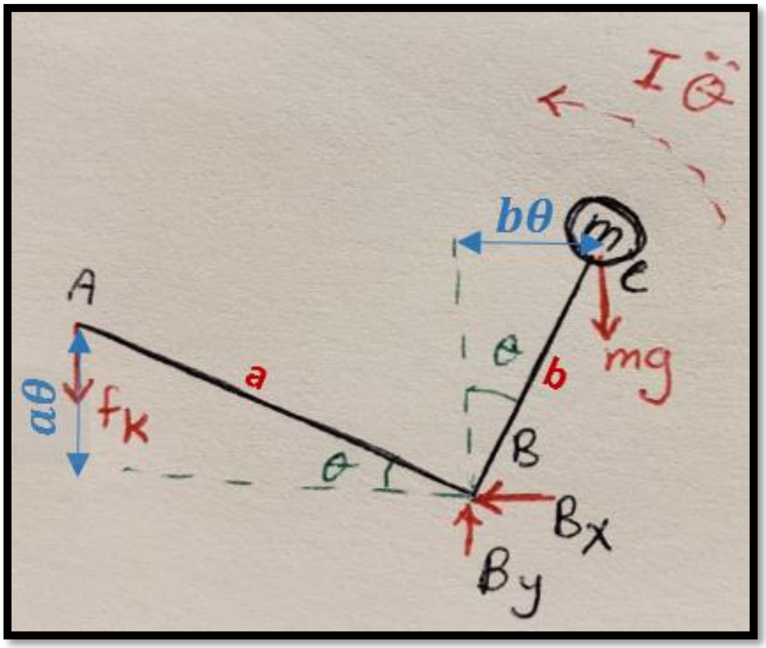
$$\boxed{\omega_n = \sqrt{\frac{\left(\frac{mgL}{2}+kL^2\right)}{\left(\frac{mL^2}{3}\right)}} = \sqrt{\frac{\left(\frac{mg}{2}+kL\right)}{\left(\frac{mL}{3}\right)}}}$$

Example 6: In the following structure, we have a massless L shaped rigid bar. A lumped mass attached to the point "C" of this bar. This bar is supported at point B and also supported at point "A" with a spring with a coefficient of "k". Other information for this structure is shown in the figure. Find the equivalent mass, equivalent stiffness, and equivalent natural frequency.



Free Body Diagram:

- Because bar assumed to be massless, so "$I\ddot{\theta}$" only related to the lumped mass.
- Remember "$I\ddot{\theta}$" is the force for D' Alembert's principle and it is always in opposite direction of motion.
- "m" is a point mass rotating about point "B", so $I = mb^2$.
- If we assume θ be small then the reaction force from the spring would be $f_k = \text{k}.(a\theta)$



$\sum M_B = 0$

$I\ddot{\theta} - mg(b\theta) + f_k(a) = 0$

$(mb^2)\ddot{\theta} - (mgb)\theta + k.a\theta(a) = (mb^2)\ddot{\theta} - (mgb)\theta + ka^2(\theta) = (mb^2)\ddot{\theta} + (ka^2 - mgb)\theta = 0$

$m\ddot{u} + ku = f(t)$

$\boxed{m_{equivalent} = mb^2}$

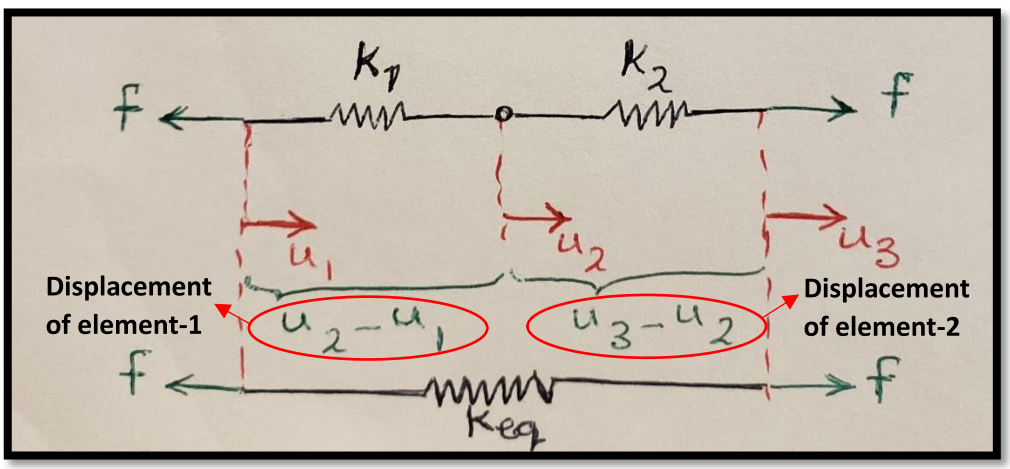
$\boxed{k_{equivalent} = ka^2 - mgb}$

$\boxed{\omega_n = \sqrt{\frac{(ka^2 - mgb)}{(mb^2)}}}$

2) Using Structural Analysis Approach

Structural analysis approach: Usually when focus is on finding equivalent stiffness of system (hard to find) and equivalent of mass is easy to find (using technics like: superposition , definition of stiffness/flexibility, combination of stiffness elements with using the concept of parallel and series springs)
In modeling a structure, the most important part is related to equivalent of stiffness element of the structure. In reality most of the structures made of many pieces (e.g. truss, frame, building, which are made of lots of loaded members) and each of these pieces have their own stiffness can be represent by a spring. For such a complex structure like that, how do you add up the effect of all of them and calculate the equivalent stiffness of the system?
In this part we will study two cases to see how we can replace all the stiffness elements in the structure with a single stiffness element.
Case 1: Spring in Series: If all stiffness elements (springs) experience the same force.

Example 7: There are two bars with different geometric and material properties connected to each other and subjected to an axial load. This can be replace by two springs with different stiffness but they are subjected to the same force. So, these two elements (springs) will be deformed differently. That is possible to replace these two elements with a single spring?



In this case, both elements are subjected to the same force. For these two springs we can write:

$f = k_1(u_2 - u_1)$ or $\frac{f}{k_1} = (u_2 - u_1)$ (1)

$f = k_2(u_3 - u_2)$ or $\frac{f}{k_2} = (u_3 - u_2)$ (2)

For the equivalent spring we will have:

$f = k_{eq}(u_3 - u_1)$ or $\frac{f}{k_{eq}} = (u_3 - u_1)$ (3)

But the summation of right side of equations 1 & 2 would be equal right side of equation 3. So, the equivalent stiffness of two parallel springs can be find from following equation:

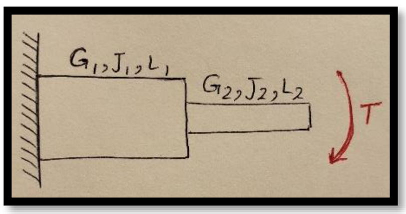
$$(u_2 - u_1) + (u_3 - u_2) = (u_3 - u_1) \Rightarrow \frac{\cancel{f}}{k_1} + \frac{\cancel{f}}{k_2} = \frac{\cancel{f}}{k_{eq}} \Rightarrow \boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}} \text{ or } \boxed{k_{eq} = \frac{k_1 k_2}{k_1 + k_2}}$$

For "n" springs in series:

$$\boxed{\frac{1}{k_{eq}} = \sum_{i=1}^{n} \frac{1}{k_i}}$$

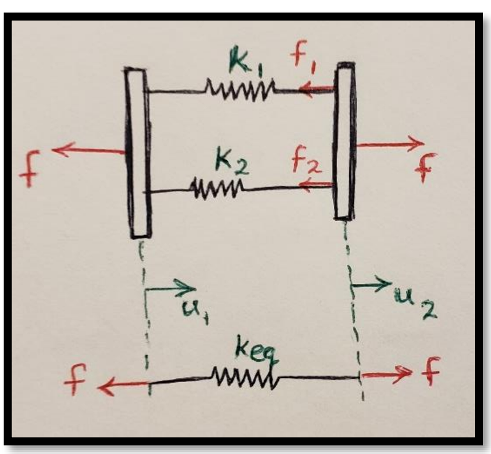
Example 8: We have two shafts with different diameters and properties, if they are subjected a same torsional deformation, what is the stiffness of each element and equivalent stiffness?

$$k_{t1} = \frac{G_1 J_1}{L_1} \qquad k_{t2} = \frac{G_2 J_2}{L_2} \qquad k_{t\_eq} = \frac{\frac{G_1 J_1}{L_1} \times \frac{G_2 J_2}{L_2}}{\frac{G_1 J_1}{L_1} + \frac{G_2 J_2}{L_2}} = \frac{\frac{G_1 G_2 J_1 J_2}{\cancel{L_1 L_2}}}{\frac{L_2 G_1 J_1 + L_1 G_2 J_2}{\cancel{L_1 L_2}}} = \boxed{\frac{G_1 G_2 J_1 J_2}{L_2 G_1 J_1 + L_1 G_2 J_2}}$$



Case 2: Spring in Parallel: If all stiffness elements (springs) experience the same deformation.

Example 9: There are two bars with different geometric and material properties and in a parallel form connected to each other. This can be replace by two springs with different stiffness. These two elements (springs) will have same deformation. That is possible to replace these two springs with a single spring?



From the free body diagram, we have:

$f = f_1 + f_2$

f_1, f_2, and f can be find from following equations:

$f_1 = k_1(u_2 - u_1)$
$f_2 = k_2(u_2 - u_1)$
$f = k_{eq}(u_2 - u_1)$

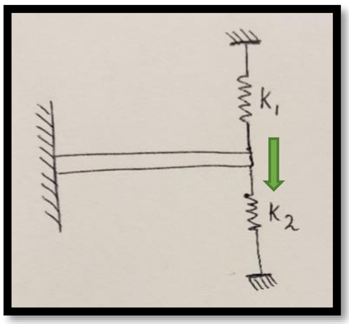
So,
$k_{eq}\cancel{(u_2 - u_1)} = k_1\cancel{(u_2 - u_1)} + k_2\cancel{(u_2 - u_1)} \Longrightarrow \boxed{k_{eq} = k_1 + k_2}$

For "n" parallel springs:

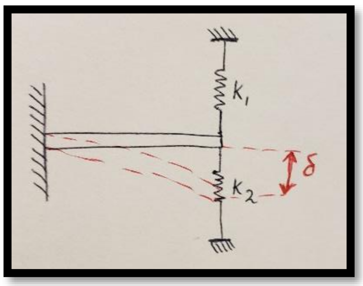
$$\boxed{k_{eq} = \sum_{i=1}^{n} k_i}$$

Right now, the main challenges is related to find out if the springs are parallel or series with each other.

Example 10: look at following Figure. There is a beam which is connected to two springs at the end. This beam start to oscillate up and down. Is this structure a parallel or series case?



It seems the springs are align with each other and experience same force so it should be a series case but it's not! Actually beam has a deformation of δ and both springs also have δ deformation too, so this structure is a parallel case!



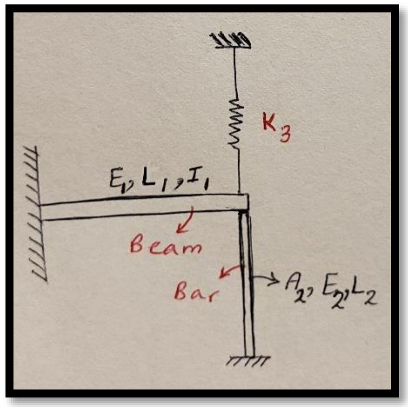
Note: That is so important to find if a structure is acting as a parallel case or series case. For instance, in the above example, if we assumed that as a series case, if $k_1$and k_2 are small values (weak stiffness), based on series case equation, increasing the stiffness of bar would not effect on equivalent stiffness (basically that part of equation goes to zero) while if it is assumed as a parallel case, then increasing the stiffness of bar will effect a lot on the equivalent stiffness.

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \overbrace{\frac{1}{\underbrace{\cancel{k_3}}_{\text{Big number}}}}^{\text{Zero}}$$

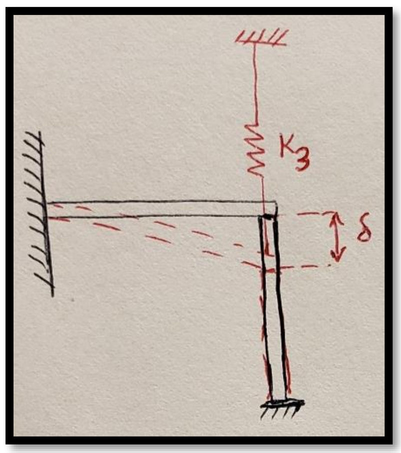
$$\overbrace{\cancel{k}_{eq}}^{\text{Big number}} = k_1 + k_2 + \overbrace{\cancel{k_3}}^{\text{Big number}}$$

2) Using Structural Analysis Approach

Example 11: Find the equivalent stiffness for the following structure.



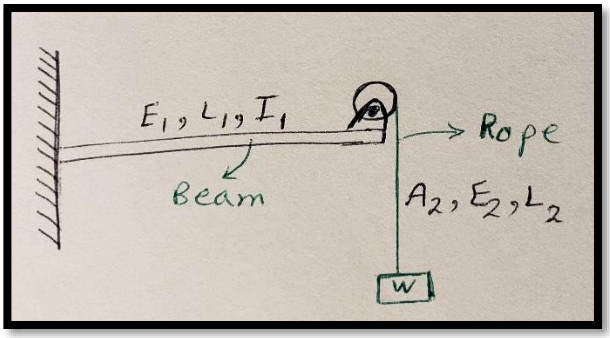
Free Body Diagram:



This is Parallel case. So, we will have:

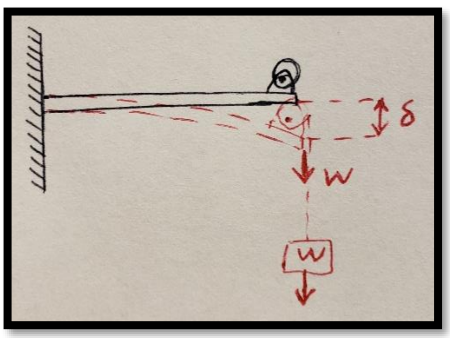
$$k_{equivalent} = \underbrace{\frac{3E_1I_1}{{L_1}^3}}_{\text{Beam}} + \underbrace{\frac{A_2E_2}{L_2}}_{\text{Bar}} + \underbrace{k_3}_{\text{Spring}}$$

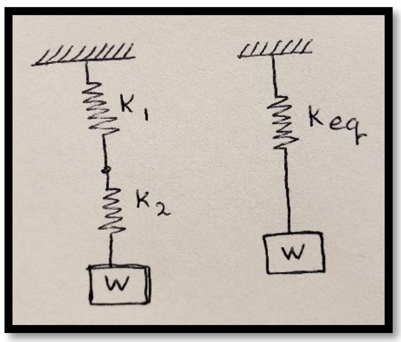
Example 12: Find the equivalent stiffness for the following structure.



Free Body Diagram:

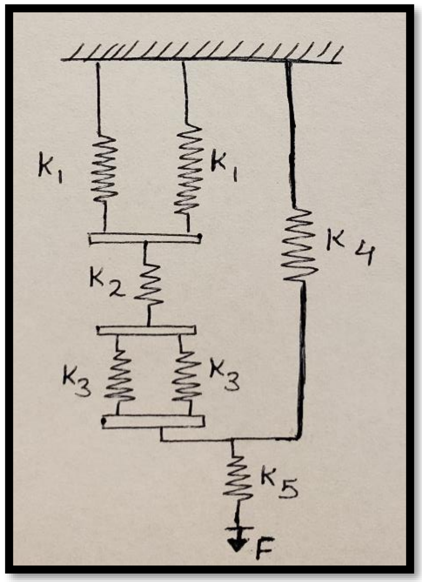
In this case, same force (W) is applied on both beam and rope but the rope and beam have different displacements. So, this structure can be equivalent to series springs.

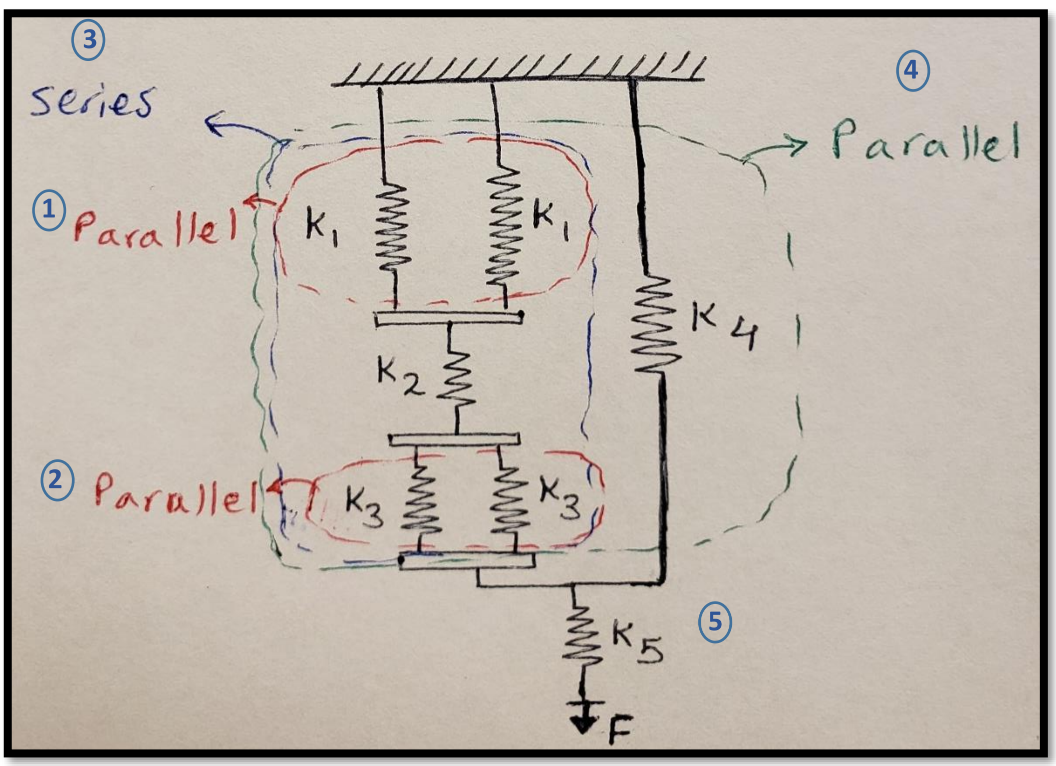




$$\boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{\frac{3E_1I_1}{{L_1}^3}} + \frac{1}{\frac{A_2E_2}{L_2}} = \frac{{L_1}^3}{3E_1I_1} + \frac{L_2}{A_2E_2}}$$

Example 13: Find the equivalent stiffness for the following structure.





Step 1: Springs with stiffness k_1 are parallel to each other.

$$k_{eq\_1} = k_1 + k_1 = 2k_1$$

Step 2: Springs with stiffness k_3 are parallel to each other.

$$k_{eq\_2} = k_3 + k_3 = 2k_3$$

Step 3: The equivalent of springs (k_{eq_1}) and spring (k_2) and equivalent of springs (k_{eq_3}) are series with each other.

$$\frac{1}{k_{eq\_3}} = \frac{1}{k_{eq\_1}} + \frac{1}{k_2} + \frac{1}{k_{eq\_2}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} = \frac{k_2k_3 + 2k_1k_3 + k_1k_2}{2k_1k_2k_3}$$

$$k_{eq\_3} = \frac{2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2}$$

Step 4: The equivalent of springs (k_{eq_3}) and spring (k_4) are parallel to each other.

$$k_{eq\_4} = k_{eq\_3} + k_4 = \frac{2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2} + k_4 = \frac{k_4(k_2k_3 + 2k_1k_3 + k_1k_2) + 2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2}$$

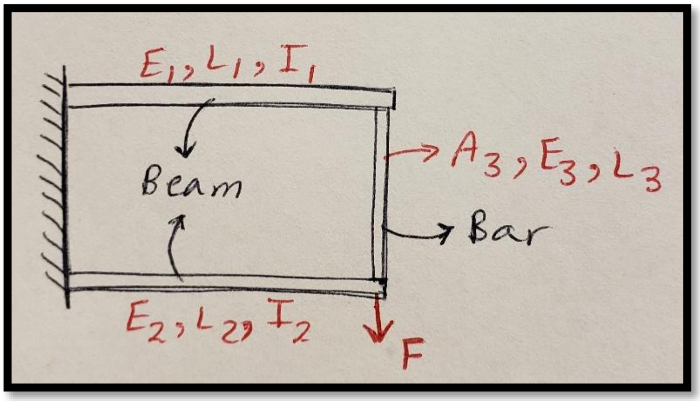
Step 5: The equivalent of springs (k_{eq_4}) and spring (k_5) are series with each other.

$$\frac{1}{k_{eq\_total}} = \frac{1}{k_{eq\_4}} + \frac{1}{k_5} = \frac{1}{\frac{k_4(k_2k_3 + 2k_1k_3 + k_1k_2) + 2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2}} + \frac{1}{k_5}$$

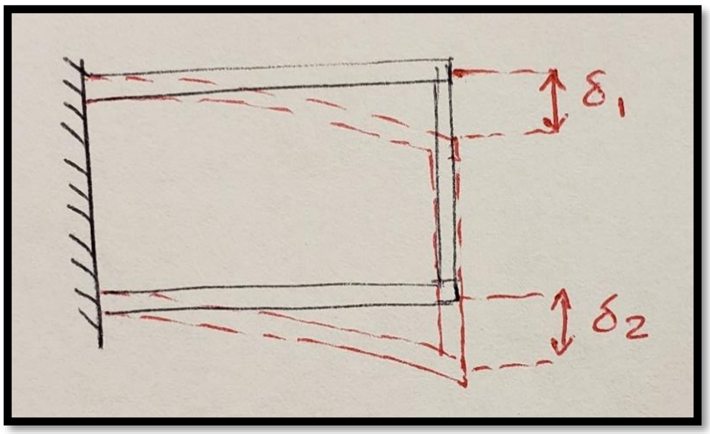
$$\boxed{k_{eq\_total} = \frac{k_2k_3k_4k_5 + 2k_1k_3k_4k_5 + k_1k_2k_4k_5 + 2k_1k_2k_3k_5}{k_2k_3k_4 + k_2k_3k_5 + k_1k_3k_4 + 2k_1k_3k_5 + k_1k_2k_4 + k_1k_2k_5 + 2k_1k_2k_3}}$$

2) Using Structural Analysis Approach

Example 14: Find the equivalent stiffness for the following structure.



Free Body Diagram:



For the beams and bar in this structure we have same force but different displacement so they are series with each other. The equivalent stiffness would be equal to:

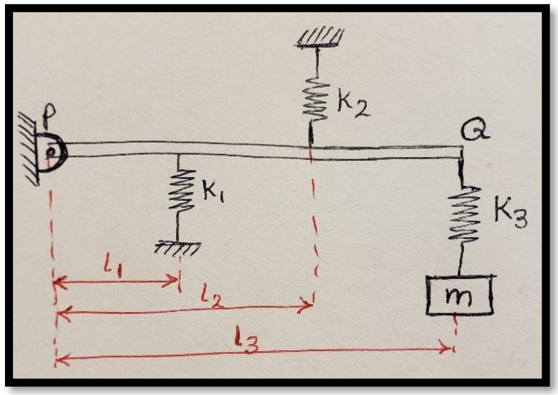
$k_1 = \frac{3E_1I_1}{{L_1}^3}$

$k_2 = \frac{3E_2I_2}{{L_2}^3}$

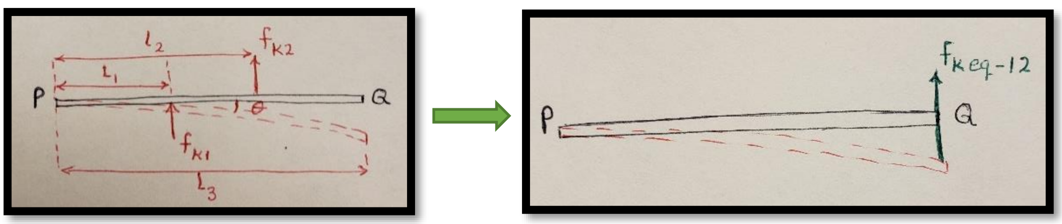
$k_3 = \frac{A_3E_3}{L_3}$

$$\boxed{\frac{1}{k_{eq}} = \frac{1}{\frac{3E_1I_1}{{L_1}^3}} + \frac{1}{\frac{3E_2I_2}{{L_2}^3}} + \frac{1}{\frac{A_3E_3}{L_3}} = \frac{{L_1}^3}{3E_1I_1} + \frac{{L_2}^3}{3E_2I_2} + \frac{L_3}{A_3E_3}}$$

Example 15: Three springs and a mass are attached to a rigid, weightless bar PQ as shown in following figure. Find the natural frequency of vibration of the system.



Free Body Diagram:



There are different methods to solve this problem. In this case because we don't know about relation between third spring and two others, we are going to use equivalent force system. Based on this method, we can combine springs 1 & 2 and put an equivalent spring for them at the end of the beam (Q). Then it possible to find the equivalent for whole structure.

For finding the equivalent of spring 1 &2, the moment respect to point P have to remain same.

$$f_{k1} = k_1(l_1\theta)$$
$$f_{k2} = k_2(l_2\theta)$$
$$f_{k\ eq-12} = k_{eq-12}\ (l_3\theta)$$

$$\sum M_P = f_{k\ eq-12}\,.\, l_3 = f_{k1}.\, l_1 + f_{k2}.\, l_2$$

$$k_{eq-12}\ (l_3\theta).\, l_3 = k_1(l_1\theta).\, l_1 + k_2(l_2\theta).\, l_2$$
$$k_{eq-12}\ l_3{}^2\cancel{\theta} = k_1 l_1{}^2\cancel{\theta} + k_2 l_2{}^2\cancel{\theta}$$

$$\boxed{k_{eq-12} = k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2}$$

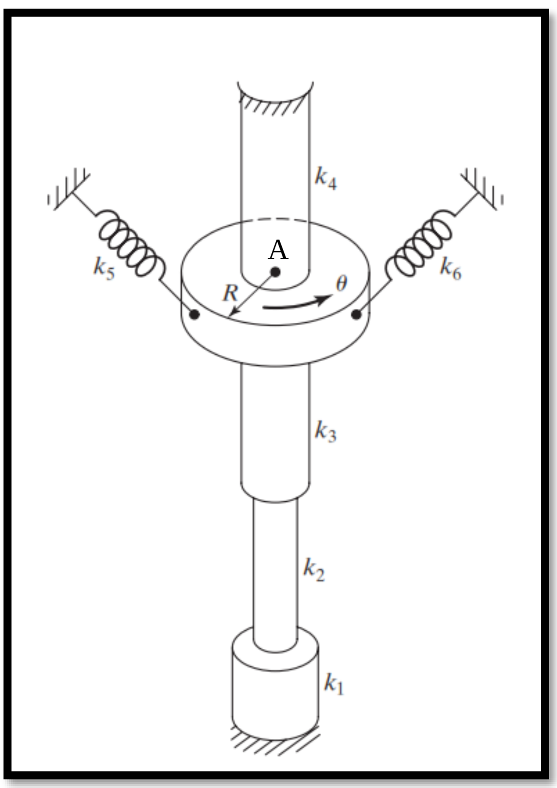
The third spring and this equivalent spring are series with each other. So, the total equivalent of these three springs would be:

$$\frac{1}{k_{eq\_total}} = \frac{1}{k_{eq-12}} + \frac{1}{k_3} = \frac{1}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2} + \frac{1}{k_3}$$

$$\boxed{k_{eq\_total} = \frac{\left[k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right]k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}}$$

$$\boxed{\omega_n = \sqrt{\frac{k_1k_3{l_1}^2 + k_2k_3{l_2}^2}{m\left(k_1{l_1}^2 + k_2{l_2}^2 + k_3{l_3}^2\right)}}}$$

Example 16: Find the equivalent stiffness for the following structure. Assume that k_1, k_2, k_3, and k_4 are torsional and k_5 and k_6 are linear spring constants.



First of all, the rotation is happened on the disk between shaft 3 and 4. So, the shaft 1, 2, and 3 will experience same force and they are series with each other. On the other hands, shaft 4 and equivalent of shafts 1-3 are in different sides of the moving disk so they will have same deformation and parallel to each other. Springs 5 and 6 both have same deformation and parallel to each other and rest of the system. However, for springs, we have to change the linear motion to rotational motion to be able to adding them to the shafts. In this case, we can use the moment relation between the spring and shafts.

Step 1:

$$\frac{1}{k_{eq\_1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \qquad k_{eq\_1} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

Step 2:

$$k_{eq\_2} = k_{eq\_1} + k_4 = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1} + k_4 = \frac{k_1 k_2 k_3 + k_4(k_1 k_2 + k_2 k_3 + k_3 k_1)}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

Step 3:

$$k_{eq\_3} = k_5 + k_6$$

Step 4:

$$\sum M = R.F_k$$

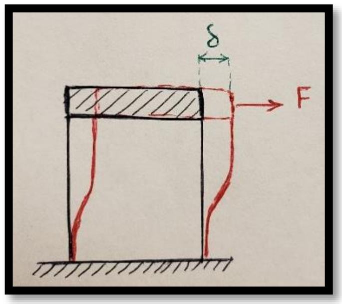
For θ rotation:

$$k_{eq\_total}.(\cancel{\theta}) = R.\overbrace{k_{eq\_3}.(R\cancel{\theta})}^{F_k} + \overbrace{k_{eq\_2}.(\cancel{\theta})}^{\textbf{Moment from the shafts}}$$

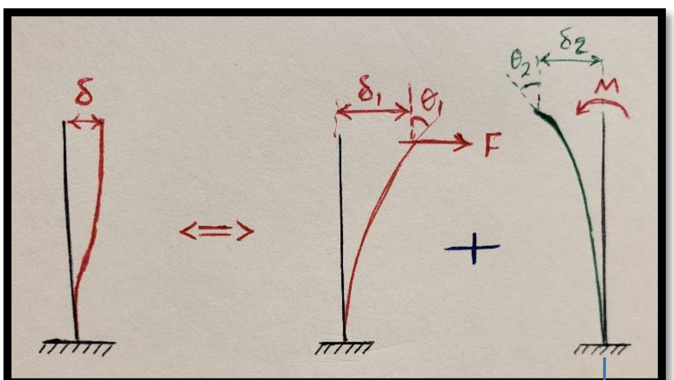
$$\boxed{k_{eq\_total} = k_{eq\_3}.R^2 + k_{eq\_2} = (k_5 + k_6).R^2 + \frac{k_1 k_2 k_3 + k_4(k_1 k_2 + k_2 k_3 + k_3 k_1)}{k_1 k_2 + k_2 k_3 + k_3 k_1}}$$

2) Using Structural Analysis Approach

Example 17: In the following figure, we have a rigid mass that rigidly connected to the ground. Find the equivalent stiffness for this structure.



If there is a rigid connection between the mass and supports, there is not any rotation happened between them ($\theta = 0$) and we only have δ displacement. For solving this problem, we will use the superposition method.



Internal moment because of rigid connection

We know $\theta = 0$, so $\theta_1 + \theta_2 = 0$ and $\delta = \delta_1 + \delta_2$.

From mechanical of material:

$\delta_1 = \frac{FL^3}{3EI}$, $\theta_1 = \frac{FL^2}{2EI}$

$\delta_2 = -\frac{ML^2}{2EI}$, $\theta_2 = -\frac{ML}{EI}$

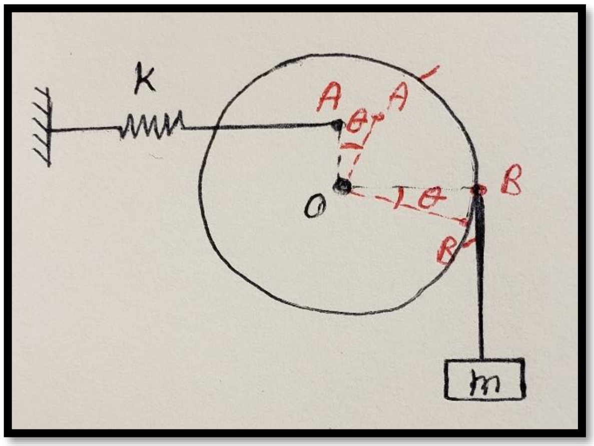
$$\theta_1 + \theta_2 = \frac{FL^2}{2EI} - \frac{ML}{EI} = 0 \Rightarrow \frac{FL^{\cancel{2}}}{2\cancel{EI}} = \frac{M\cancel{L}}{\cancel{EI}} \Rightarrow \boxed{M = \frac{FL}{2}}$$

$$\delta = \delta_1 + \delta_2 = \frac{FL^3}{3EI} - \frac{ML^2}{2EI} = \frac{FL^3}{3EI} - \frac{\left(\frac{FL}{2}\right)L^2}{2EI} = \frac{FL^3}{3EI} - \frac{FL^3}{4EI} = \frac{FL^3}{12EI} \Rightarrow \boxed{\delta = \frac{FL^3}{12EI}}$$

So, stiffness for each support element would be equal: $\overset{1}{\delta} = \frac{\overset{k}{F}L^3}{12EI} \Rightarrow \boxed{k = \frac{12EI}{L^3}}$

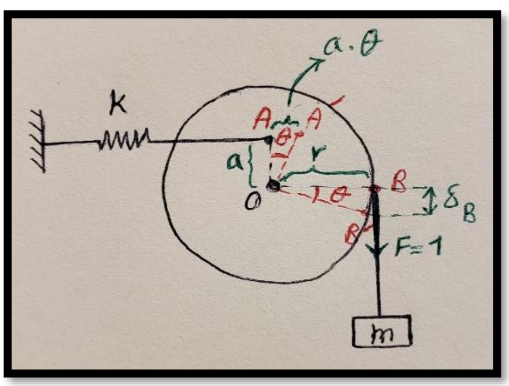
Total stiffness of structure have to multiply by two. $\boxed{k_{eq} = \frac{24EI}{L^3}}$

Example 18: In the following figure, we have a weightless disk with a spring with coefficient of "k" attached to the wall. This disk can rotate around point "O". A mass with a non-deformable cable is connected to this disk. Find the equivalent stiffness and natural frequency for this structure.

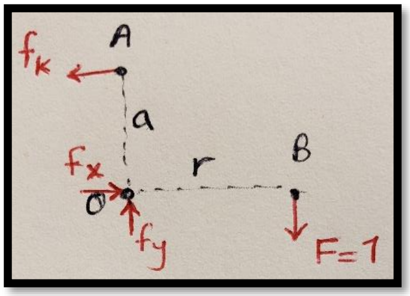


If we want to solve this problem from the main definition of stiffness which is force based on unit displacement, it would be hard to identify the unit displacement. So, in this case we will use a method is named Flexibility.

Flexibility (the displacement for unit force) is the inverse of equivalent stiffness $\left(\frac{1}{k_{eq}}\right)$ or in other words, it is the displacement based on the unit force.



Free Body Diagram:



$\sum M_O = 0$

$$\left.\begin{array}{ll} f_k.a = r.1 & k(a\theta) = r \\ r\theta = \delta_B & \theta = \frac{\delta_B}{r} \end{array}\right] \Longrightarrow k\left(a\frac{\delta_B}{r}\right)a = r \Longrightarrow \boxed{\delta_B = \frac{r^2}{k.a^2}}$$

As it mentioned before in the flexibility method, the displacement for unit force (flexibility) equal $\left(\frac{1}{k_{eq}}\right)$. So, the equivalent stiffness and natural frequency for this example would be:

$$\frac{1}{k_{eq}} = \frac{r^2}{k.a^2} \Longrightarrow \boxed{k_{eq} = \frac{k.a^2}{r^2}} \Longrightarrow \boxed{\omega_n = \frac{a}{r}\sqrt{\frac{k}{m}}}$$

3) Energy method

Energy method is based on:

A) Conservation of energy

   Summation of kinetic and potential energy is constant: $Ke + Pe = E$

B) Rayleigh's method

C) Lagrange's approach

A) Conservation of energy

In conservation of energy, we say the total energy of system is made of two major source kinetic energy (Ke) and potential energy (Pe).

$$\boxed{Pe + Ke = E_{total}}$$

Also, $E_{total} = constant$ so:

$$\frac{d}{dt}E_{total} = 0 \quad \Longrightarrow \quad \boxed{\frac{d}{dt}(Pe + Ke) = 0}$$

If you remember, the kinetic and potential energy of a spring respectively equal to:

$$Ke = \frac{1}{2}m\dot{u}^2 \qquad Pe = \frac{1}{2}ku^2$$

- Let's check this method for the general solution of displacement (based on initial conditions) of an undamped system. From above equations if velocity become zero, the kinetic energy become zero and potential energy become maximum. On the other hands, if displacement become zero the potential energy become zero and kinetic energy become maximum.

First, we try the energy method for the case that system has initial displacement but its initial velocity is zero. ($u_0 \neq 0\,, \dot{u}_0 = 0$)

$$u(t) = u_0\cos(\omega_n\, t) + \frac{\dot{u}_0}{\omega_n}\ sin\,(\omega_n\, t) \Longrightarrow u(t) = u_0\cos(\omega_n\, t) \ \ \& \quad \dot{u}(t) = -\omega_n u_0 \sin(\omega_n\, t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$E = \frac{1}{2}ku^2 + \frac{1}{2}m\dot{u}^2 = \frac{1}{2}k[u_0\cos(\omega_n\, t)]^2 + \frac{1}{2}m[-\omega_n u_0\sin(\omega_n\, t)]^2 = \frac{1}{2}k{u_0}^2\cos^2(\omega_n\, t) + \frac{1}{2}m\left(\frac{k}{m}\right){u_0}^2\sin^2(\omega_n\, t) = \frac{1}{2}k{u_0}^2[\cos^2(\omega_n\, t) + \sin^2(\omega_n\, t)] = \boxed{\frac{1}{2}k{u_0}^2} \Longrightarrow$$ **Max potential energy**

Now, we try the energy method for the case that system has initial velocity but its initial displacement is zero. ($\dot{u}_0 \neq 0\,, u_0 = 0$)

$\Longrightarrow$

$\Longrightarrow$ **Max kinetic energy**

$u(t) = u_0 \cos(\omega_n\, t) + \frac{\dot{u}_0}{\omega_n}\ sin\ (\omega_n\, t) \qquad u(t) = \frac{\dot{u}_0}{\omega_n}\ sin\ (\omega_n\, t)\ \ \&\ \ \dot{u}(t) = \dot{u}_0\ cos\ (\omega_n\, t)$

$E = \frac{1}{2}ku^2 + \frac{1}{2}m\dot{u}^2 = \frac{1}{2}k\left[\frac{\dot{u}_0}{\omega_n}\ sin\ (\omega_n\, t)\right]^2 + \frac{1}{2}m[\dot{u}_0\ cos\ (\omega_n\, t)]^2 = \frac{1}{2}k{\dot{u}_0}^2\left(\frac{m}{k}\right)\sin^2(\omega_n\, t) + \frac{1}{2}m\,{\dot{u}_0}^2\cos^2(\omega_n\, t) = \frac{1}{2}m{\dot{u}_0}^2[\sin^2(\omega_n\, t) + \cos^2(\omega_n\, t)] = \frac{1}{2}m{\dot{u}_0}^2$

As it mentioned before, $E = constant$, so: $\boxed{\frac{1}{2}k{u_0}^2 = \frac{1}{2}m{\dot{u}_0}^2 \text{ or } Pe_{max} = Ke_{max}}$

- We can write the conservation of energy equation for the general solution of displacement (in the term of single sinusoidal) for an undamped system. In this equation "*A*" is amplitude which is equal maximum displacement. If we replace the conservation energy with the values of *Pe* and *Ke* for the spring:

$\overset{\text{Max}}{u(t)} = A\ \overset{1}{\sin(\omega_n\, t + \theta)} \Longrightarrow u_{max} = A$

$\overset{\text{Max}}{\dot{u}(t)} = A\omega_n\ \overset{1}{\cos(\omega_n\, t + \theta)} \Longrightarrow \dot{u}_{max} = A\omega_n \Longrightarrow \dot{u}_{max} = \omega_n u_{max}$

We have $\dot{u}_{max}$ for maximum kinetic energy and u_{max} for maximum potential energy. From previous calculation we have:

$\cancel{\frac{1}{2}}m{\dot{u}_{max}}^2 = \cancel{\frac{1}{2}}k{u_{max}}^2 \Longrightarrow \dot{u}_{max} = \sqrt{\frac{k}{m}}u_{max}$

If we replace it in previous equation:

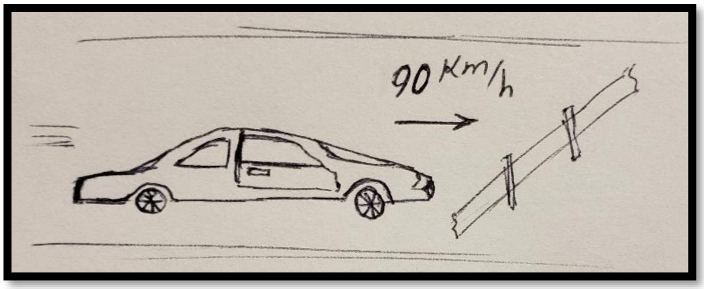
$\sqrt{\frac{k}{m}}\cancel{u_{max}} = \omega_n \cancel{u_{max}} \Longrightarrow \boxed{\omega_n = \sqrt{\frac{k}{m}}}$

- We can write the conservation of energy equation for an undamped system. If we replace the conservation energy with the values of *Pe* and *Ke* for the spring:

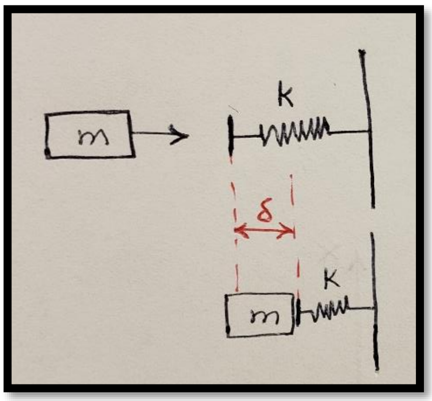
$\frac{d}{dt}(Pe + Ke) = \frac{d}{dt}\left(\frac{1}{2}ku^2 + \frac{1}{2}m\dot{u}^2\right) = 0$

$k u\cancel{\dot{u}} + m\cancel{\dot{u}}\ddot{u} = 0 \Longrightarrow \boxed{ku + m\ddot{u} = 0}$ Equation of motion for a system with spring (no damped)

Example 19: A vehicle with speed of 90 km/h hitting the guardrails in high way. This vehicle has the weight of 1000kg and assume the guardrails have a displacement of $\delta = 0.25\ m$. Find the stiffness of this guardrails (N/m). The friction would be negligible in this problem.



Free Body Diagram:



First we have to change everything in a uniform unit.

$$\dot{u} = 90\ \frac{km}{h} \times \frac{1\ h}{3600\ s} \times \frac{1000\ m}{1\ km} = 25\frac{m}{s}$$

We know $Ke_{max} = Pe_{max}$ and from the velocity of vehicle we can find the maximum kinetic energy.

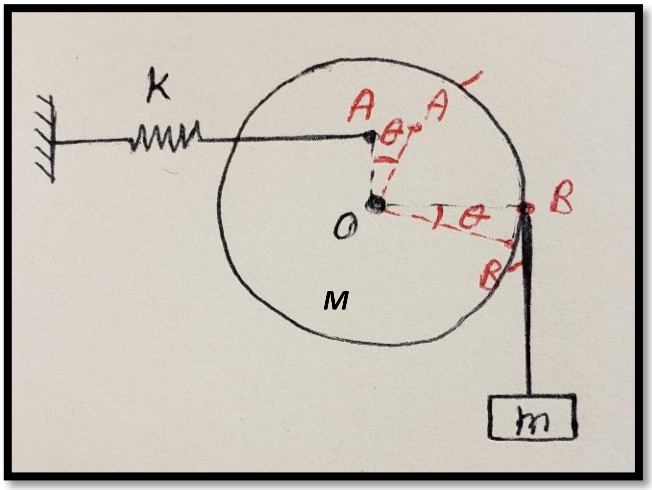
$$Ke_{max} = \frac{1}{2}m\dot{u}_0{}^2 = \frac{1}{2} \times 1000\text{ kg} \times \left(25\frac{m}{s}\right)^2 = 312500\ N.m = Pe_{max}$$

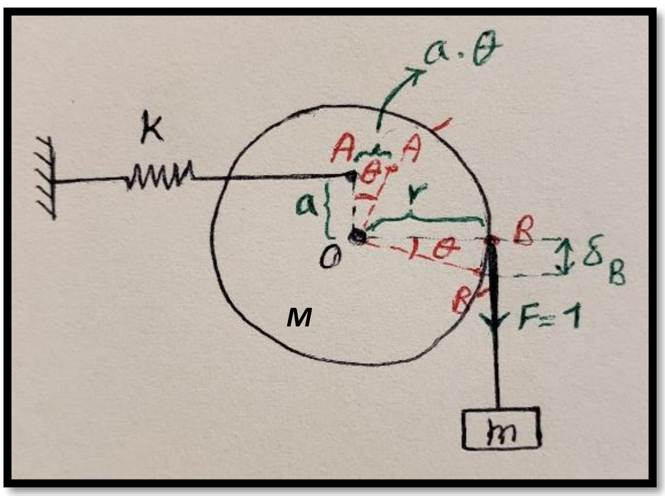
$$Pe_{max} = \ \frac{1}{2}ku_0{}^2 = \frac{1}{2}k(0.25m)^2 = 312500\ N.m$$

$$\boxed{k = \frac{312500}{0.0625} = 10{,}000{,}000\,\frac{N}{m}}$$

3) Energy method

Example 20: In the following figure, a spring with coefficient of "k" from one side attached to a disk and from other side is attached to the wall. This disk can rotate around point "O". A mass with a non-deformable cable is connected to this disk. Find the equivalent stiffness and natural frequency for this structure.





$\delta_B = r\theta \quad \& \quad I = \frac{1}{2}Mr^2$

$Ke = \frac{1}{2}m(\dot{\delta_B})^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$

$Pe = \frac{1}{2}k(a\theta)^2 = \frac{1}{2}ka^2\theta^2$

$\frac{d}{dt}(Pe + Ke) = 0$

$\frac{d}{dt}(Pe) = ka^2\theta\dot{\theta}$

$$\frac{d}{dt}(Ke) = mr^2\dot{\theta}\ddot{\theta} + I\dot{\theta}\ddot{\theta}$$

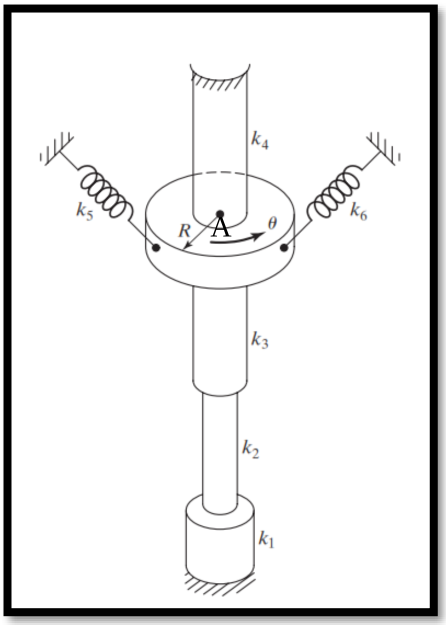
$$ka^2\theta\cancel{\dot{\theta}} + mr^2\cancel{\dot{\theta}}\ddot{\theta} + I\cancel{\dot{\theta}}\ddot{\theta} = 0 \quad \Longrightarrow \quad ka^2\theta + mr^2\ddot{\theta} + I\ddot{\theta} = (ka^2)\theta + (mr^2 + I)\ddot{\theta} = 0$$

$$\boxed{k_{eq} = ka^2}$$

$$\boxed{m_{eq} = (mr^2 + I) = \left(mr^2 + \frac{1}{2}Mr^2\right)}$$

$$\boxed{\omega_n = \sqrt{\frac{ka^2}{\left(mr^2 + \frac{1}{2}Mr^2\right)}}}$$

Example 21: Find the equivalent stiffness for the following structure. Assume that k_1, k_2, k_3, and k_4 are torsional and k_5 and k_6 are linear spring constants.



Step 1:

$$\frac{1}{k_{eq\_1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \qquad k_{eq\_1} = \frac{k_1k_2k_3}{k_1k_2 + k_2k_3 + k_3k_1}$$

Step 2:

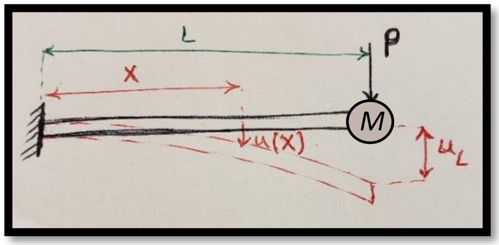
$$\cancel{\frac{1}{2}}k_{eq\_total}\,.\cancel{\theta^2} = \frac{1}{2}k_{eq\_1}\,.\theta^2 + \frac{1}{2}k_4\,.\theta^2 + \frac{1}{2}k_5\,.(R\theta)^2 + \frac{1}{2}k_6\,.(R\theta)^2 = \cancel{\frac{1}{2}}[k_{eq\_1}\,.\cancel{\theta^2} + k_4\,.\cancel{\theta^2} + k_5\,.R^2\cancel{\theta^2} + k_6\,.R^2\cancel{\theta^2}]$$

$$\boxed{k_{eq\_total} = k_{eq\_1} + k_4 + k_5\,.R^2 + k_6\,.R^2}$$

$$\boxed{k_{eq\_total} = \frac{k_1k_2k_3}{k_1k_2 + k_2k_3 + k_3k_1} + k_4 + k_5\,.R^2 + k_6\,.R^2 = (k_5 + k_6).R^2 + \frac{k_1k_2k_3 + k_4(k_1k_2 + k_2k_3 + k_3k_1)}{k_1k_2 + k_2k_3 + k_3k_1}}$$

B) Rayleigh's method

Let's imagine a simple system with only a beam with a mass at one end but this time beam has weight (distributed through the beam). In other words, we can't simply replace the weight by a lumped force and basically it includes infinite lumped forces.



The static deflection of the beam:

$$u(x) = \frac{Px^2}{6EI}(3L - x)$$

$$u_{max} = u(L) = \frac{PL^2}{6EI}(3L - L) = \frac{PL^3}{3EI}$$

$$u(x) = \frac{u_{max}\, x^2}{2L^3}(3L - x) = \frac{u_{max}}{2L^3}(3\, x^2 L - \, x^3)$$

The total kinetic energy of the beam:

$$\dot{u}(x) = \frac{\dot{u}_{max}}{2L^3}(3\, x^2 L - \, x^3)$$

The maximum kinetic energy of the beam itself (Ke_{max}) where m is the total mass and $\left(\frac{m}{L}\right)$ is the mass per unit length of the beam.

$$Ke_{max} = \int_0^L \frac{1}{2} . \frac{m}{L} . (\dot{u}(x))^2 dx = \frac{m}{2L}\int_0^L \left(\frac{\dot{u}_{max}}{2L^3}(3\, x^2 L - \, x^3)\right)^2 dx$$

$$= \frac{m}{2L}\frac{\dot{u}^2{}_{max}}{4L^6}\int_0^L 9\, x^4 L^2 + \, x^6 - 6\, x^5 L \, dx = \frac{m\, \dot{u}^2{}_{max}}{8\cancel{L^7}}\left(\frac{33\cancel{L^7}}{35}\right) = \left(\frac{33}{280}m\right)\dot{u}^2{}_{max}$$

We know $Ke_{max} = \frac{1}{2}m\dot{u}^2{}_{max}$, so:

$$Ke_{max} = \frac{1}{2}m_{eq}\, \cancel{\dot{u}^2}{}_{max} = \left(\frac{33}{280}m\right)\cancel{\dot{u}^2}{}_{max} \qquad m_{eq} = \frac{33}{140}m$$

Thus the total effective mass acting at the end of the beam would be:

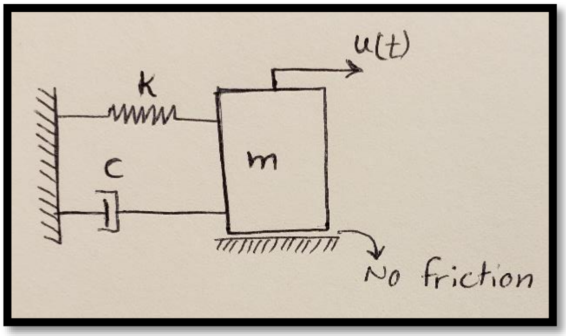
$$M_{total} = M + m_{eq}$$

Natural frequency of the system would be:

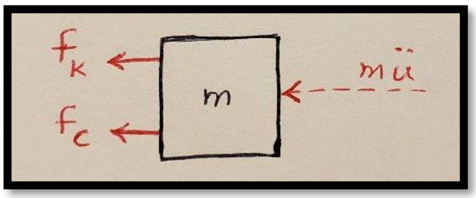
$$\boxed{\omega_n = \sqrt{\frac{k}{M_{total}}} = \sqrt{\frac{k}{\left(M + \frac{33}{140}m\right)}}}$$

Damped Free Vibration S.D.O.F System

Look at this system:



Free Body Diagram:



$f_k = ku$, $\quad f_c = c\dot{u}$

Equation of motion: $m\ddot{u} + c\dot{u} + ku = 0$

Let's solve this differential equation:

$$mx^2 + cx + k = 0$$

$$\boxed{x_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}}$$

It has two roots, so we will have two solutions

Two solutions:

$$u_1(t) = c_1 e^{x_1 t}$$

$$u_2(t) = c_2 e^{x_2 t}$$

General solution:

$$u(t) = c_1 e^{x_1 t} + c_2 e^{x_2 t}$$

Critical damping coefficient (c_c): The critical damping is defined as value of damping which makes $c^2 - 4mk = 0$.

$$c_c = \sqrt{4mk} = 2\sqrt{mk} \xrightarrow{\times \frac{\sqrt{m}}{\sqrt{m}}} \boxed{c_c = 2m\omega_n}$$

In this case:

$$x_{1,2} = \frac{-c_c}{2m} = \frac{-\cancel{2m}\omega_n}{\cancel{2m}} = \boxed{-\omega_n}$$

That is why it's named critical damping!

Damping ratio (ζ): Damping ratio is defined as the ratio of current damping over critical damping $\left(\frac{c}{c_c}\right)$.

$$\boxed{\zeta = \frac{c}{c_c}}$$

Based on values of ζ and ω_n, the equation of motion can be rewrite:

$$m\ddot{u} + c\dot{u} + ku = 0 \xrightarrow{\times \frac{1}{m}} \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0 \longrightarrow \boxed{\ddot{u} + 2\zeta\omega_n\dot{u} + {\omega_n}^2 u = 0}$$

$$\frac{c}{m} \longrightarrow \frac{c}{m} = \underbrace{\frac{c}{c_c}}_{\zeta} \times \frac{c_c}{m} = \boxed{2\zeta\omega_n}$$

$$c_c = 2m\omega_n \longrightarrow \frac{c_c}{m} = 2\omega_n$$

Also, we can rewrite the equation of roots:

$$x_{1,2} = \frac{\overbrace{-c}^{A} \pm \overbrace{\sqrt{c^2 - 4mk}}^{B}}{2m}$$

A:

$$\boxed{-\frac{c}{2m} = -\zeta\omega_n}$$

B:

$$\frac{\sqrt{c^2 - 4mk}}{2m} = \sqrt{\frac{c^2 - 4mk}{4m^2}} = \sqrt{\zeta^2{\omega_n}^2 - {\omega_n}^2} = i\omega_n\sqrt{1 - \zeta^2}$$

$$x_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \longrightarrow \boxed{x_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1 - \zeta^2}\right)}$$

Different solution cases based on damping values (ζ):

Case1: Underdamped system ($\zeta < 1$ or $c < c_c$ or $\frac{c}{2m} < \sqrt{\frac{k}{m}}$): We will have two complex conjugate roots.

$$x_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right)$$

$$u(t) = c_1 e^{\left(-\zeta+i\sqrt{1-\zeta^2}\right)\omega_n t} + c_2 e^{\left(-\zeta-i\sqrt{1-\zeta^2}\right)\omega_n t} = e^{-\zeta\,\omega_n t}\left[c_1 e^{i\sqrt{1-\zeta^2}\,\omega_n t} + c_2 e^{-i\sqrt{1-\zeta^2}\,\omega_n t}\right]$$

$$u(t) = e^{-\zeta\,\omega_n t}\left[(c_1+c_2)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + i(c_1-c_2)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right]$$

Note: $e^{i\theta} = cos\theta + isin\,\theta$

Damped natural frequency (ω_d): The frequency of underdamped case. $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

The rest of simplifications and steps can be find in the book but final solution will be:

$u(t) = A_1 e^{-\zeta\,\omega_n t}\cos(\omega_d t - \emptyset)$ or $u(t) = A_2 e^{-\zeta\,\omega_n t}\sin(\omega_d t + \emptyset_0)$

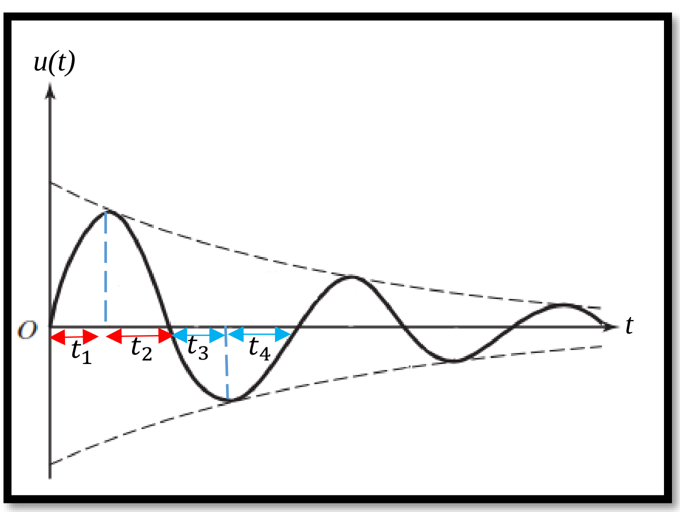
Note: $A_1, A_2, \emptyset, \emptyset_0$ will be find from initial conditions.

In terms of initial displacement and velocity:

$t = 0$ $\quad u(0) = u_0$ $\quad$ & $\quad \dot{u}(0) = \dot{u}_0$

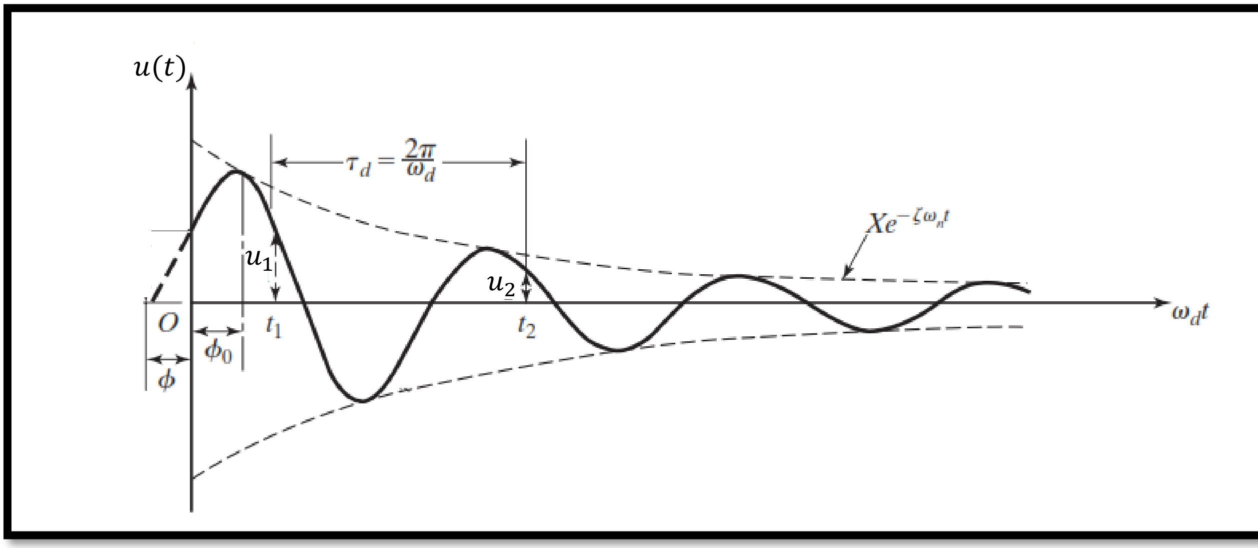
$$u(t) = e^{-\zeta\,\omega_n t}\left[u_0 cos\omega_d t + \frac{\dot{u}_0 + \zeta\,\omega_n u_0}{\omega_d} sin\omega_d t\right]$$

Note: The above equation show the $u(t)$ is a **sinusoidal function** or in other words, it is a **harmonic function**. However, the amplitude of this function because of $e^{-\zeta\,\omega_n t}$will exponentially **decrease** with time.



Note: As it is shown in above graph, for underdamped case, always going from zero to maximum displacement is much faster than return ($t_1 < t_2$ & $t_3 < t_4$, ...) like any shock absorber.

We can plot the function of $u(t)$ for a system from experimental tests (like following figure). For an underdamped system, we will be able to calculate the damping ratio (ζ) of the system from the amplitude of graph at time t_1& t_2 when t_2=t_1+T$_d$ (T$_d$: Period of damped vibration).



$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ & $\omega_d = \frac{2\pi}{T_d}$

$\frac{2\pi}{T_d} = \sqrt{1-\zeta^2}\,\omega_n \Rightarrow \boxed{T_d = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}}$

$\Delta = \frac{u_1}{u_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_2}} = \frac{\cancel{e^{-\zeta\omega_n t_1}}}{\cancel{e^{-\zeta\omega_n (t_1}}\,{}^{+T_d)}} = e^{\zeta\omega_n T_d}$

If we use natural logarithm from both side of above equation we can make it simpler:

$$\delta = ln\frac{u_1}{u_2} = \zeta\omega_n T_d = \zeta\cancel{\omega_n}\frac{2\pi}{\sqrt{1-\zeta^2}\,\cancel{\omega_n}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

In the most of the case in real structures and systems the ζ is very small. So, if that is the case the $\sqrt{1-\zeta^2} \cong 1$ and we can rewrite above equation:

$$\boxed{\delta = 2\pi\zeta} \qquad (\zeta \ll 1)$$

Damped Free Vibration S.D.O.F System

Case2: Critically damped system ($\zeta = 1$ or $c = c_c$ or $\frac{c}{2m} = \sqrt{\frac{k}{m}}$): We will have two equal real roots.

$$x_{1,2} = \omega_n\left(\cancelto{-1}{-\zeta} \pm i\cancelto{0}{\sqrt{1-\zeta^2}}\right) \Longrightarrow \boxed{x_{1,2} = -\omega_n}$$

General solution:

$$\boxed{u(t) = \underbrace{(c_1 + c_2 t)}_{\text{Linear part}}e^{-\omega_n t}}$$

Exponential part

The above equation show the $u(t)$ is combination of **linear and exponential function** and it means the system will be damped but not very quickly! Also, it is not a harmonic function, so it doesn't have repetition (non-periodic).

For finding the constant values c_1 & c_2 we need to use two initial conditions:

$t = 0$ $\qquad u(0) = u_0 \qquad$ & $\qquad \dot{u}(0) = \dot{u}_0$

$$\boxed{c_1 = u_0}$$

$$\boxed{c_2 = \dot{u}_0 + \omega_n u_0}$$

Case3: Overdamped system ($\zeta > 1$ or $c > c_c$ or $\frac{c}{2m} > \sqrt{\frac{k}{m}}$): This We will have two real roots.

$$x_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right) \Longrightarrow \boxed{x_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)}$$

General solution:

$$u(t) = c_1 e^{x_1 t} + c_2 e^{x_2 t}$$

$$\boxed{u(t) = c_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + c_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}}$$

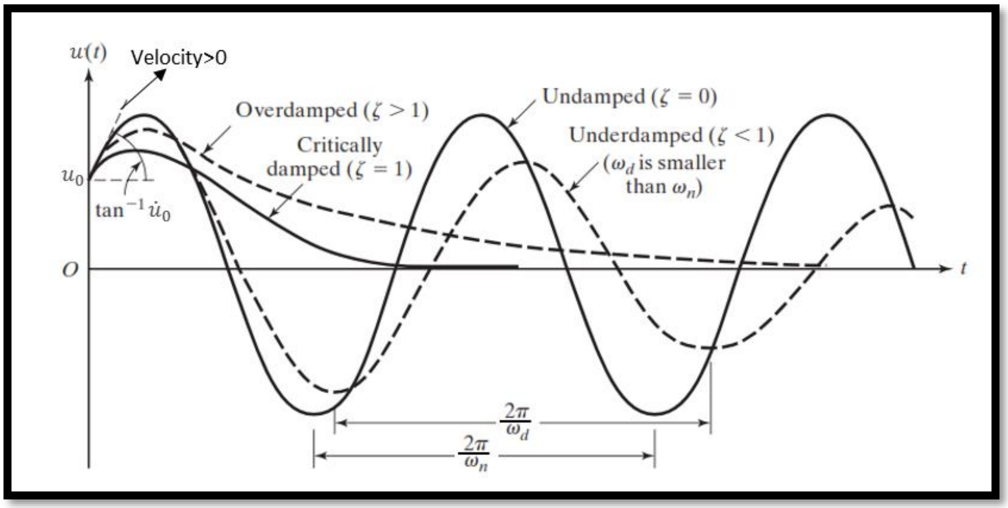
The above equation show the $u(t)$ is an **exponential function** and it means the system take infinite time to be completely damped! Also, it is not a harmonic function, so it doesn't have repetition (non-periodic).

For finding the constant values c_1 & c_2 we need to use two initial conditions:

$t = 0$ $\qquad u(0) = u_0 \qquad$ & $\qquad \dot{u}(0) = \dot{u}_0$

$$\boxed{c_1 = \frac{u_0\ \omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{u}_0}{2\ \omega_n \sqrt{\zeta^2 - 1}}}$$

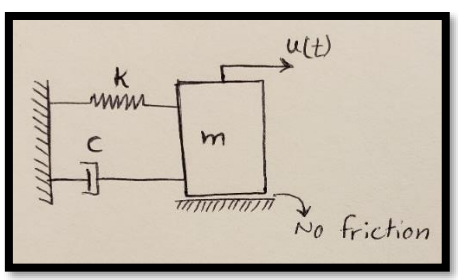
$$\boxed{c_2 = \frac{-u_0\ \omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{u}_0}{2\ \omega_n \sqrt{\zeta^2 - 1}}}$$



Watch this video for comparison between overdamped, critical damped, and underdamped systems: https://www.youtube.com/watch?v=99ZE2RGwqSM&list=LL&index=2

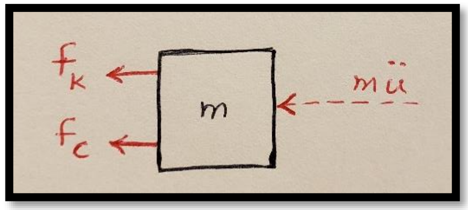
Example 22: The following system is given. The mass of the system *m=2.5kg*, coefficient of spring *k=10N/m.* If the initial conditions of the system are $u_0 = 0.05m$ & $\dot{u}_0 = 0$ .Find the response of system for following cases.

a) *If C=5* $^{N-s}/_{m}$
b) *If C=10* $^{N-s}/_{m}$
c) *If C=12* $^{N-s}/_{m}$



This is a free vibration S.D.O.F damped system. For this problem or similar cases we have to do the following steps:

Step 1: Free Body Diagram:



Step 2: Finding the natural frequency of system (ω_n):

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{2.5}} = 2\ \frac{rad}{s}$$

Step 2: Finding the critical damping coefficient of system (c_c):

$$c_c = 2m\omega_n = 2 \times 2.5 \times 2 = 10\ {}^{N-s}/_{m}$$

Step 3: Finding the damping ratio of system (ζ) and find that is related to which damping case:

a) $\zeta_1 = \frac{c_1}{c_c} = \frac{5}{10} = 0.5$ Underdamped

b) $\zeta_2 = \frac{c_2}{c_c} = \frac{10}{10} = 1$ Critically damped

c) $\zeta_3 = \frac{c_3}{c_c} = \frac{12}{10} = 1.2$ Overdamped

Step 4: Write the equation of motion (only for under-damped case, first you have to calculate damped natural frequency (ω_d)):

a) Underdamped

$$\omega_d = \sqrt{1-\zeta^2}\ \omega_n = 2 \times \sqrt{1-0.5^2} = 1.73\ \frac{rad}{s}$$

$$u(t) = e^{-\zeta\,\omega_n t}\left[u_0 cos\omega_d t + \frac{\dot{u}_0 + \zeta\,\omega_n u_0}{\omega_d} sin\omega_d t\right]$$ General solution for underdamped case

$$u(t) = e^{-0.5\times 2t}\left[0.05\cos(1.73t) + \frac{0.5\ \times 2 \times 0.05}{1.73}\sin(1.73t)\right]$$

$$\boxed{u(t) = e^{-t}[0.05\cos(1.73t) + 0.029\sin(1.73t)]}$$

b) Critically damped

$$u(t) = (c_1 + c_2 t)e^{-\omega_n t}$$ General solution for critically damped case

$$\boxed{c_1 = u_0 = 0.05}$$

$$\boxed{c_2 = \dot{u}_0 + \omega_n u_0 = 2 \times 0.05 = 0.1}$$

$$\boxed{u(t) = (0.05 + 0.1t)e^{-2t}}$$

c) Overdamped

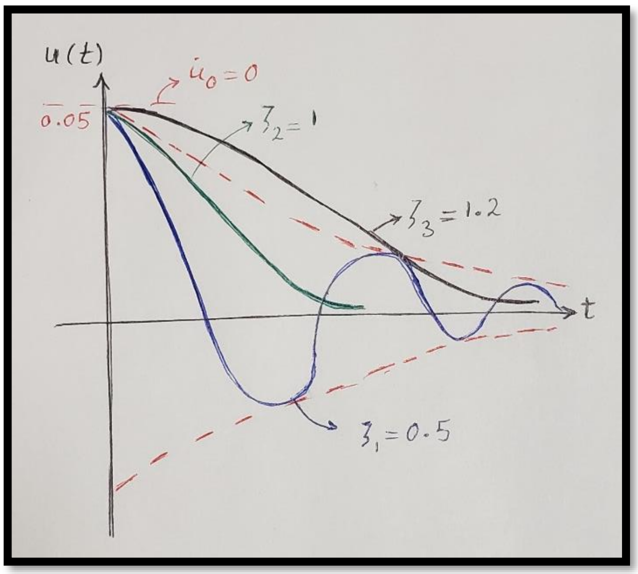
$u(t) = c_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + c_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$ General solution for overdamped case

$$c_1 = \frac{u_0\, \omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + \dot{u}_0}{2\,\omega_n\sqrt{\zeta^2-1}} = \frac{0.05\times\ 2\left(1.2+\sqrt{(1.2)^2-1}\right)}{2\times 2\sqrt{(1.2)^2-1}} = \boxed{0.07}$$

$$c_2 = \frac{-u_0\, \omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - \dot{u}_0}{2\,\omega_n\sqrt{\zeta^2-1}} = \frac{-0.05\times\ 2\left(1.2-\sqrt{(1.2)^2-1}\right)}{2\times 2\sqrt{(1.2)^2-1}} = \boxed{-0.02}$$

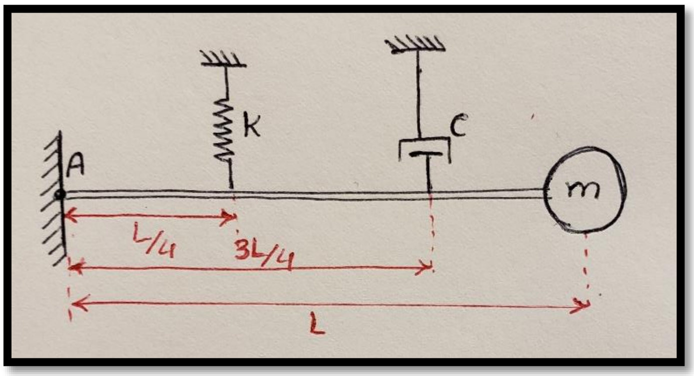
$$u(t) = 0.07e^{\left(-1.2+\sqrt{(1.2)^2-1}\right)2t} - 0.02e^{\left(-1.2-\sqrt{(1.2)^2-1}\right)2t}$$

$$\boxed{u(t) = 0.07e^{-1.073t} - 0.02e^{-3.727t}}$$

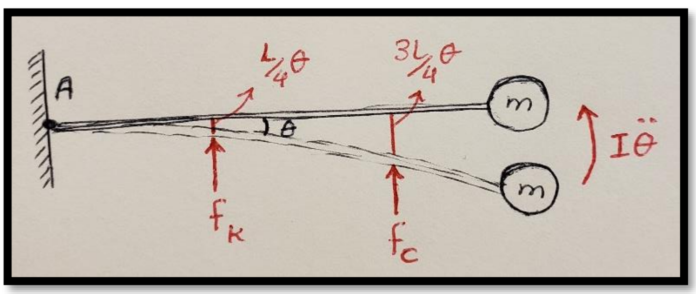


Damped Free Vibration S.D.O.F System

Example 23: The following system is given. This system includes a massless **rigid** beam attached to a lumped mass at the end of the beam, a spring with coefficient of “k” and damping system with coefficient of “c”. The length of the beam L=*2m*, *m=5kg*, coefficient of spring *k=20N/m*, damping coefficient c=8 $^{N-s}/_{m}$. Find the equation of motion and response of system for initial condition of $u(0) = 0.03m$ & $\dot{u}(0) = 0.2\ m/s$.



Step 1: Free Body Diagram:



This system is on static equilibrium position, so we can ignore the force of mg in free body diagram.
We can write the moment for point “A”.

$$\sum M_A = 0$$
$$I\ddot{\theta} + f_c\left(\frac{3L}{4}\right) + f_k\left(\frac{L}{4}\right) = 0$$
$$f_c = c\dot{u} = c\left(\frac{3L}{4}\right)\dot{\theta} = 8\left(\frac{3\times 2}{4}\right)\dot{\theta} = 12\,\dot{\theta}$$
$$f_k = ku = k\left(\frac{L}{4}\right)\theta = 20\left(\frac{2}{4}\right)\theta = 10\,\theta$$
$$I = mL^2 = 5\times 2^2 = 5\times 4 = 20$$

$$20\,\ddot{\theta} + 12\,\dot{\theta}\left(\frac{3\times 2}{4}\right) + 10\,\theta\left(\frac{2}{4}\right) = 0 \Rightarrow \boxed{20\,\ddot{\theta} + 18\,\dot{\theta} + 5\,\theta = 0} \text{ Equation of motion}$$

$$m\ddot{u} + c\dot{u} + ku = 0$$

$m_{eq} = 20$, $c_{eq} = 18$, $k_{eq} = 5$

Step 2: Finding the natural frequency of system (ω_n):

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{5}{20}} = \frac{1}{2}\,\frac{rad}{s}$$

Step 2: Finding the critical damping coefficient of system (c_c):

$$c_c = 2m_{eq}\omega_n = 2 \times 20 \times \frac{1}{2} = 20\ {}^{N-s}/_{m}$$

Step 3: Finding the damping ratio of system (ζ) and find that is related to which damping case:

$$\zeta = \frac{c_{eq}}{c_c} = \frac{18}{20} = 0.9 \quad \text{underdamped}$$

Step 4: Because it is under-damped case, first you have to calculate damped natural frequency (ω_d)):

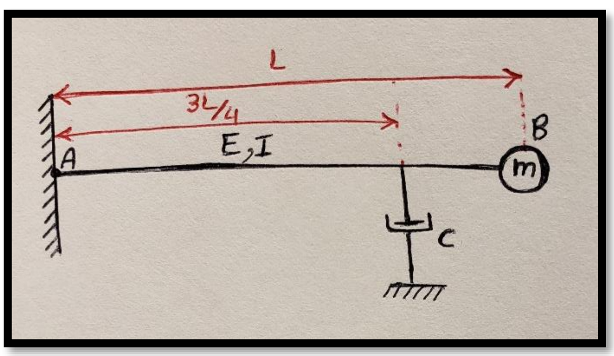
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \frac{1}{2} \times \sqrt{1-0.9^2} = 0.218\ \frac{rad}{s}$$

$$u(t) = e^{-\zeta\,\omega_n t}\left[u_0 cos\omega_d t + \frac{\dot{u}_0+\zeta\,\omega_n u_0}{\omega_d} sin\omega_d t\right] \quad \text{General solution for underdamped case}$$

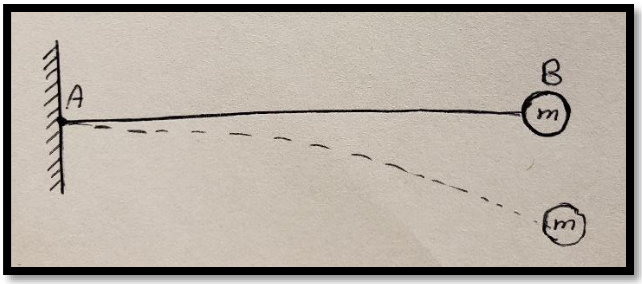
$$u(t) = e^{-0.9\times\frac{1}{2}t}\left[0.03\cos(0.218t) + \frac{0.2+0.9\ \times\frac{1}{2}\times 0.03}{0.218}\sin(0.218t)\right]$$

$$\boxed{u(t) = e^{-0.45t}[0.03\cos(0.218t) + 0.98\sin(0.218t)]} \quad \text{Response of system}$$

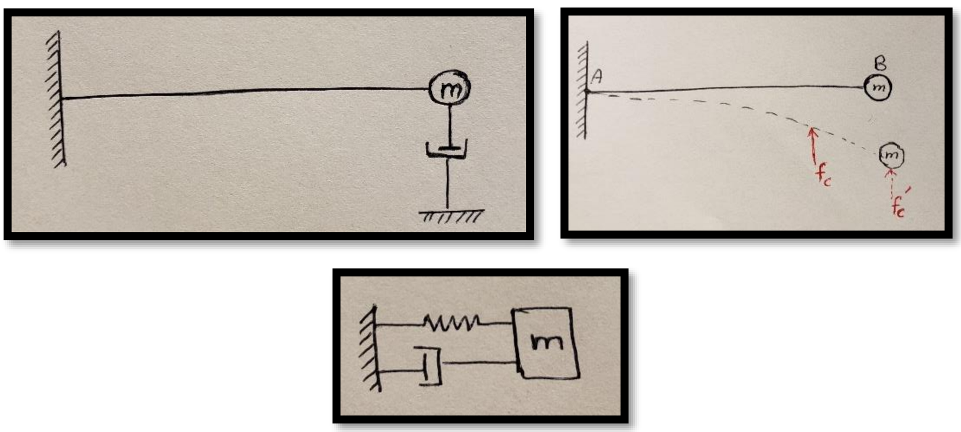
Example 24: The following system is given. This system includes a massless **flexible** beam attached to a lumped mass at the end of the beam and damping system with coefficient of "c". The length of the beam *L=2m*, *m=5kg*, damping coefficient *c*=8 ${}^{N-s}/_{m}$ and the beam has *E=207*$\times 10^9\ Pa$, *I=5*$\times 10^{-11} m^4$. Find the equation of motion and response of system for initial condition of $u(0) = 0.03m$ & $\dot{u}(0) = 0.2\ m/s$.



In this case, we have a flexible beam and that means there is not any more relation between angle θ, distance from the origin and beam displacement (beam has curve displacement).



Also, we don't have an obvious spring but the beam act as a vertical spring at point "B". So, if we can move the damping system to the point "B", then we will have a simple damped system which both spring and damper are connected to the mass.



In this case, we can use the static deflection of the beam same as what we did in Rayleigh's method (The dynamic displacement would be similar to static displacement).

From before we had:

$$u(x) = \frac{u_{max}}{2L^3}(3\,x^2L - \,x^3) = \frac{u_L}{2L^3}(3\,x^2L - \,x^3)$$

$$\dot{u}(x) = \frac{\dot{u}_{max}}{2L^3}(3\,x^2L - \,x^3) = \frac{\dot{u}_L}{2L^3}(3\,x^2L - \,x^3)$$

And now:

$f_c = c\dot{u}\left(\frac{3L}{4}\right)$ & $f_c{}' = c'\dot{u}(L)$

For $\sum M_A$ we can write:

$$c\dot{u}\left(\frac{3L}{4}\right)\times\frac{3L}{4} = c'\dot{u}(L)\times L$$

$$\dot{u}\left(\frac{3L}{4}\right) = \frac{\dot{u}_L}{2L^3}\left(3\left(\frac{3L}{4}\right)^2 L - \left(\frac{3L}{4}\right)^3\right) = \frac{81}{128}\dot{u}_L$$

$$\dot{u}(L) = \frac{\dot{u}_L}{2L^3}(3\,L^3 - L^3) = \dot{u}_L$$

$$c' = \frac{\frac{3\cancel{L}}{4}\times c\times\frac{81}{128}\cancel{\dot{u}}_L}{\cancel{L}\times\cancel{\dot{u}}_L} \qquad \boxed{c' = \frac{243}{512}c}$$

$$m\ddot{u} + c'\dot{u} + \frac{3EI}{L^3}u = 0 \Rightarrow m\ddot{u} + \left(\frac{243}{512}\right)c\dot{u} + \frac{3EI}{L^3}u = 5\ddot{u} + \left(\frac{243}{512}\right)\times 8\dot{u} + \frac{3\times207\times10^9\times5\times10^{-11}}{(2)^3}u = 0$$

$\boxed{5\ddot{u} + 3.8\,\dot{u} + (3.9)u = 0}$ Equation of motion

$m_{eq} = 5$, $c_{eq} = 3.8$, $k_{eq} = 3.9$

Step 2: Finding the natural frequency of system (ω_n):

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{3.9}{5}} = 0.88\,\frac{rad}{s}$$

Step 2: Finding the critical damping coefficient of system (c_c):

$$c_c = 2m_{eq}\omega_n = 2\times5\times0.88 = 8.8\ {}^{N-s}/_{m}$$

Step 3: Finding the damping ratio of system (ζ) and find that is related to which damping case:

$\zeta = \frac{c_{eq}}{c_c} = \frac{3.8}{8.8} = 0.43$ underdamped

Step 4: Because it is under-damped case, first you have to calculate damped natural frequency (ω_d)):

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 0.88\times\sqrt{1-0.43^2} = 0.79\,\frac{rad}{s}$$

$u(t) = e^{-\zeta\,\omega_n t}\left[u_0 cos\omega_d t + \frac{\dot{u}_0+\zeta\,\omega_n u_0}{\omega_d}sin\omega_d t\right]$ General solution for underdamped case

$$u(t) = e^{-0.43\times0.88t}\left[0.03\cos(0.79t) + \frac{0.2 + 0.43\times0.88\times0.03}{0.79}\sin(0.79t)\right]$$

$\boxed{u(t) = e^{-0.38t}[0.03\cos(0.79t) + 0.267\sin(0.79t)]}$ Response of system

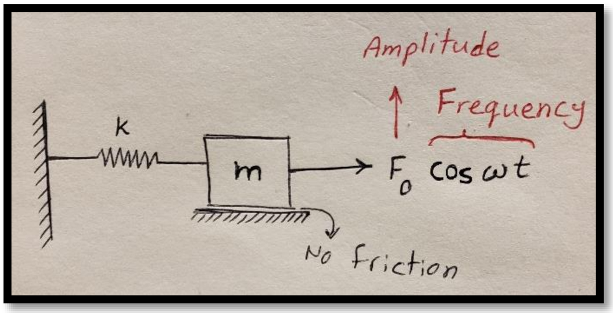
Forced Vibration S.D.O.F System

As it mentioned before, we have different type of forces: A) Harmonic force B) Periodic force C) General force (which can be divided to Deterministic and Random forces). In this part we will take a look how each of these types of forces act on an Undamped & Damped forced systems.

A) Harmonic Force

A.1) Analyzing an Undamped System That Excited by a Harmonic Force

Look at the following system.



If a force $F(t) = F_0\, cos\omega t$ act on the mass m of an Undamped system, the equation of motion would be:

$$\boxed{m\ddot{u} + ku = F_0\, cos\omega t} \qquad \text{Equation of motion}$$

The total solution for this equation of motion includes two parts: 1) homogeneous solution (called transient) which already find for free vibration, and 2) particular solution (called steady state).

$$u(t)_{total} = u_h(t) + u_p(t)$$

$$u_h(t) = A_1 \cos(\omega_n\, t) + A_2\ sin\,(\omega_n\, t)$$

Because in this case we have a harmonic force, from differential equation we know the particular solution also would be **harmonic with same frequency of harmonic force ($\boldsymbol{\omega}$).**

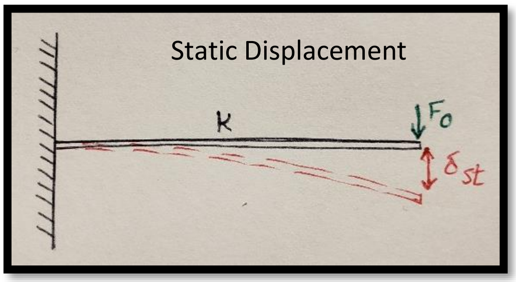
$$u_p(t) = A_3 \cos \omega t$$

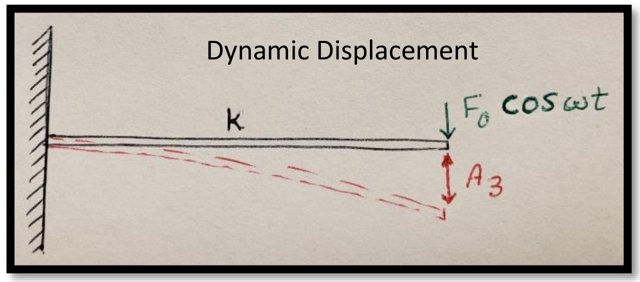
We have to find the value of amplitude for steady state response (A_3). By substituting u_p in the equation of motion:

$$\left.\begin{aligned} m\ddot{u}_p + ku_p &= F_0\, cos\omega t \\ \ddot{u}_p &= -A_3\omega^2 \cos\omega t \end{aligned}\right\} \Longrightarrow -mA_3\omega^2 \cos\omega t + kA_3 \cos\omega t = F_0\, cos\omega t$$

$$A_3\, \cancel{\cos\omega t}\,(-m\omega^2 + k) = F_0\, \cancel{cos\omega t} \Longrightarrow A_3 = \frac{F_0}{(k - m\omega^2)} \xrightarrow{\times\frac{1}{k}} \boxed{A_3 = \frac{\frac{F_0}{k}}{(1 - \frac{\omega^2}{\omega_n{}^2})}}$$

In above equation, let's name the $\delta_{st} = \frac{F_0}{k}$ static deflection (displacement) which is the deflection of the mass under the force F_0 . Also, let's name the $r = \frac{\omega}{\omega_n}$ frequency ratio.





$$\boxed{A_3 = \frac{\delta_{st}}{(1 - r^2)}}$$

So, the particular solution would be:

$$\boxed{u_p = \frac{\delta_{st}}{(1 - r^2)} \cos\omega t}$$

Note: The initial condition for particular solution: $u_p(0) \neq 0$ & $\dot{u}_p(0) = 0$

And, the total solution would be:

$$\boxed{u(t) = \underbrace{A_1 \cos(\omega_n\, t) + A_2\, sin\,(\omega_n\, t)}_{\text{Depends on natural frequency } \omega_n} + \underbrace{\frac{\delta_{st}}{(1 - r^2)} \cos\omega t}_{\text{Depends on natural frequency } (\omega_n) \text{ and force frequency } (\omega)}}$$

For the initial condition $u(0) = u_0$ & $\dot{u}(0) = \dot{u}_0$, A_1 & A_2 would be equal to:

$$A_1 = u_0 - \frac{\delta_{st}}{(1 - r^2)} \qquad \& \qquad A_2 = \frac{\dot{u}_0}{\omega_n}$$

$$u(t) = \left(u_0 - \frac{\delta_{st}}{(1-r^2)}\right)\cos(\omega_n\, t) + \frac{\dot{u}_0}{\omega_n}\ \ sin\,(\omega_n\, t) + \frac{\delta_{st}}{(1-r^2)}\cos\omega t$$

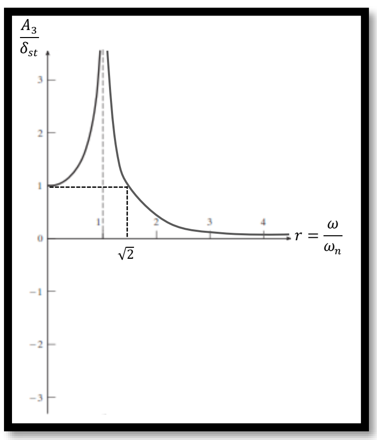
For the initial condition $u(0) = 0$ & $\dot{u}(0) = 0$, the total response:

$$u(t) = \frac{\delta_{st}}{(1-r^2)}[\cos\omega t - \cos\omega_n\, t]$$

The relation between maximum dynamic deformation (A_3) and maximum static deformation (δ_{st}) would be:

$$\boxed{\frac{A_3}{\delta_{st}} = \frac{1}{(1-r^2)}}$$

Note: $M = \frac{1}{(1-r^2)}$ is named magnification factor and the absolute value of that is named frequency response function: $\mathrm{H(r)} = \frac{1}{(1-r^2)}$



This graph shows, the value of frequency is growing till frequency become close to natural frequency of system and $r = 1$, then dynamic response goes to infinity (resonance). Then if the frequency still continue growing at $r = \sqrt{2}$ ($\omega = \sqrt{2}\omega_n$) the dynamic response and static response become equal and static and dynamic displacement would be equal (This would be happened in $r = 0$ too but that means $\omega = 0$ so basically we don't have any dynamic forces!). With growing r more than $\sqrt{2}$, frequency response function $\mathrm{H(r)}$ become closer and closer to zero. This means for a **large** value of frequency of harmonic force (ω) the dynamic force will make **less** displacement than static force! So, that means: "**Not always the response from a dynamic force is worse than the response from a static force.**"

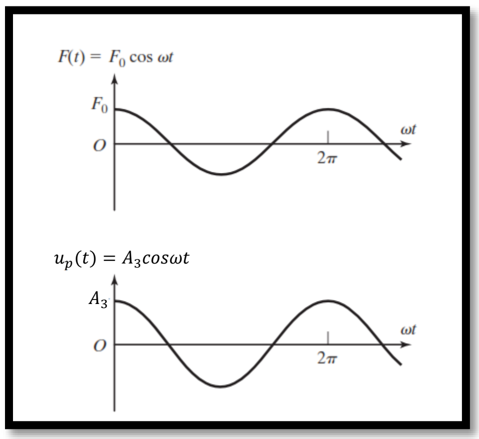
Forced Vibration S.D.O.F System

Based on value of the frequency ratio of "r" there are three cases:

Case 1: $0 < r < 1$

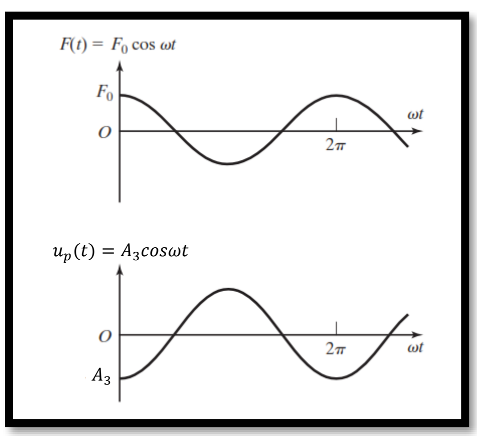
That means $A_3 = \frac{\delta_{st}}{(1-r^2)}$ become positive and the particular (steady state) response of system (u_p) and external force are in the same phase and they just have different amplitudes.

$$u_p(t) = A_3 cos\omega t$$
$$F(t) = F_0\, cos\omega t$$



Case 2: $r > 1$

That means $A_3 = \frac{\delta_{st}}{(1-r^2)}$ become negative and the particular (steady state) response of system (u_p) and external force are in the opposite phase (180° out of phase).

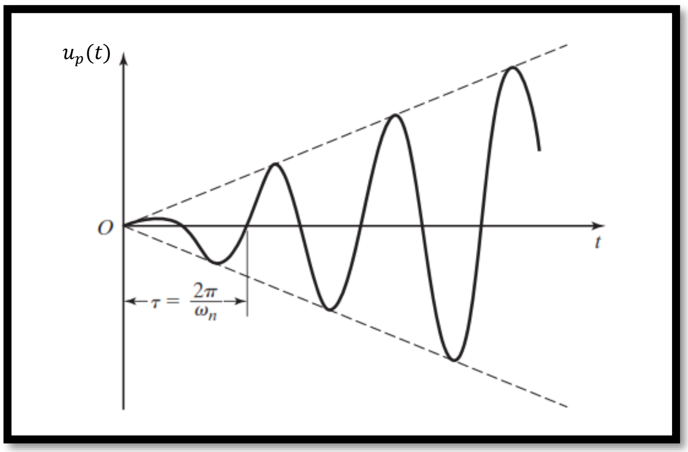


Case 3: $r = 1$ and that means $\omega = \omega_n$

In this case, the response of system $u(t)$ indeterminate form of $\left(\frac{0}{0}\right)$, so we have to use L'Hopital's rule to solve it.

$$\boxed{u(t) = \frac{\delta_{st}\omega_n\, t}{2}\sin\omega_n\, t}$$

Here is the plot of this response function. As you can see, the amplitude of this response grows linearly and goes to infinity this phenomenon is named ***resonance***.



Case 3′: $r \cong 1$ or $\omega \cong \omega_n$, if the harmonic forcing frequency is very close to natural frequency of system but they are exactly same. This phenomenon is named ***beating***.

The total response for initial conditions equal zero we had:

$$u(t) = \frac{\delta_{st}}{(1 - r^2)}[\cos\omega t - \cos\omega_n\, t] = \frac{F_0}{m({\omega_n}^2 - \omega^2)}[\cos\omega t - \cos\omega_n\, t]$$

So, for the case of $\omega \cong \omega_n$, we can write the following equation (ε is very small value):

$$\omega_n - \omega = 2\varepsilon \quad \& \quad \omega_n + \omega \cong 2\omega \;\Longrightarrow\; {\omega_n}^2 - \omega^2 = (\omega_n - \omega)(\omega_n + \omega) = 2\varepsilon \times 2\omega = 4\varepsilon\omega$$

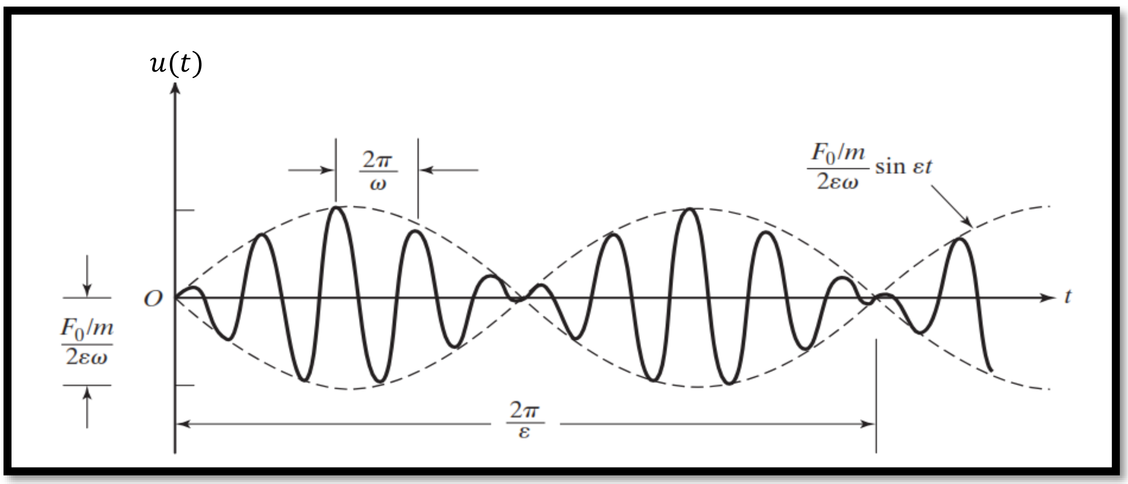
So,

$$u(t) = \frac{F_0}{m({\omega_n}^2 - \omega^2)}\left[-2\sin\frac{\omega+\omega_n}{2}t\,.\sin\frac{\omega - \omega_n}{2}\;t\right] = \frac{F_0}{m({\omega_n}^2 - \omega^2)}\left[2\sin\frac{\omega+\omega_n}{2}t\,.\sin\frac{\omega_n - \omega}{2}\;t\right]$$

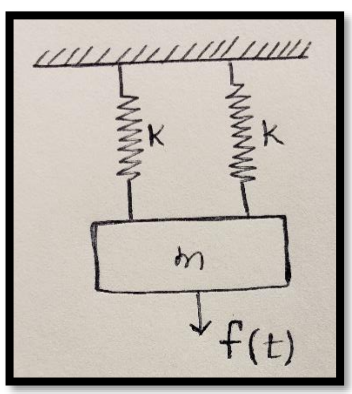
$$u(t) = \frac{F_0}{m(\underset{2}{\cancel{4}}\varepsilon\omega)}\left[\cancel{2}\sin\frac{\cancel{2}\omega}{\cancel{2}}t\,.\sin\frac{\cancel{2}\varepsilon}{\cancel{2}}\;t\right] = \frac{F_0}{2m\varepsilon\omega}[\sin\omega\, t\,.\sin\varepsilon t]$$

$$u(t) = \underbrace{\boxed{\frac{F_0\ \sin \varepsilon t}{2m\varepsilon\omega}}}_{\text{Variable Amplitude}}\underbrace{\boxed{(\sin \omega\, t)}}_{\text{Phase}}$$

The period for $\sin \varepsilon t$ would be $T_1 = \frac{2\pi}{\varepsilon}$ and the period for $\sin \omega t$ would be $T_2 = \frac{2\pi}{\omega}$. Because ε is an very small number, $T_1 \gg T_2$. Basically, in this case, we have two signals and when they are in phase, their amplitudes add to each other and when they are out of phase they cancel each other and this process repeat for each $(\frac{\pi}{\varepsilon})$.



Example 25: In the following system, the springs has coefficient of $k = 10N/m$ and mass of the system is equal to $m = 5kg$. A dynamic force, $f(t) = 10\ cos\ 5t$ is applying to this system. For initial conditions $u(0) = 0\ \ \&\ \dot{u}(0) = 0$, find the equation of motion and plot the response.



$$u(t) = A_1 \cos(\omega_n\, t) +\ A_2\ \ sin\ (\omega_n\, t) + \frac{\delta_{st}}{(1-r^2)} \cos \omega t$$

Step 1: Let's find the amplitude of the steady state part of response:

Springs are parallel, so:

$k_{eq} = k + k = 2k = 2 \times 10 = 20N/m\,, \quad F_0 = 10N\,, \quad \omega = 5\,rad/s$

$\omega_n = \sqrt{\frac{20}{5}} = 2\,rad/s$

$\delta_{st} = \frac{F_0}{k_{eq}} = \frac{10}{20} = 0.5\,m$

$r = \frac{\omega}{\omega_n} = \frac{5}{2} = 2.5 > 1 \Longrightarrow$ Case 2

$$\boxed{\frac{\delta_{st}}{(1-r^2)} = \frac{0.5}{(1-(2.5)^2)} = -0.095}$$

Step 2: Finding A_1 & A_2 from initial condition.

$\dot{u}(t) = -A_1\omega_n \sin(\omega_n\, t) + A_2\, \omega_n \cos(\omega_n\, t) - \omega \frac{\delta_{st}}{(1-r^2)} \sin \omega t$

$u(t) = A_1 \cos 2t + A_2 \sin 2t - 0.095 \cos 5t$

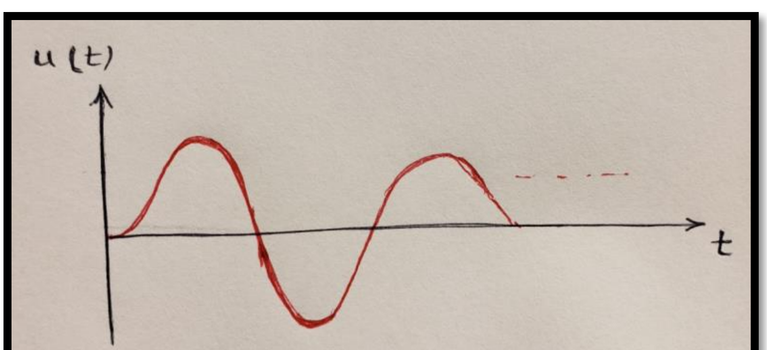
$u(0) = A_1 - 0.095 = 0 \Longrightarrow \boxed{A_1 = 0.095}$

$\dot{u}(0) = A_2\, \omega_n = 0 \Longrightarrow \boxed{A_2 = 0}$

Step 3: Total response would be equal to:

$$\boxed{u(t) = 0.095 \cos 2t - 0.095 \cos 5t}$$

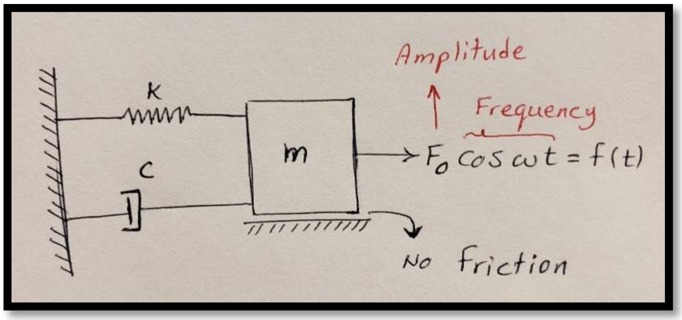
Step 4: Two cos function the plot would be look like this:



But for more exact sketch, you can plot it in MATLAB.

A.2) Analyzing a Damped System That Excited by a Harmonic Force

Look at the following system.



If a force $F(t) = F_0\, cos\omega t$ act on the mass m of a Damped system, the equation of motion would be:

$$\boxed{m\ddot{u} + c\dot{u} + ku = F_0\, cos\omega t}$$

Or

$$\boxed{\ddot{u} + 2\zeta\, \omega_n \dot{u} + {\omega_n}^2 u = \frac{F_0}{m} cos\omega t} \qquad \text{Equation of motion}$$

The total solution for this equation of motion includes two parts: 1) homogenous solution (called transient) which already find for free vibration, and 2) particular solution (called steady state).

$$u(t)_{total} = u_h(t) + u_p(t)$$

$$u_h(t) = A_1 e^{-\zeta\, \omega_n t} \cos(\omega_d t - \emptyset_0)$$

Because in this case we have a harmonic force, from differential equation we know the particular solution also would be **harmonic with but not in phase with frequency of the harmonic force ($\boldsymbol{\omega}$).**

$$u_p(t) = A_3 \cos(\omega t - \emptyset)$$

So in this case, we have to find the value of amplitude for steady state response (A_3) and phase angle of response ($\emptyset$). By substituting u_p in the equation of motion:

$$\left.\begin{aligned} m\ddot{u}_p + c\dot{u}_p + ku_p &= F_0\, cos\omega t \\ \ddot{u}_p &= -A_3\omega^2 \cos(\omega t - \emptyset) \\ \dot{u}_p &= -A_3\omega \sin(\omega t - \emptyset) \end{aligned}\right\} \Rightarrow -mA_3\omega^2 \cos(\omega t - \emptyset) - cA_3\omega \sin(\omega t - \emptyset) + kA_3 \cos(\omega t - \emptyset) = F_0\, cos\omega t$$

$$A_3[(k - m\omega^2)\cos(\omega t - \emptyset) - c\omega \sin(\omega t - \emptyset)] = F_0\, cos\omega t$$

$\cos(\omega t - \emptyset) = \cos\omega t \cos\emptyset + \sin\omega t \sin\emptyset$
$\sin(\omega t - \emptyset) = \sin\omega t \cos\emptyset - \cos\omega t \sin\emptyset$

$A_3\big[(k - m\omega^2)[\cos\omega t \cos\emptyset + \sin\omega t \sin\emptyset\,] - c\omega[\sin\omega t \cos\emptyset - \cos\omega t \sin\emptyset\,]\big] = F_0\,cos\omega t$

Anything in the left side of above equation which is coefficient of $\cos\omega t$ would be equal to F_0 and rest of them (parts Not include*s* $\cos\omega t$) would be equal to zero. So, we will have:

1) $A_3[(k - m\omega^2)\cos\emptyset + c\omega\sin\emptyset\,] = F_0$

2) $A_3[(k - m\omega^2)\sin\emptyset - c\omega\cos\emptyset] = 0$

For finding A_3, square both equations and add them with each other:

$$A_3{}^2[(k - m\omega^2)^2\cancel{\cos^2\emptyset} + c^2\omega^2\cancel{\sin^2\emptyset} + \cancel{2(k - m\omega^2)c\omega\cos\emptyset\sin\emptyset}] = F_0{}^2$$

$$+\quad A_3{}^2[(k - m\omega^2)^2\cancel{\sin^2\emptyset} + c^2\omega^2\cancel{\cos^2\emptyset} - \cancel{2(k - m\omega^2)c\omega\cos\emptyset\sin\emptyset}] = 0$$

$$A_3{}^2[(k - m\omega^2)^2 + c^2\omega^2] = F_0{}^2$$

$$\boxed{A_3 = \frac{F_0}{\sqrt{[(k - m\omega^2)^2 + c^2\omega^2]}} = \frac{\frac{F_0}{k}}{\sqrt{[(1 - r^2)^2 + (2\zeta r)^2]}}}$$

$$\boxed{\emptyset = tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \emptyset = tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)}$$

Note: ($r = \omega/\omega_n$, $\zeta = \frac{c}{c_c}$, $c_c = 2m\omega_n$)

Let's check this values for an Undamped case ($\zeta = 0$):

Amplitude of the steady state response:

$$A_3 = \frac{\frac{F_0}{k}}{(1 - r^2)}$$ Static displacement

$M = \frac{1}{(1-r^2)}$ $\quad(Magnification\ Factor) \Rightarrow H(r) = \left|\frac{1}{(1-r^2)}\right|$ *Frequency response function*

However this values for a Damped case ($\zeta \neq 0$):

$$A_3 = \frac{\frac{F_0}{k}}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}$$

Static displacement

$M = \frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$ $(Magnification\ Factor)$ $\Rightarrow$ $|H(r)| = \frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$ *Frequency response function*

In this case we have “complex frequency response function:

$$\frac{1}{(1-r^2) + i2\zeta r} \quad Complex\ frequency\ response\ function$$

The magnitude of this function would be equal to the Magnification factor:

Proof:

$$\frac{1}{(1-r^2)+i2\zeta r} \times \frac{(1-r^2)-i2\zeta r}{(1-r^2)-i2\zeta r} = \overbrace{\left(\frac{(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}\right)}^{A} - i\overbrace{\left(\frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2}\right)}^{B}$$

The magnitude of it can be find from: $\sqrt{A^2+B^2}$

$$\sqrt{\left(\frac{(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}\right)^2 + \left(\frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2}\right)^2} = \boxed{\frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}}$$

The *Frequency response function* $H(r)$ can give us lots of information. In this part, we will talk about them one by one.

$|H(r)| = \frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$

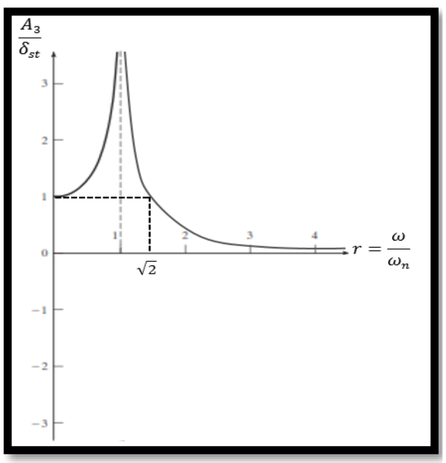
1) Undamped system ($\zeta = 0$)

$r = 0 \Rightarrow H(r) = 1$

$r = 1 \Rightarrow H(r) = \infty$

$r = \sqrt{2} \Rightarrow H(r) = 1$

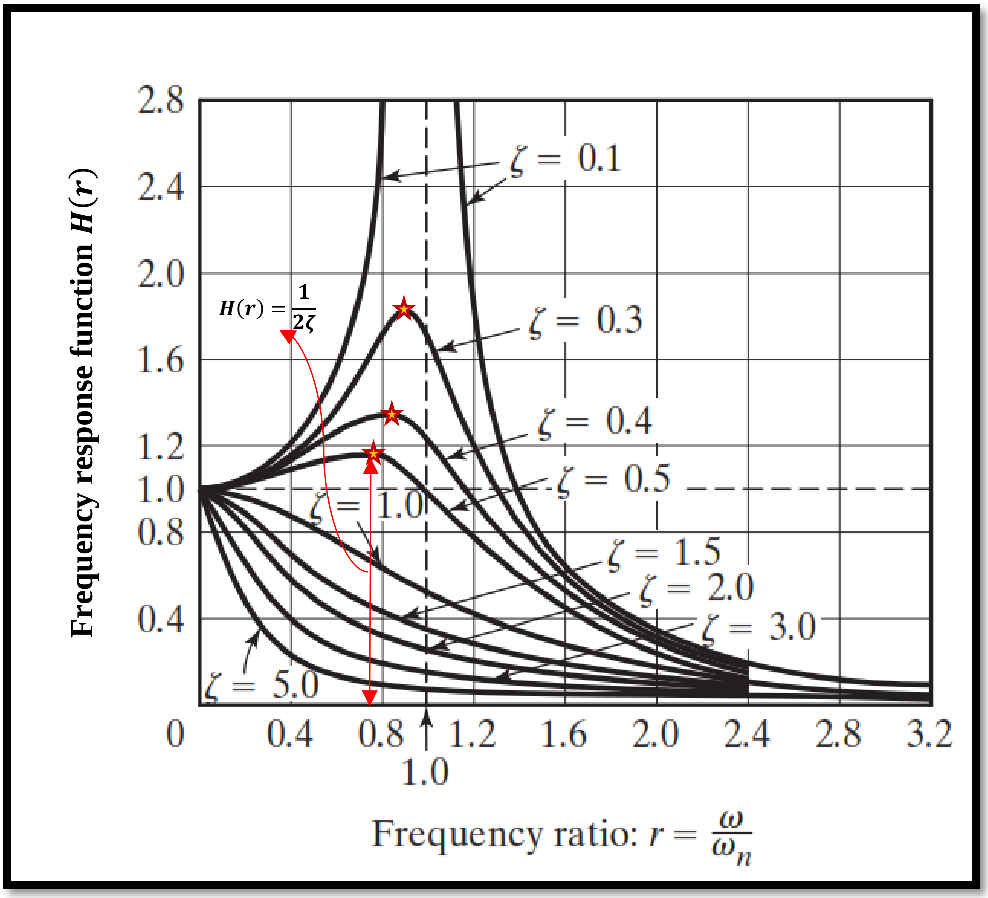
$r = \infty \Rightarrow H(r) = 0$



2) For Damped system ($\zeta \neq 0$), the *Frequency response function* $H(r)$ is dependent on two parameters ζ & r.

2a) For ($0 < \zeta < 0.707$)

The maximum value of the curves of *Frequency response function* $H(r)$ will be change between 0-1 and as much as the value of ζ increases, the maximum of the curve will be shift more to the left of the line $r = 1$ on the graph.



2b) For ($0.707 < \zeta$)

The maximum value of the curves of *Frequency response function* $H(r)$ for all value of ζ will be on $r = 0$!

Important Note: **Only** for $\mathbf{0 < \zeta < 0.707}$ for any system, the maximum dynamic response can be larger than corresponding static response! For $\mathbf{0.707 < \zeta}$, always dynamic response would be less than corresponding static response (except at $r = 0$ which means we have a static case). So, in order to decrease the effect of dynamic motion, you need to increase the damping which is not always a good idea because the consequence of it would be waste of energies, generating heat, etc.

A.2) Analyzing a Damped System That Excited by a Harmonic Force

Mathematical analysis of the graphs:

For finding the maximum value of the curves, we have to find the roots of "r" for derivative of $H(r)$ equal to zero. For constant value of damping we will have:

$$\frac{d|H(r)|}{dr} = 0 \Rightarrow \frac{r(1-r^2-2\zeta^2)}{[(1-r^2)^2+4\zeta^2 r^2]^{3/2}} = 0 \Rightarrow r(1-r^2-2\zeta^2) = 0$$

So,

$r = 0$ or $(1-r^2-2\zeta^2) = 0$

1) For $r = 0$ can be maximum or minimum of the graph. For finding if it is the maximum or minimum, we need to take the second derivative of $H(r)$ to see if it is negative (maximum) or positive (minimum)!

$$\frac{d^2H(r)}{dr^2} < 0 \qquad \text{Maximum}$$

$$\frac{d^2H(r)}{dr^2} > 0 \qquad \text{Minimum}$$

If you take the second derivative and put $r = 0$, you will find:

$$\frac{d^2H(r=0)}{dr^2} = 1 - 2\zeta^2$$

For $\zeta > \frac{\sqrt{2}}{2} \cong 0.707$ we will have $\frac{d^2H(r=0)}{dr^2} < 0$ Maximum

For $\zeta < \frac{\sqrt{2}}{2} \cong 0.707$ we will have $\frac{d^2H(r=0)}{dr^2} > 0$ Minimum

Note: Therefore, for all systems with $\zeta > 0.707$, the dynamic response will never get to corresponding static displacement $\left(\frac{F_0}{k}\right)$.

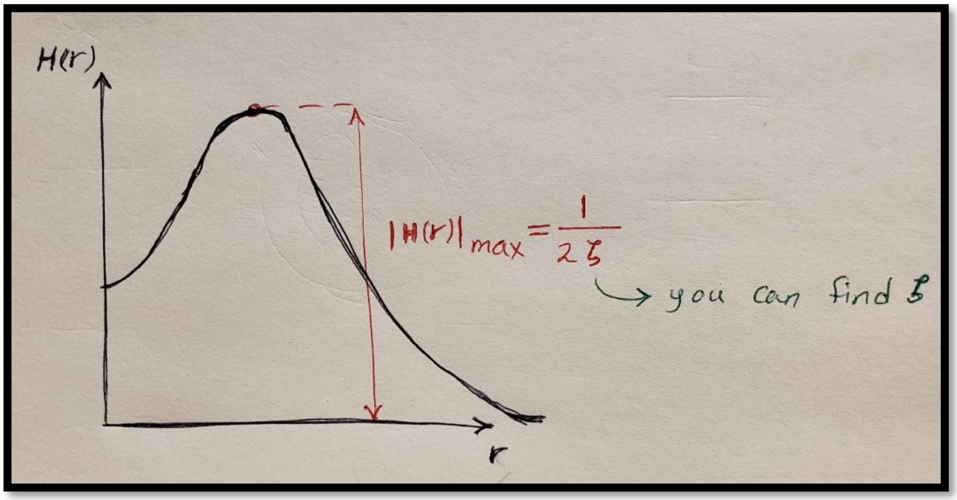
2) For $(1-r^2-2\zeta^2) = 0$ ($r \geq 0$ always):

$$\boxed{r = \sqrt{1-2\zeta^2}} \Rightarrow H(r) = \frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}} = \frac{1}{\sqrt{[(\cancel{1}-\cancel{1}+2\zeta^2)^2+4\zeta^2(1-2\zeta^2)]}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\xrightarrow{\textbf{For } \zeta \leq 0.707} \zeta^2 \text{ become very small number} \Rightarrow \sqrt{1-\zeta^2} \approx 1 \Rightarrow \boxed{|H(r)|_{max} \approx \frac{1}{2\zeta}}$$

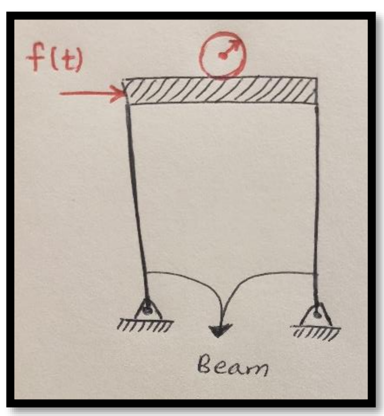
So, the maximum value of $H(r)$ for $\zeta \leq 0.707$ would be almost equal to $\left(\frac{1}{2\zeta}\right)$! Therefore, if you find and plot $H(r)$ for a system from experiment and measure the height of the maximum of the function, then you can calculate and find ζ from $\left(\frac{1}{2\zeta}\right)$. So, one of the benefit of frequency response function $H(r)$ is finding damping ratio. $H(r)_{max}$ is named "*Quality*

Factor" and is showing with *"Q"*. *High Q* system means systems with **low damping** and *low Q* systems means systems that have **high damping**.



Example 26: The following figure showing a structure which is hinged to the floor and supported rotating machinery that exerts a force *F= 900 cos 5t N*. This system has a damping ratio of $\zeta = 0.05$ and mass of $m = 6800kg$. For the beams in this structure, $= 207 \times 10^9\ Pa$, $I = 2.88 \times 10^{-5}\ m^4$, $l = 5m$ and section modulus $\left(\frac{I}{c}\right) = 35 \times 10^{-5}\ m^3$.

a) Find the maximum steady-state displacement.
b) Find the maximum dynamic bending stress.



a)

First let find the equivalent of stiffness for the beams. The beams are parallel, so:

$$k = \frac{3EI}{l^3} \Longrightarrow k_t = 2 \times \frac{3EI}{l^3} = 2 \times 3 \times \frac{207 \times 10^9 \times 2.88 \times 10^{-5}}{125} = \boxed{286.157 \times 10^3\ \frac{N}{m}}$$

Then find the equivalent static displacement (A_{st})

$$A_{st} = \frac{F_0}{k_t} = \frac{900}{286.157 \times 10^3} = 0.003m$$

Now, we can calculate the amplitude or maximum steady-state displacement from following equation:

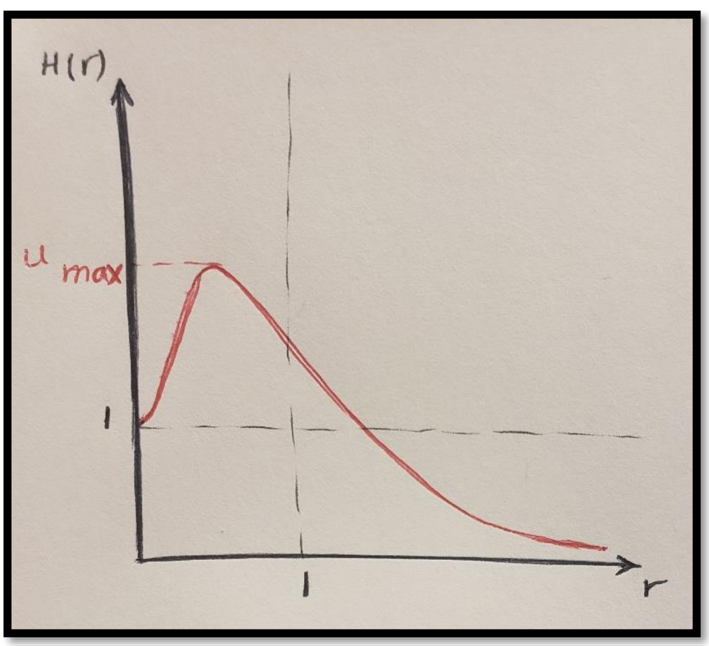
$$U_{max} = |H(r)|.A_{st} = \frac{\frac{F_0}{k}}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}$$

$$\omega_n = \sqrt{\frac{286.157 \times 10^3}{6800}} = \boxed{6.5\ \frac{r}{s}} \quad \& \quad \omega = 5\frac{r}{s} \Rightarrow \boxed{r = \frac{\omega}{\omega_n} = \frac{5}{6.5} = 0.77}$$

$$U_{max} = \frac{0.003m}{\sqrt{[(1-(0.77)^2)^2 + (2 \times 0.05 \times 0.77)^2]}} = \boxed{0.007m}$$

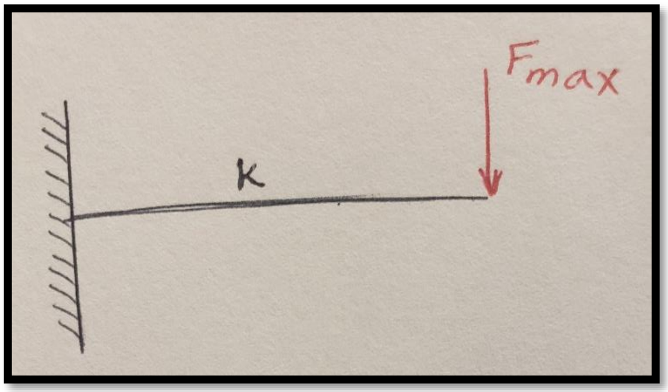
As you can see the maximum steady-state dynamic displacement is larger than static displacement in this case.

$$U_{max} > A_{st}$$



b)

What is this force on the beam? Because the force is perpendicular to the beam, so it is a shear force (V)!

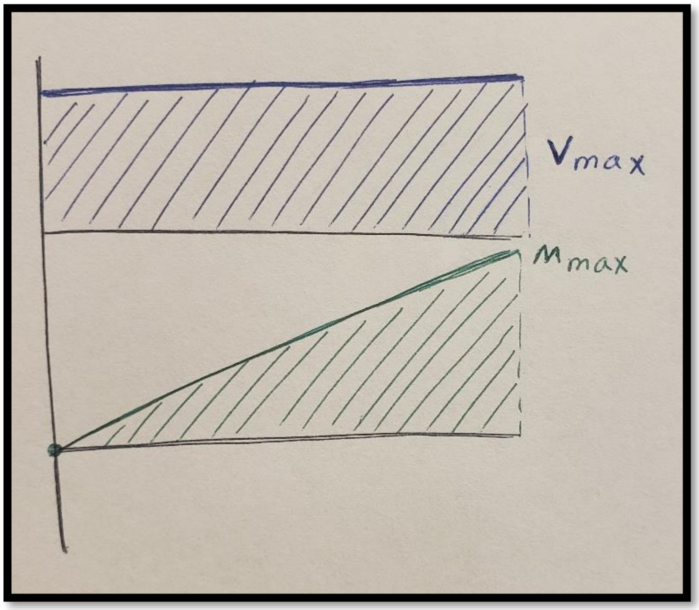


Maximum shear force can be find from following equation:

$$F_{max} = V_{max} = k \times U_{max} = \frac{3EI}{l^3} \times U_{max}$$

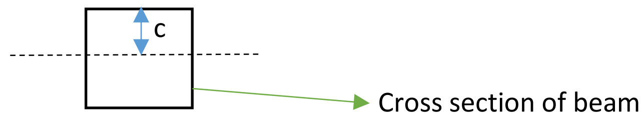
$$V_{max} = 143.078 \times 10^3 \times 0.007 = \boxed{1001.5\ N}$$

Note: From mechanic of material: $M_{max} = V_{max} \times l$



$$M_{max} = V_{max} \times l = 1001.5 \times 5 = \boxed{5007.5\ N.m}$$

From mechanics of material bending stress can be find from $\sigma = \frac{M \times y}{I} = \frac{M \times c}{I}$



Maximum bending stress would be equal to:

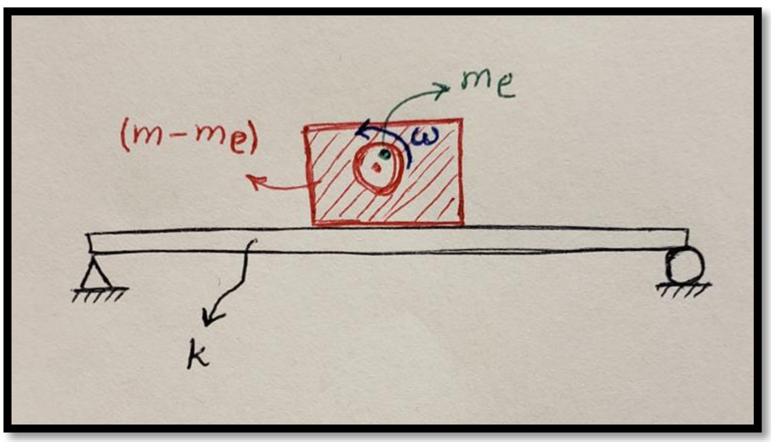
$$\sigma_{max} = \frac{M_{max}}{I/c} = \frac{5007.5}{35 \times 10^{-5}} = \boxed{1.43 * 10\text{^}7\ Pa}$$

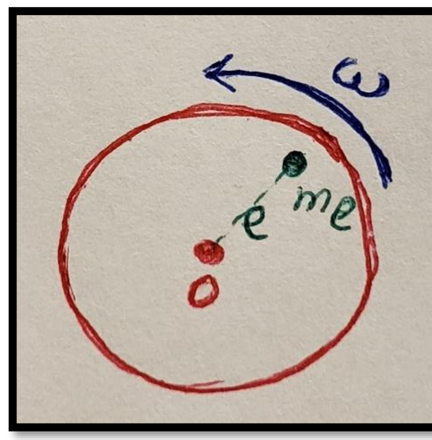
A.2) Analyzing a Damped System That Excited by a Harmonic Force

Rotating Unbalance

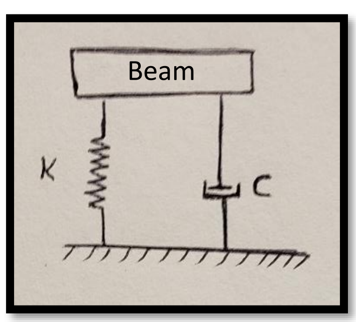
Rotating unbalance damped system is one of the special cases of the harmonically excited mechanical systems. In general, most of the systems are subjected to some kind of harmonic excitation (e.g. engine of the cars, jet engines, rotating machineries, etc.). If the center of mass and center of rotation do not coincide, that will be the rotating unbalance case. To know about importance of this topic, for instance the breakage of only one turbine blade of a jet engine causes a huge rotating unbalance and causes the total failure of the system!

Example 27: The following figure is showing a supported beam that holding a rotating machinery (has a rotating component). The rotating part has a frequency of ω. There is a small eccentric mass (m_e) on the rotating part when the center of rotation and center of this mass are not coincide (they have a distance of "e"). The total mass of the rotating part is ($m - m_e$). (The **beam** is assumed **massless**)

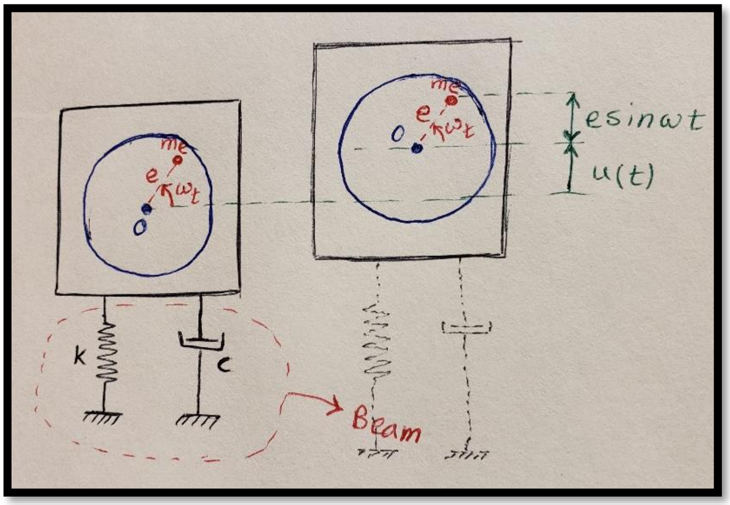




Without (m_e) this example was a harmonic excited damped system which we studied before. This beam can be shown with spring-damp system, where the k is the stiffness of the beam and c is given.

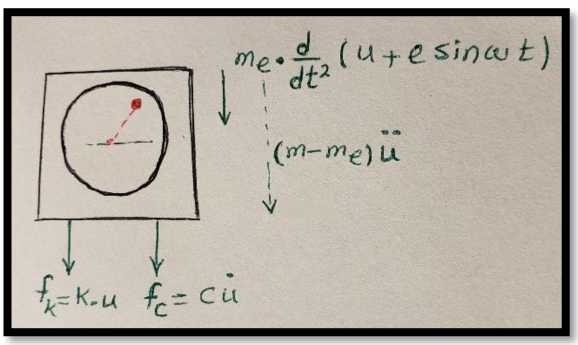


So, whole structure can be model like this:



In this case additional to the rotation part which experience a displacement of $u(t)$, (m_e) is experiences a displacement of $d = u(t) + e\ sin\ \omega t$

Free body diagram would be:



Write the equation of motion for the system:

$$(m - m_e)\ddot{u} + c\dot{u} + ku + m_e \frac{d^2}{dt^2}(u + e\ sin\ \omega t) = 0$$

$$m\ddot{u} - \cancel{m_e}\ddot{u} + c\dot{u} + ku + \cancel{m_e}\ddot{u} - m_e e\omega^2\ sin\ \omega t = 0$$

$$\boxed{m\ddot{u} + c\dot{u} + ku = m_e e\omega^2\ sin\ \omega t}$$ Equation of motion

There are some observation from the equation of motion for rotating unbalanced system. $m_e e\omega^2$ = constant, so the right side of this equation is very similar to F_0 sin ωt. We can say that is a harmonically excited system with amplitude of forcing function equal to $(m_e e\omega^2)$ or the amplitude of the forcing function is dependent on the frequency (ω).

If we write $F_0 = m_e e\omega^2$, then U_{max} would be:

$$U_{max} = \frac{\frac{F_0}{k}}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}} = \boxed{\frac{\frac{m_e e\omega^2}{k}}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}}$$

Or

$$U_{max} = \frac{\frac{m_e e\omega^2}{k} \times \frac{{\omega_n}^2}{{\omega_n}^2}}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}} = \boxed{\frac{\left(\frac{m_e e}{m}\right) r^2}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}}$$

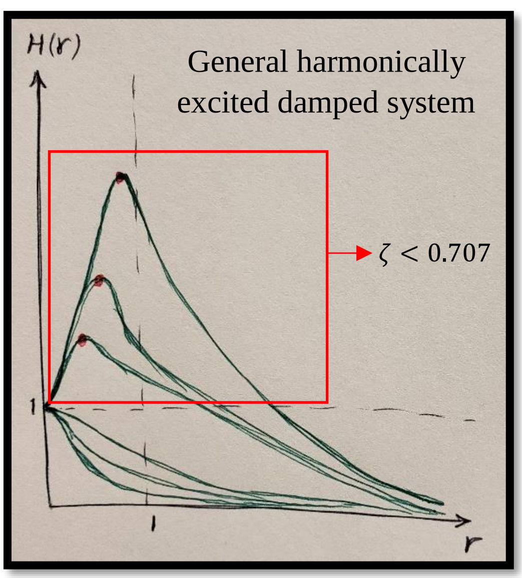
If we compare the results with a damped harmonic excited system:

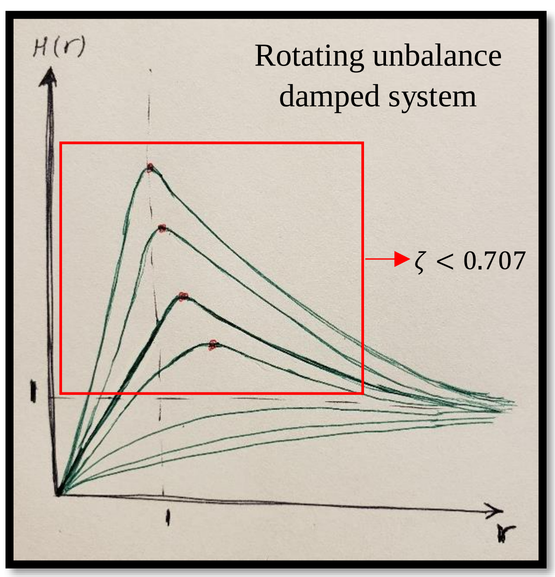
For a **general harmonically excited damped system**:

Forcing function: $f(t) = F_0 \, sin \, \omega t$
Frequency response function: $|H(r)| = \frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$
Maximum steady-state response: $U_{max} = |H(r)| \times \frac{F_0}{k}$

For a **rotating unbalance damped system**:

Forcing function: $f(t) = \, m_e e\omega^2 \, sin \, \omega t$ (*The amplitude of forcing function is frequency dependent in this case*)
Frequency response function: $|H(r)| = \frac{r^2}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$
Maximum steady-state response: $U_{max} = \frac{\frac{m_e e\omega^2}{k}}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}} = \frac{\left(\frac{m_e e}{m}\right) r^2}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$





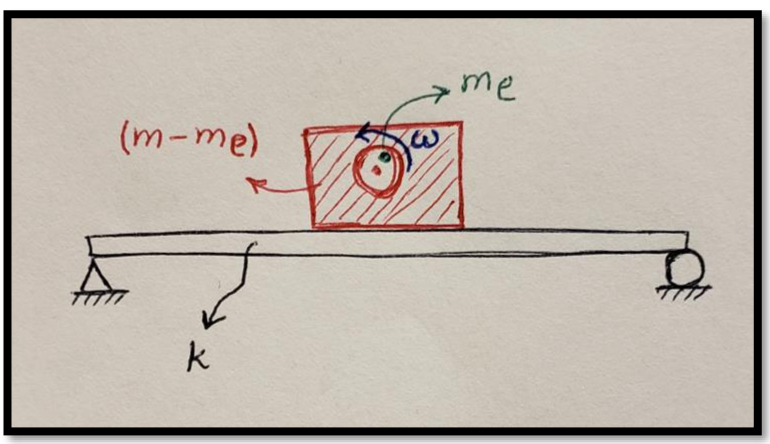
As you can see from the graphs, the general harmonically excited damped system & the rotating unbalance damped system are completely different.

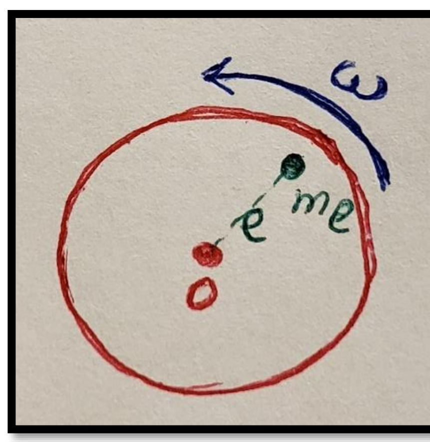
In the rotating unbalance damped system:

1) In this case, the peaks of the functions instead of being on the left of the r=1 are on the right of the r=1. So, if the peak is on the right of r=1, that means there is an unbalance in the system! That is how they find the unbalance in the tires/wheels of vehicle.
2) As damping ratio (ζ) increases, the peaks are shift further to the right of the $r = 1$.
3) If $r \to \infty \Longrightarrow |H(r)| \to \frac{r^2}{r^2} = 1$, This means (for $\zeta < 0.707$ & $r > 1$) there is no way under rotation unbalance system that you can get below the maximum static displacement! At the time that the forcing frequency become so much larger then natural frequency of system ($\omega \gg \omega_n$), at best you would approach the static displacement!
4) For $r = 0 \Longrightarrow |H(r)| = 0$ (no matter what is the value of ζ)

Exercise: Find the maximum values for the peaks on the rotating unbalance damped system and show why they are shifting to the right of $r = 1$ with increasing value of ζ (for $\zeta < 0.707$).

Example 28: The following figure is showing a supported beam that holding a rotating machinery (laundry machine). The rotating part has a frequency of ω and mass of m =7250kg. For the beam in this structure, $= \; 207 \times 10^9 \; Pa$, $I = 5 \times 10^{-5} \; m^4$, $l = 3.5m$. The motor has a speed of 300 rpm. There is a small eccentric mass ($m_e = 20kg$) on the rotating part that the center of rotation and center of this mass are not coincide (they have a distance of $e = 0.25m$). The damping ratio of the system is $\zeta = 10\%$. The beam is assumed massless. Find the maximum displacement of the system U_{max}?





U_{max} =?

$$U_{max} = \frac{\frac{m_e e \omega^2}{k}}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}}$$

r =?

$\omega = 300\ rpm \times \frac{2\pi}{60} = 31.4\ \frac{rad}{s}$

$k = \frac{48\ EI}{l^3} = \frac{48\times207\times10^9\times\ 5\times10^{-5}}{(3.5)^3} = \frac{49680\times10^4}{42.875} = 11.6\times10^6\ \frac{N}{m}$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{11.6\times10^6}{7250}} = 40\ \frac{rad}{s}$

$r = \frac{\omega}{\omega_n} = \frac{31.4}{40} = \boxed{0.785}$

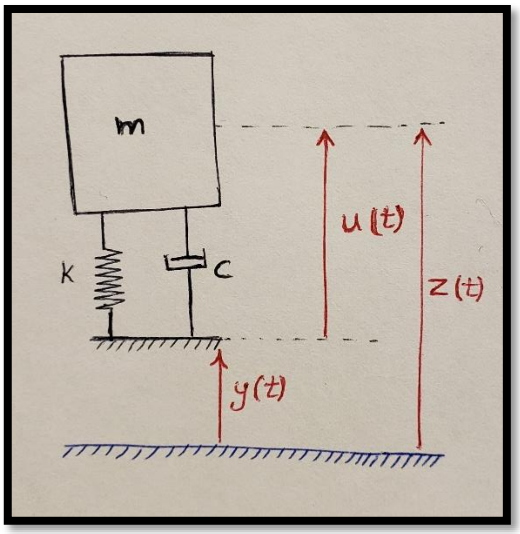
$$U_{max} = \frac{\frac{m_e e\omega^2}{k}}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}} = \frac{\frac{20\times0.25\times31.4^2}{11.6\times10^6}}{\sqrt{[(1-(0.785\ )^2)^2+(2\times0.1\times0.785\ )^2]}} = \frac{4.25\times10^{-4}}{0.414} = \boxed{0.001\ m}$$

So, for just a very small unbalance mass compare to the total mass causes 1mm displacement! ($\frac{20}{7250}\times100 = 0.27\%$ of total weight)

A.2) Analyzing a Damped System That Excited by a Harmonic Force

Time Dependent Input Displacement

This is a special case of damped (or Undamped) systems and instead of system be subjected to an external force, the base of system is subjected to a displacement (e.g. earthquake). Input displacement doesn't have to be necessarily harmonic but here we are discussing the **harmonic** and later we will discuss a general case.



$z = u + y$

We can write the equation of motion in term of "u":

$$m\ddot{z} + c\dot{u} + ku = 0 \Longrightarrow m\ddot{u} + m\ddot{y} + c\dot{u} + ku = 0 \Longrightarrow \boxed{m\ddot{u} + c\dot{u} + ku = -m\ddot{y}}$$

This way would be good for civil engineers because usually the earthquake input measure respect to the ground moving acceleration.

Also, we can write the equation of motion in term of "z":

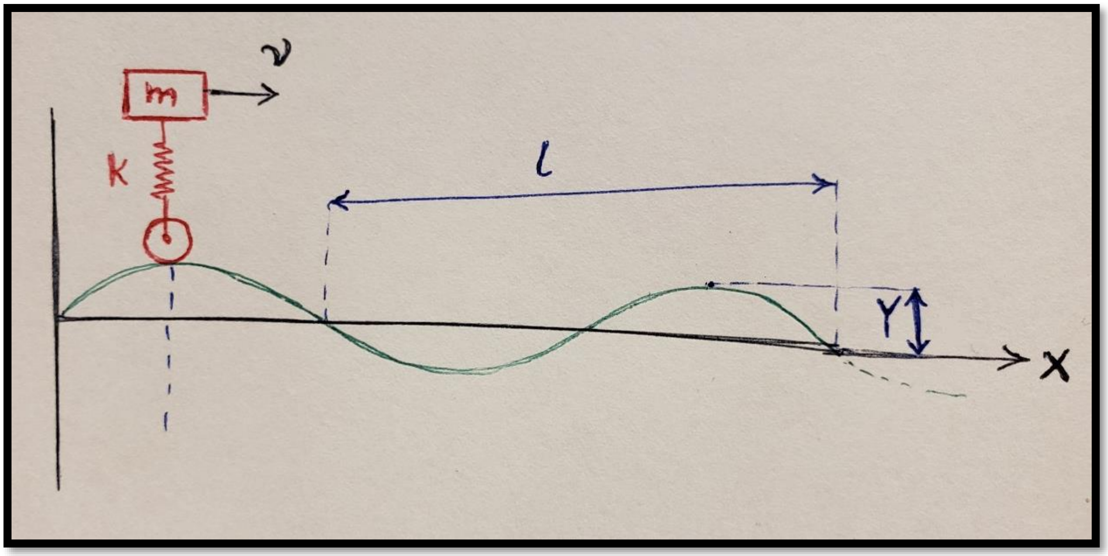
$u = z - y$

$$m\ddot{z} + c\dot{u} + ku = 0 \Longrightarrow m\ddot{z} + c\dot{z} - c\dot{y} + kz - ky = 0 \Longrightarrow \boxed{m\ddot{z} + c\dot{z} + kz = c\dot{y} + ky}$$

This way would be good for mechanical cases where usually we are dealing with an input displacement.
If we want to write these equations as the form of the forced vibration, we will have:

$f(t) = \begin{cases} -m\ddot{y} \\ c\dot{y} + ky \end{cases}$ or

Example 29: Let's consider that a car is travelling on a road with roughness in shape of sinusoidal function. The amplitude of the roughness for the surface of the road is $y = Y \sin\ 2\pi \frac{x}{l}$ ("l" is one period for the wave of the road). The car is modeled as an Undamped SDOF system ($c = 0$). This car travelling in x direction with constant speed "v". Find the most undesirable speed.



When the most undesirable speed will be happened? In case of resonance (When the input frequency (ω) and natural frequency of the system (ω_n) become equal and $r = 1$)

The equation of motion would be:

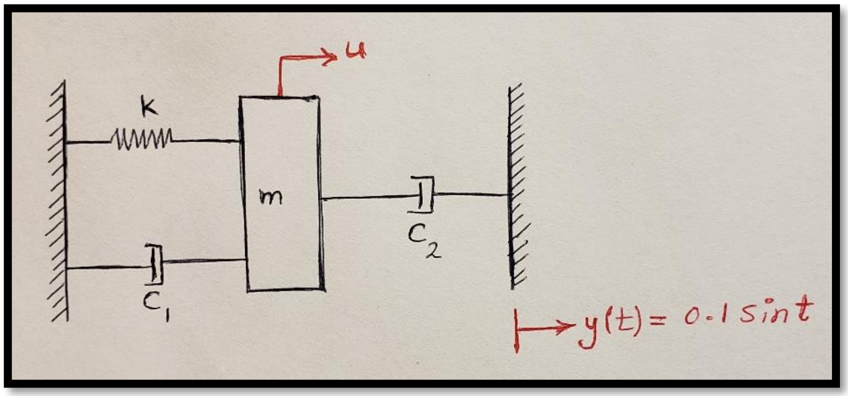
$$m\ddot{z} + kz = ky = kY \sin\ 2\pi \frac{x}{l}$$

Velocity is constant $\Longrightarrow x = v.t \Longrightarrow 2\pi \frac{x}{l} = \frac{2\pi v t}{l} = \omega t \Longrightarrow \boxed{\omega = \frac{2\pi v}{l}}$

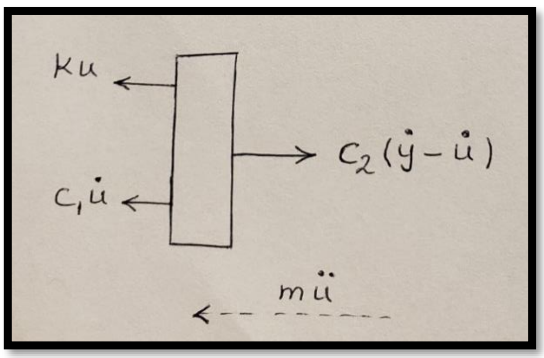
$$\boxed{\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}}$$

$\omega_n = \omega \Longrightarrow \sqrt{\frac{kg}{W}} = \frac{2\pi v}{l} \Longrightarrow \boxed{v = \frac{l}{2\pi}\sqrt{\frac{kg}{W}}} \Longrightarrow$ The most undesirable speed

Example 30: The following system is supported at both sides with walls. The wall at the right side subjected to a displacement $y(t) = 0.1\, sin\, t$. In this system, $m = 1\, kg, k = 1\frac{N}{m}$, $c_2 = 2\, {}^{N-s}/_{m}$, and $\zeta = 0.2$. Find the maximum displacement that the mass experience (U_{max}).



Free body diagram:



Equation of motion:

$$-m\ddot{u} + c_2(\dot{y} - \dot{u}) - ku - c_1\dot{u} = 0$$

$$\boxed{m\ddot{u} + (c_1 + c_2)\dot{u} + ku = c_2\dot{y}}$$

We don't have value of c_1 but damping ratio is given ($\zeta = 0.2$). So, we can divide both side of equation by "m" and we will have:

(Note: $c_{total} = c_1 + c_2 \Longrightarrow \zeta = \frac{c}{2m\omega_n} \Longrightarrow \frac{c}{m} = 2\zeta\omega_n$)

$$\ddot{u} + 2\zeta\omega_n\dot{u} + {\omega_n}^2 u = \frac{c_2}{m}\dot{y}$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{1}} = 1$, $\zeta = 0.2$, $c_2 = 2$, $m = 1$, $y = 0.1\, sin\, t \Longrightarrow \dot{y} = 0.1\, cos\, t$

$$\boxed{\ddot{u} + 0.4\dot{u} + 1u = 0.2\, cos\, t}$$

Equation of motion

$$f(t) = 0.2\, cos\, t$$

F_0

$$U_{max} = \frac{F_0}{k}|H(r)|_{max} = \frac{F_0}{k} \times \frac{1}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}} = \frac{0.2}{1} \times \frac{1}{\sqrt{[\cancel{(1-1)^2} + (2 \times 0.2 \times 1)^2]}}$$

$$\boxed{U_{max} = \frac{0.2}{0.4} = 0.5\, m}$$

A.2) Analyzing a Damped System That Excited by a Harmonic Force

Transmissibility Ratio (TR)

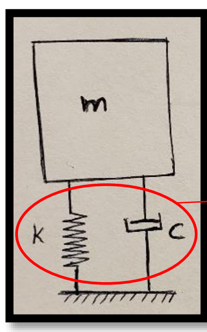
Till now we discussed about systems which are subjected to a force or a displacement. But we want to know, what is the effect of this force or displacement on our system to protect the system from undesirable vibrations.

Transmissibility Ratio (TR): It is a quantity that measures the impact of the **Force** or **Displacement** acting on a system.

Case 1: Force Transmitted

If we have a force acting on the system, it causes the system displacement, so the structure supports which are *"k"* and *"c"* as a result of that displacement would be subjected the transmitted force (f_T).

$$\boxed{f_T = ku + c\dot{u}}$$



Structure supports

In the case of force acting on the system, we define Transmissibility Ratio (TR) as:

For a harmonic force:

$$\boxed{TR = \frac{Max\ Force\ Transmitted}{Max\ Force\ Applied} = \frac{ku + c\dot{u}}{F_0}}$$

$$f(t) = F_0 \sin \omega t$$

$$u(t) = \frac{F_0}{k} H(r) \sin(\omega t - \theta)$$

$$\dot{u}(t) = \frac{F_0}{k} H(r)\, \omega \cos(\omega t - \theta)$$

$$f_T = k \times u_{max} + c \times \dot{u}_{max} = \cancel{k} \times \frac{F_0}{\cancel{k}} H(r) \sin(\omega t - \theta) + c \times \frac{F_0}{k} H(r)\, \omega \cos(\omega t - \theta)$$

$$f_T = \underbrace{F_0\, H(r)}_{A_1} \sin(\omega t - \theta) + \underbrace{F_0 H(r)\, \frac{c\omega}{k}}_{A_2} \cos(\omega t - \theta)$$

$$f_T = \sqrt{{A_1}^2 + {A_2}^2}\left(\frac{A_1}{\sqrt{{A_1}^2 + {A_2}^2}} sin\,(\omega t - \theta) + \frac{A_2}{\sqrt{{A_1}^2 + {A_2}^2}}\ cos\,(\omega t - \theta)\right)$$

$(f_T)_{max}$

$$\boxed{(f_T)_{max} = F_0\, H(r)\sqrt{1 + \frac{c^2\omega^2}{k^2}}}$$

$$TR = \frac{(f_T)_{max}}{F_0} = \frac{\cancel{F_0}\, H(r)\sqrt{1 + \frac{c^2\omega^2}{k^2}}}{\cancel{F_0}} = H(r)\sqrt{1 + \frac{c^2\omega^2}{k^2}}$$

$H(r) = \frac{1}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}$

$\frac{c^2\omega^2}{k^2} \times \frac{{\omega_n}^2}{{\omega_n}^2} \times \frac{4}{4} = 4\frac{c^2}{4km} \times r^2 = 4\zeta^2 r^2$

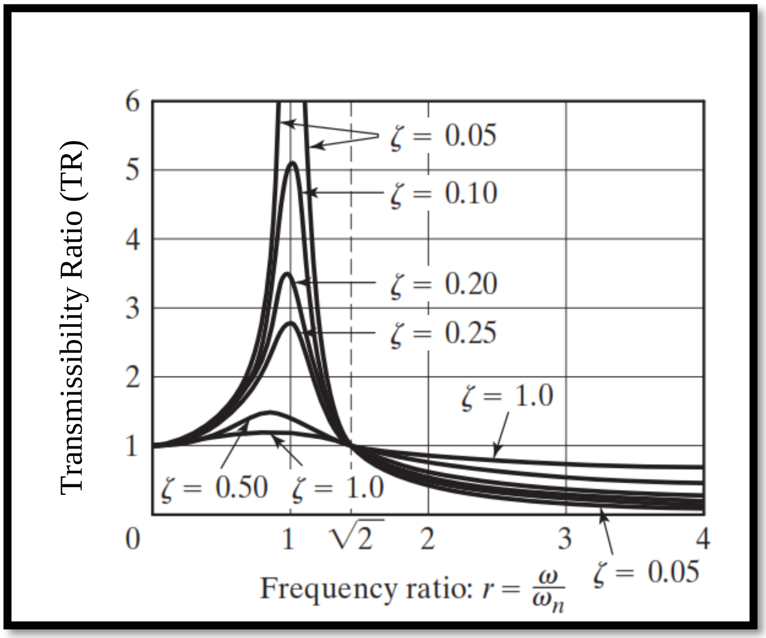
ζ^2

Note: $\zeta = \frac{c}{2\sqrt{km}}$

$$\boxed{TR = \sqrt{\frac{1+4\zeta^2 r^2}{(1-r^2)^2 + (2\zeta r)^2}}}$$

This is showing the ratio of maximum force that the supports of a system experience to the maximum harmonic force that acts on the system.

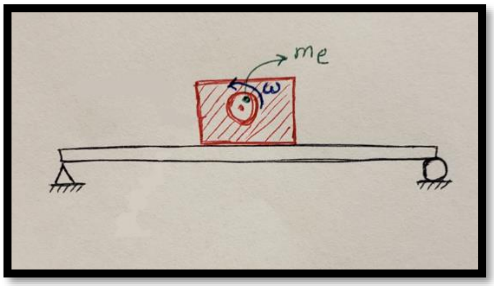
The following plot showing the Force Transmissibility Ratio function. We care about the case $\zeta < 0.707$ because for $\zeta > 0.707$ we know the dynamic displacement is always less than static displacement.

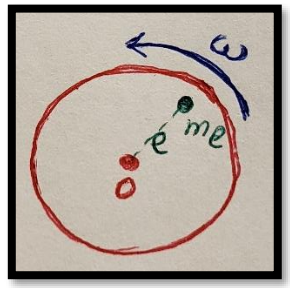


Following observation from this plot:

- All curves go through $r = \sqrt{2}$ and intersect on that point
- $TR < 1$ **Only if** $r > \sqrt{2}$, That means in order to reduce the unpleasant effect of a forcing function on your system, you have to make sure always $\omega > \sqrt{2}\,\omega_n$
- For $r > \sqrt{2}$ as damping ratio increases, TR increases too. That means, we want to have a system less damped in order to less force transmitted to the system.
- For having $r > \sqrt{2}$, the best way is having system with smaller natural frequency ($\omega_n = \sqrt{\frac{k}{m}}$). Because in most of the cases, we don't have much control over the mass so we need system with smaller stiffness (k). So, softer system better absorb the force but then you will have larger static displacement ($\delta = \frac{W}{k}$)! Therefore, vibration isolation is a balancing act that you need to find out what is your optimum stiffness (k) value.

Example 31: The following figure is showing a supported beam that holding a rotating machinery with a weight of $M = 500\ kg$ and frequency of $\omega = 7200\ rpm$. This system has an unbalanced mass with $m_e = 1\ g$ and $e = 20\ cm$.The damping ratio of the system is negligible ($\zeta = 0$). Design an isolation system that assures the force transmitted (f_T) is less than 250N?





$$\zeta = 0 \Longrightarrow TR = \frac{Max\ Force\ Transmitted}{Max\ Force\ Applied} = \left|\frac{1}{r^2 - 1}\right|$$

$Max\ Force\ Applied$: This an unbalanced system, so, maximum applied force would be:

$$F_0 = m_e e\omega^2$$

$Max\ Force\ Transmitted$: That is given by problem equal to 250 N.

So, we will have:

$$\frac{250}{m_e e\omega^2} = \left|\frac{1}{r^2 - 1}\right|$$

$\omega = 7200\ rpm \times \frac{2\pi}{60} = 754\ \frac{rad}{s}\ ,\ m_e = 0.001\ \text{kg}\ ,\ e = 0.2\ m$

$\frac{250}{0.001\times0.2\times(754)^2} = \frac{250}{113.7} = 2.2 = \left|\frac{1}{r^2-1}\right| \Longrightarrow \boxed{r = 1.2}$

$r = \frac{\omega}{\omega_n} \Longrightarrow 1.2 = \frac{754}{\omega_n} \Longrightarrow \boxed{\omega_n = 628.3\frac{rad}{s}}$

So, for having force transmitted (f_T) less than 250N, the support system (beam) need to have a natural frequency of $\omega_n = 628.3\frac{rad}{s}$. Based on this natural frequency, we can find the stiffness of the system.

$\omega_n = \sqrt{\frac{k}{m}} \Longrightarrow 628.3 = \sqrt{\frac{k}{500}} \Longrightarrow \boxed{k = 1.9 \times 10^8\ \frac{N}{m}}$

However, we have to make sure, system with this stiffness not make a large static displacement.

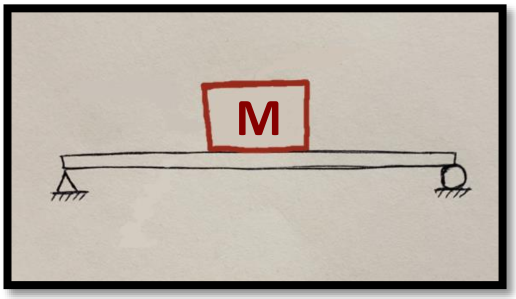
$$\boxed{\delta = \frac{W}{k} = \frac{500 \times 9.81}{1.9 \times 10^8} = 2.6 \times 10^{-5}\ m}$$

The displacement is very small and almost negligible. That means this stiffness (k) is acceptable for the system.

A.2) Analyzing a Damped System That Excited by a Harmonic Force

Example 32: The following figure is showing a supported beam that holding a cylinder-piston machinery with a weight of $M = 1750\ kg$ and frequency of $\omega = 60\ \frac{rad}{s}$. The properties of the beam are, $= 207 \times 10^9\ Pa$, $I = 5 \times 10^{-5}\ m^4$, $l = 3\ m$. This machinery generate a harmonic force with amplitude $F_0 = 32000\ N$. The damping ratio of the system is $\zeta = 10\%$.

a) Find the amplitude (maximum displacement) of the motion
b) Find the force transmitted to the beam



a)
This is simply supported beam, so:

$$k = \frac{48\ EI}{l^3} = \frac{48\ \times 207 \times 10^9 \times 5 \times 10^{-5}}{27} = 18.4 \times 10^6\ \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{18.4 \times 10^6}{1750}} = 102.5\ \frac{rad}{s}$$

$$\zeta = 0.1 \quad , \quad r = \frac{\omega}{\omega_n} = \frac{60}{102.5} = 0.585$$

$$U_{max} = \left(\frac{F_0}{k}\right) H(r) = \left(\frac{F_0}{k}\right) \frac{1}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}} = \left(\frac{32000}{18.4 \times 10^6}\right) \frac{1}{\sqrt{[(1-0.585^2)^2 + (2 \times 0.1 \times 0.585)^2]}} = \boxed{2.6 \times 10^{-3}\ m}$$

b)

$$TR = \sqrt{\frac{1+4\zeta^2 r^2}{(1-r^2)^2 + (2\zeta r)^2}} = \sqrt{\frac{1+4 \times 0.1^2 \times 0.6^2}{(1-0.6^2)^2 + (2 \times 0.1 \times 0.6)^2}} = 1.547$$

$$TR = \frac{Max\ Force\ Transmitted}{Max\ Force\ Applied} = \frac{(f_T)_{max}}{F_0}$$

$$(f_T)_{max} = TR \times F_0 = 1.547 \times 32000 = \boxed{49504\ N}$$

As you can see, this machinery with weight of $1750\ kg$ ($\cong 17500N$), if it operates at the frequency of $\omega = 60\ \frac{rad}{s}$, the system experiences almost three times higher force than the weight of machine!

Case 2: Input Displacement Transmitted

$$TR = \frac{Max\ Displacement\ Transmitted}{Max\ Input\ Displacement}$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{y}$$ Equation of motion

Let's assume the system experiences a harmonic displacement:

$$y = Y \sin\ \omega t$$

$$m\ddot{u} + c\dot{u} + ku = m\omega^2\ Y \sin\ \omega t$$

Maximum Input Displacement (F_0)

If we solve for $u(t)$, we will have:

$$u(t) = \left(\frac{F_0}{k}\right) H\ sin\ (\omega t - \theta) = \left(\frac{m\omega^2\ Y}{k}\right) H\ sin\ (\omega t - \theta)$$

The absolute displacement would be:

$$\text{Absolute displacement} = u(t) + y(t)$$

If you are going to the same process as we did for the force transmitted case, you can see the transmissibility ratio will be exactly same as previous case and it has an identical expression!

$$\boxed{TR = \sqrt{\frac{1+4\zeta^2r^2}{(1-r^2)^2+(2\zeta r)^2}}}$$

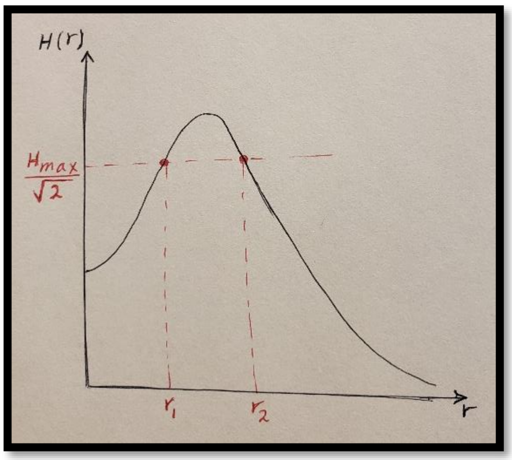
Note: The goal is keep the "TR" **less** than "1" for the systems!

Half Power Method

If you remember, we used $H(r)$ plot and quality (Q) (which was equal to $H(r)_{max} = \frac{1}{2\zeta}$) to find damping ratio (ζ). The "Half Power Method" is another approach for finding the damping ratio (ζ).

The frequency response function is not smooth like that is showing in the graphs and usually it is full of noises. So, finding the exact maximum value of the graph would be hard and using estimation is not very accurate.

Let's assume we can find $\frac{H_{max}}{\sqrt{2}}$ and draw a line to cross the graph (where the graph is smoother). Find the frequency ratio (r) for two intersection points. In this method, we are using these (r) values to calculate and find the peak of the graph and damping ratio.



For finding r_1 & r_2, we can solve the following equation and roots of that equation will be r_1 & r_2 values:

$$H_{max} = \frac{1}{2\zeta} \Rightarrow \boxed{\frac{1}{\sqrt{2}} \times \frac{1}{2\zeta} = \frac{1}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}}$$

$$\boxed{\begin{array}{l} {r_1}^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1+\zeta^2} \\ {r_2}^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1+\zeta^2} \end{array}}$$

We know the damping is very small ($\zeta \ll 1$), so, r_1 & r_2 can write as:

$${r_1}^2 \cong 1 - 2\zeta^2 - 2\zeta$$
$${r_2}^2 \cong 1 - 2\zeta^2 + 2\zeta$$

With a good approximation r_1 & r_2 will be equal to:

$$\boxed{\begin{aligned} r_1 &\cong 1 - \zeta^2 - \zeta \\ r_2 &\cong 1 - \zeta^2 + \zeta \end{aligned}}$$

The value of ζ can be find from subtracting these two equations from each other.

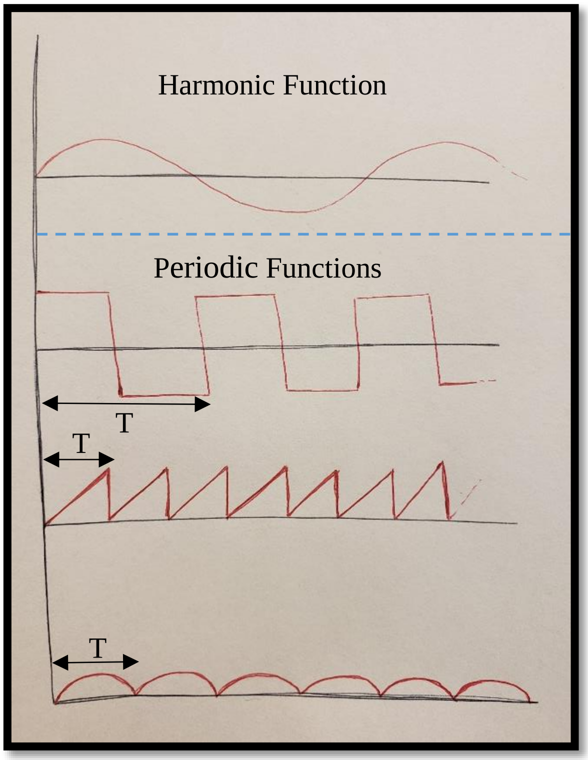
$$r_2 - r_1 = 2\zeta$$

$$\boxed{\zeta = \frac{1}{2}(r_2 - r_1)}$$

B) Periodic Force

In the real life there are many cases that system is subjected to a periodic force. In reality a true harmonic excitation is very rare and mostly exist and study on the laboratories.

In reality you can see many situations that the forcing function is periodic and repeated by period of *"$\boldsymbol{T}$"* but it is not harmonic!



For any periodic functions, we can find a proper Fourier series (include summation of large number of harmonic terms) that reproduce the periodic function. In other words, a periodic function can be find by summation of lots of harmonic functions as a Fourier series.

In general, if we have a periodic function with period of ($\boldsymbol{T}$) and frequency of $\omega = \frac{2\pi}{T}$, Fourier series represent this function can be write as follow:

$$f(t) = a_0 + a_1 cos\ \omega t + a_2 cos\ 2\omega t + a_3 cos\ 3\omega t + \cdots + a_n cos\ n\omega t\ + b_1 sin\ \omega t + b_2 sin\ 2\omega t + b_3 sin\ 3\omega t + \cdots + b_n sin\ n\omega t$$

For any periodic function, we only need to find all proper coefficients for above Fourier series (a_0, a_1, a_2, …, a_n, b_1, b_2, …, b_n). Then, that periodic function would be converted to a summation of series of harmonic functions.

This Fourier series can be write as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n cos\ n\omega t\ + b_n sin\ n\omega t\ ]$$

For the Fourier series, a_0, a_n, b_n would be:

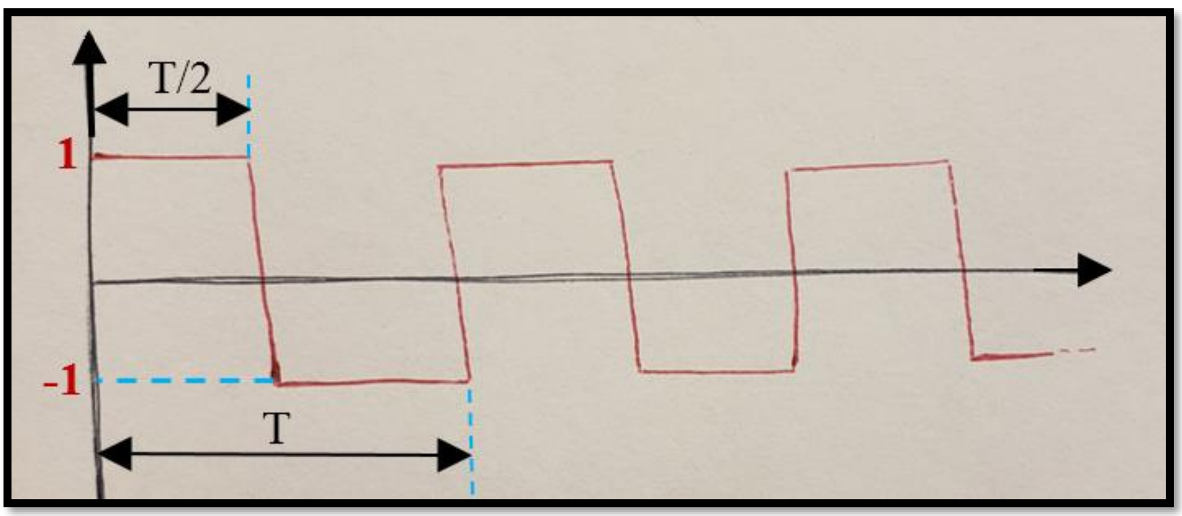
$a_0 = \frac{1}{T} \int_0^T f(t)\, dt$

$a_n = \frac{2}{T} \int_0^T f(t) cos\ n\omega t\ \ dt$

$b_n = \frac{2}{T} \int_0^T f(t) sin\ n\omega t\ \ dt$

So, for any problem, only we need to insert any given periodic function and period of (T) in these equations, find these coefficients and apply them in Fourier series.

Example 33: The following graph showing a periodic function. Find the proper Fourier series which is representing this function.



Step1: The first step in solving a problem like this is finding the $f(t)$ from the given graph.

$$f(t) = \begin{cases} 1 & 0 < t < T/2 \\ -1 & T/2 < t < T \end{cases}$$

Step2: find the coefficients

As this function is discontinues, we have to break the integral.

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left[\int_0^{T/2} 1\,dt + \int_{T/2}^{T} -1\,dt\right] = \frac{1}{T}\left[\frac{T}{2} - \left(T - \frac{T}{2}\right)\right] = \boxed{0}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\;dt = \frac{2}{T}\left[\int_0^{T/2} 1.\cos n\omega t\,dt + \int_{T/2}^{T} -1\,.\cos n\omega t\,dt\right]$$

$$a_n = \frac{2}{T}\left[\frac{1}{n\omega}\sin n\omega t\Big|_0^{T/2} - \frac{1}{n\omega}\sin n\omega t\Big|_{T/2}^{T}\right]$$

$$a_n = \frac{2}{T}\left[\frac{1}{n\omega}\sin n\omega\frac{T}{2} - \frac{1}{n\omega}\left(\sin n\omega T - \sin n\omega\frac{T}{2}\right)\right] = \frac{2}{n\omega T}\left[2\sin n\omega\frac{T}{2} - \sin n\omega T\right]$$

$$a_n = \frac{2}{n\omega T}\left[2\sin n\frac{2\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}} - \sin n\frac{2\pi}{\cancel{T}}\cancel{T}\right]$$

$$a_n = \frac{4}{n\omega T}\underbrace{\cancel{\sin n\pi}}_{0} - \frac{2}{n\omega T}\underbrace{\cancel{\sin 2n\pi}}_{0} = \boxed{0}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\;dt = \frac{2}{T}\left[\int_0^{T/2} 1.\sin n\omega t\,dt + \int_{T/2}^{T} -1\,.\sin n\omega t\,dt\right]$$

$$b_n = \frac{2}{T}\left[-\frac{1}{n\omega}\cos n\omega t\Big|_0^{T/2} + \frac{1}{n\omega}\cos n\omega t\Big|_{T/2}^{T}\right]$$

$$b_n = \frac{2}{T}\left[-\frac{1}{n\omega}\cos n\omega\frac{T}{2} + \frac{1}{n\omega} + \frac{1}{n\omega}\cos n\omega T - \frac{1}{n\omega}\cos n\omega\frac{T}{2}\right]$$

$$b_n = \frac{2}{n\omega T}\left[-2\cos n\omega\frac{T}{2} + 1 + \cos n\omega T\right] = \frac{\cancel{2}}{n\frac{\cancel{2}\pi}{\cancel{T}}\cancel{T}}\left[-2\cos n\frac{2\pi}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}} + 1 + \cos n\frac{2\pi}{\cancel{T}}\cancel{T}\right]$$

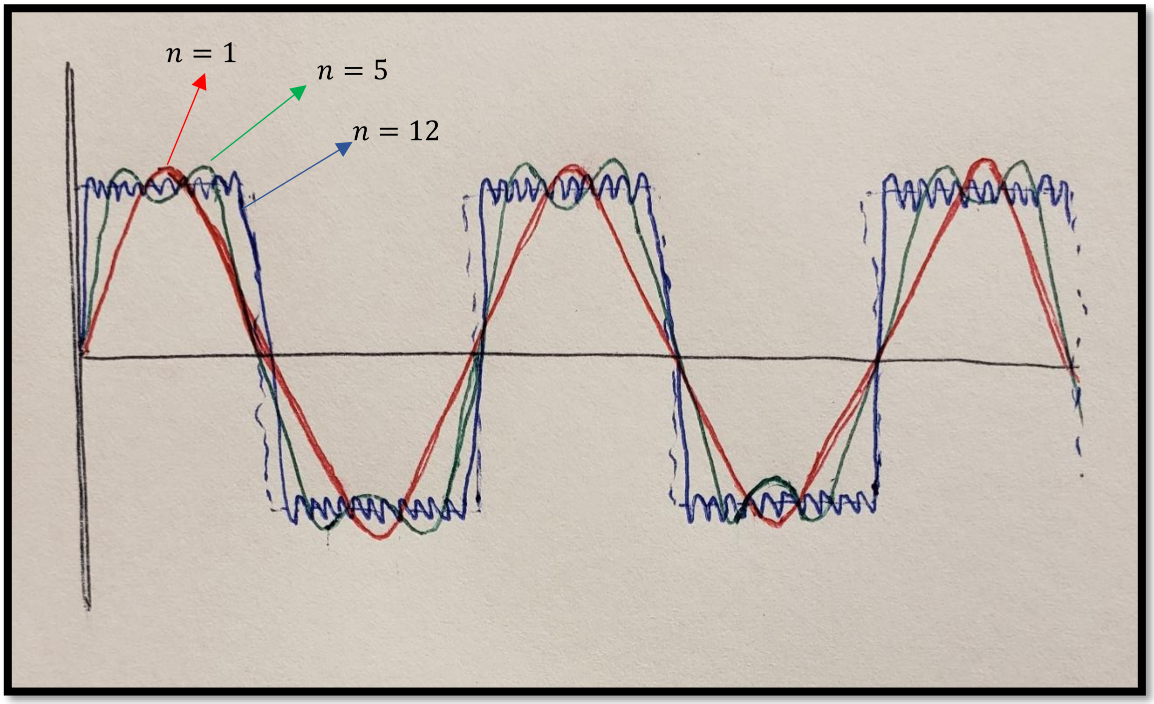
$$b_n = \frac{1}{n\pi}\left[-2\cos n\pi + 1 + \cos 2n\pi\right]$$

$$\boxed{b_n = \begin{cases} b_n = 0 & n = even\ numbers\ (2,4,6,\dots) \\ b_n = \dfrac{4}{n\pi} & n = odd\ numbers\ (1,3,5,\dots) \end{cases}}$$

Step3: write the periodic function in form of Fourier series

$$\boxed{f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{T}t}$$

If we plot this function, for $n = 1$, $n = 5$, $n = 12$ terms of this Fourier series and compare it with the original function, you will see, with increasing the number of terms in Fourier series, the results will be closer to the original function. Usually for $n > 100$ this function become very close (almost same) to the original function with the advantage of working with series of harmonic terms.



Application for using Fourier series in vibration:

If we having forcing function in the form of periodic function, we can write it in form of Fourier series, find the solution for each term in series and add them with each other (use superposition).

$$m\ddot{u} + c\dot{u} + ku = a_0 + \sum_{n=1}^{\infty}[a_n cos\ n\omega t\ + b_n sin\ n\omega t\ ]$$

$a_0 \to u_{static}$ $\longrightarrow$ Find the solution for the static term

$\sum_{n=1}^{\infty} a_n cos\ n\omega t \to u_{Cn}(\cos terms)$ $\longrightarrow$ Find the solution for all of the cos terms

$\sum_{n=1}^{\infty} b_n sin\ n\omega t \to u_{sn}(\sin terms)$ $\longrightarrow$ Find the solution for all of the sin terms

Add all solutions to get final solution. It would be hard with hand calculation but you can use some computational software (e.g. Matlab) to do this.

In general form we have from before:

$$m\ddot{u} + c\dot{u} + ku = F_0\, cos\omega t = \left(\frac{F_0}{k}\right)\frac{1}{\sqrt{[(1-r^2)^2 + (2\zeta r)^2]}}$$

$$u(t) = u_{static} + \sum_{n=1}^{\infty}(u_{Cn} + u_{sn})$$

$u_{static} = \frac{F_0}{k} = \frac{a_0}{k}$ $\quad (\frac{1}{\sqrt{[(1-r^2)^2+(2\zeta r)^2]}} = 0)$

$$\boxed{u(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty}\frac{a_n}{k}\frac{1}{\sqrt{[(1-n^2r^2)^2 + (2\zeta nr)^2]}}\cos(n\omega t - \theta_n) + \sum_{n=1}^{\infty}\frac{b_n}{k}\frac{1}{\sqrt{[(1-n^2r^2)^2 + (2\zeta nr)^2]}}\sin(n\omega t - \theta_n)}$$

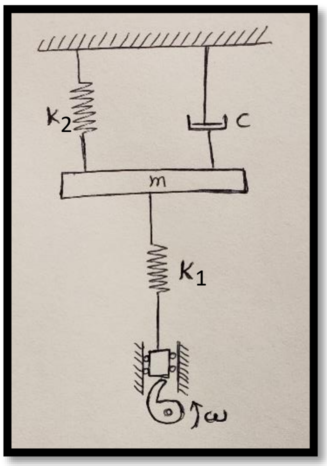
General Response

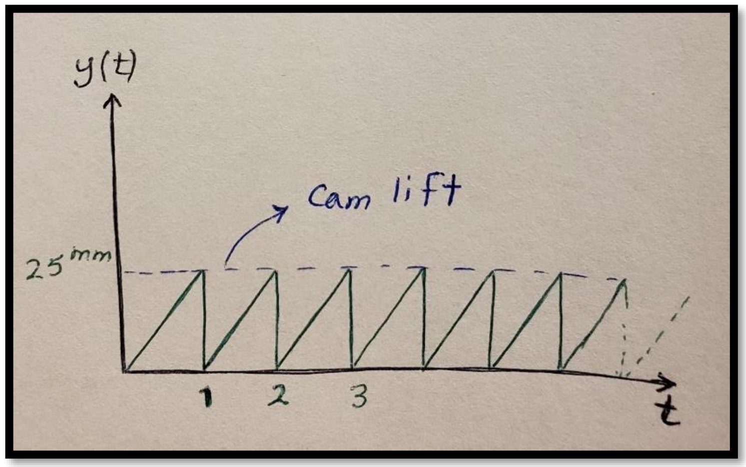
When for harmonically excited system θ_n would be:

$$\boxed{\theta_n = tan^{-1}\frac{2\zeta nr}{1-n^2r^2}}$$

B) Periodic Force

Example 34: A SODF system is shown in following figure. This system includes two springs with same stiffness of $k = 3.5\ kN/m$ and a damper with damping coefficient of $c = 0.2\ kN.sec/m$ and a mass $m = 20\ kg$. A cam drive mechanism is attached to the spring and move the system up. The cam operating at a frequency of $\omega = 60\ rpm$. The input displacement function from the cam to the system is showing on the following graph. Find the response of the system u(t) based on this input displacement function.





The input displacement is just related to spring (1), so the equation of motion would be:

$$m\ddot{u} + c\dot{u} + ku = k_1\, y(t)$$

$k_1 = 3500\, {}^{N}/_{m}$, $\omega = 60 \times \frac{2\pi}{60} = 2\pi\ \frac{rad}{s}$, $y_{max} = 25\ mm = 0.025m$, $T = 1\ sec$

$f(t) = k_1\, y(t) = (3500)(0.025t) = 87.5t \quad (0 < t < 1)$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \int_0^1 87.5t\,dt = \frac{87.5}{2}\ t^2\ |_0^1 = \boxed{\frac{87.5}{2}}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\ \ dt = 2\int_0^1 87.5t\cos n\omega t\ \ dt$$

$$a_n = 2 \times 87.5\left[\frac{1}{(n\omega)^2}\cos n\omega t + \frac{1}{n\omega}t\sin n\omega t\right]|_0^1$$

$$a_n = 2 \times 87.5\left[\cancel{\frac{1}{(2\pi n)^2}} + 0 - \cancel{\frac{1}{(2\pi n)^2}} - 0\right] = \boxed{0}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\ \ dt = 2\int_0^1 87.5t\sin n\omega t\ \ dt$$

$$b_n = 2 \times 87.5 \left[\frac{1}{(n\omega)^2}\ sin\ n\omega t - \frac{1}{n\omega} t\ cos\ n\omega t\right]\Big|_0^1$$
$$b_n = 2 \times 87.5 \left[0 - \frac{1}{2\pi n} - 0 + 0\right] = \boxed{-\frac{87.5}{\pi n}}$$

Note: $\int u dv = uv - \int v du$, For example:

$\int t\ cos\ (t) dt$, $u = t, dv = cos\ (t)$ $\Longrightarrow$ $v = sin\ (t)$, $du = 1$
$\int u dv = uv - \int v du = t\ sin\ (t) - \int sin\ (t) dt = t\ sin\ (t) + cos\ (t)$

The springs in this system are going through same deformation, so they are parallel with each other. So total stiffness of the system would be equal to:

$$k_{eq} = k_1 + k_2 = 3500 + 3500 = \boxed{7000\ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{k.m}} = \frac{0.2\times10^3}{2\sqrt{7000\times20}} = \boxed{0.267}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7000}{20}} = \boxed{18.71\ \frac{rad}{s}}$$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi}{18.71} = \boxed{0.11\pi}$$

$$u(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty} \frac{a_n}{k} \frac{1}{\sqrt{[(1-(rn)^2)^2 + (2\zeta nr)^2]}} \cos(n\omega t - \theta_n) + \sum_{n=1}^{\infty} \frac{b_n}{k} \frac{1}{\sqrt{[(1-(rn)^2)^2 + (2\zeta nr)^2]}} sin(n\omega t - \theta_n)$$

$$\theta_n = tan^{-1} \frac{2\zeta nr}{1 - n^2 r^2}$$

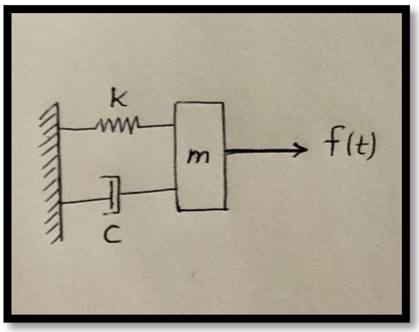
$$\boxed{u(t) = \frac{87.5}{2(7000)} - \frac{87.5}{\pi(7000)} \sum_{n=1}^{\infty} \frac{1}{n} \frac{sin(2\pi nt - \theta_n)}{\sqrt{[(1-(0.11n\pi)^2)^2 + (2 \times 0.267 \times n \times 0.11\pi)^2]}}}$$

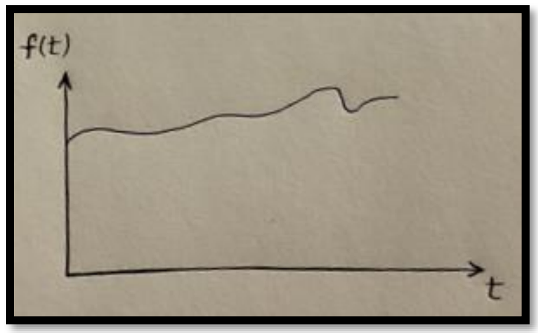
$$\boxed{\theta_n = tan^{-1} \frac{0.0587 n\pi}{1 - (0.11n\pi)^2}}$$

C) General Forcing Function

As the last type of forcing function, we study the general forcing function without any limitation. The force function can be impulse force, impact load, blast load, earthquake load, etc.

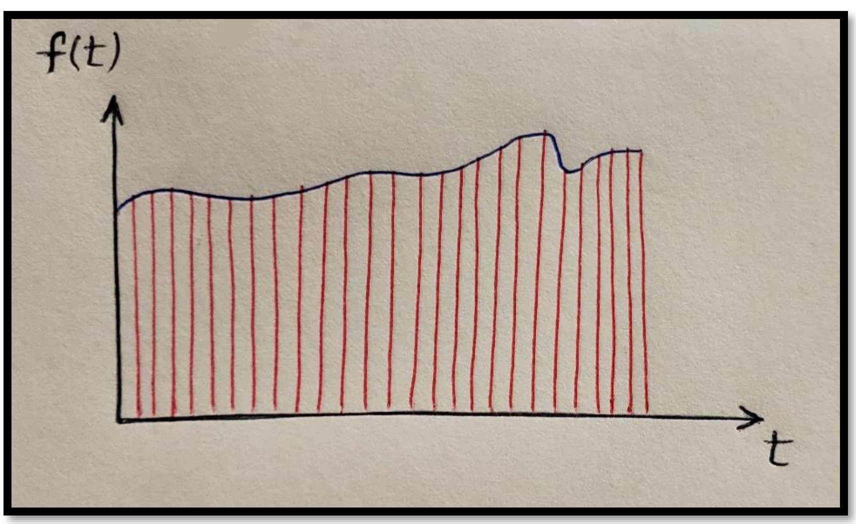
Look at the response of the mass under the action of a general function $f(t)$. In this case the approach to drive the response is done in two different ways: 1) Convolution Theorem 2) Duhamel's Integral. With using both approaches, we reach the same results. However, in this class, we are only working on Duhamel's Integral method.



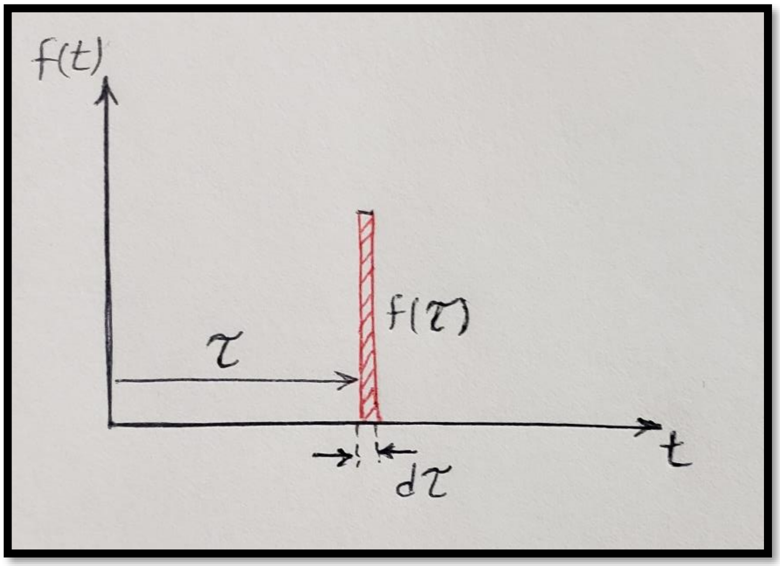


First of all, we assume system is **linear** and that means if you have several loads/forces acting on the system, if you find the response to the system under the action of each **individual load** and add them up, that will give you **total response**.

For the previous function, we can break down that function to infinite number of loads.



If we can find the response of the system under the load of $f(\tau)$ for any arbitrary point at the time of (τ) with duration of $(d\tau)$, then we can apply it throughout the entire period and find total solution.



Let's introduce some terminologies:

1) Impulse loading: Load with a short duration.

$Impulse = f(\tau).d\tau$

2) Impulse apply to a system causes a change in the momentum

$f_i(\tau).d\tau \xrightarrow{\text{For simplicity}} f.d\tau = d(m\dot{u}) \longrightarrow \boxed{f = \frac{d}{d\tau}(m\dot{u})}$

If the mass is constant, we can say the **impulse** causes a change in **velocity**.

$$f.d\tau = d(m\dot{u}) = md(\frac{du}{dt}) \longrightarrow \boxed{\frac{f.d\tau}{m} = \frac{d}{dt}(du)}$$

In the other words, from initial condition for system would be:

$$\begin{cases} u_0 = 0 \\ \dot{u}_0 = \dfrac{f.d\tau}{m} \end{cases}$$

This concept is the general form of the Newton's second law!

Newton's second law: Rate of change of momentum of the mass is equal to the force acting on it.

For finding the response for general force, we assume we don't have a force vibration but we have a system which due to an impulse, it is subjected to an initial velocity. So, that means we have a free vibration, due to initial conditions (just initial velocity).

From before we have the steady state response for free vibration of a damped system would be:

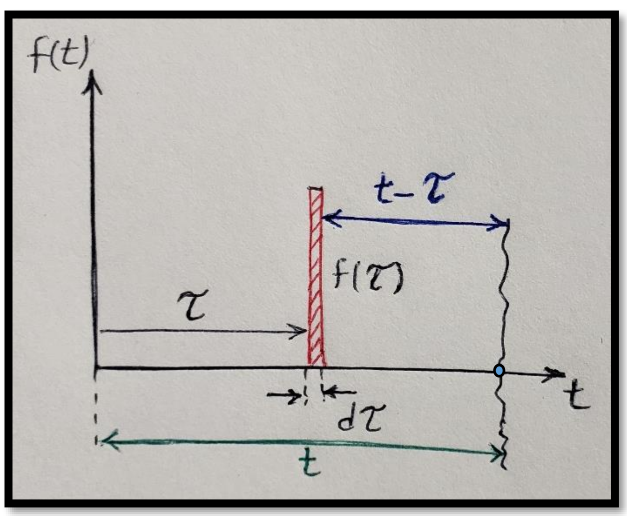
$$u(t) = e^{-\zeta\omega_n t}\left[\left(\frac{\zeta\omega_n}{\omega_d}\, sin\ \omega_d t + cos\omega_d t\right)u_0 + \frac{sin\ \omega_d t}{\omega_d}\dot{u}_0\right]$$

But in this case and from initial conditions $u_0 = 0$ & $\dot{u}_0 = \frac{f.d\tau}{m}$, so we have:

$$u(t) = e^{-\zeta\omega_n t}\left[\cancelto{0}{\left(\frac{\zeta\omega_n}{\omega_d}\, sin\ \omega_d t + cos\omega_d t\right)u_0} + \frac{sin\ \omega_d t}{\omega_d}\cancelto{\frac{f.d\tau}{m}}{\dot{u}_0}\right]$$

$$\boxed{u(t) = \frac{f.d\tau}{m} \times \frac{1}{\omega_d} e^{-\zeta\omega_n t}\, sin\ \omega_d t}$$

However, we are looking to find the response of system at any point (moment) after this impulse was applied (a general point).



The response of the system on that point would be:

$$\boxed{u(t-\tau) = \frac{f.d\tau}{m} \times \frac{1}{\omega_d} e^{-\zeta\omega_n (t-\tau)}\, sin\ \omega_d (t-\tau)}$$

However, this is the response of the system to one of these impulses! Now, we need to find the total response of system by adding the responses for all small impulses. The total response would be:

$$\boxed{u(t) = \frac{1}{m\omega_d}\int_0^t f(\tau)\, e^{-\zeta\omega_n(t-\tau)}\, sin\, \omega_d(t-\tau)\, d\tau}$$

Duhamel's Integral

τ: A dummy variable.

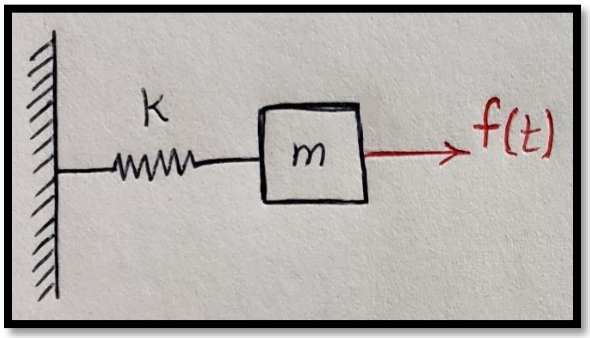
For an Undamped system($\zeta = 0$), we will have:

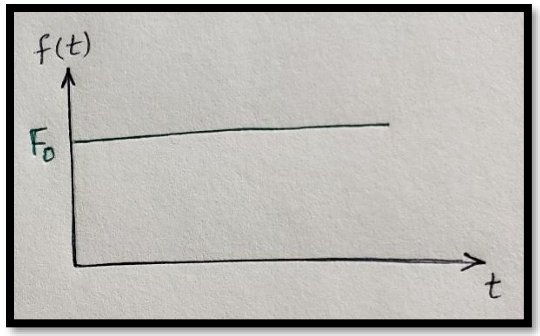
$$\boxed{u(t) = \frac{1}{m\omega_n}\int_0^t f(\tau)\; sin\, \omega_n(t-\tau)\, d\tau}$$

C) General Forcing Function

Let's see some application of this concept.

Example 35: We have an undamped system which is subjected to an impact load (F_0). Find the maximum displacement due to this load.





Impact load: Whenever a machine members are subjected to load with a sudden impact due to falling or hitting one object on another (**zero** force at time zero but suddenly change to some **constant force**).

$$f(t) = \begin{cases} 0 & t \le 0 \\ F_0 & t > 0 \end{cases}$$

Note: There is **big difference** between the **impact load** and **static load**. In static load, the load adding to the system gradually during process but for impact load, we dropping an object on system and from zero load suddenly adding a constant load to system (that causes initial conditions on system).

$$u(t) = \frac{1}{m\omega_n}\int_0^t f(\tau)\ \sin \omega_n(t-\tau)\ d\tau = \frac{F_0}{m\omega_n}\int_0^t \sin \omega_n(t-\tau)\ d\tau$$

$$u(t) = \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n}\cos \omega_n(t-\tau)\right]\Big|_0^t = \frac{F_0}{m{\omega_n}^2}(1-\cos \omega_n t) = \frac{F_0}{\cancel{m}\frac{k}{\cancel{m}}}(1-\cos \omega_n t)$$

$$\boxed{u(t) = \frac{F_0}{k}(1-\cos \omega_n t)}$$

$\delta_{static} = \frac{F_0}{k}$ (Static displacement)

The maximum of $u(t)$ would be happened on $cos\ \omega_n t = -1$

$$\boxed{u_{max} = 2\frac{F_0}{k} = 2\,\delta_{static}}$$

Dynamic Factor (DF)

Dynamic Load Factor (DLF)

That is why for an impact load in machine design we always consider factor of safety of 2 for a dynamic load!

Now let's look at the **damped case** but with very small damping ratio ($\zeta \neq 0$ & $\zeta \ll 1$)

So, we have to assume τ as a dummy variable and solve the Duhamel's integral.

$$u(t) = \frac{1}{m\omega_d}\int_0^t f(\tau)\, e^{-\zeta\omega_n(t-\tau)}\ sin\ \omega_d(t-\tau)\, d\tau$$

$$u(t) = \underbrace{\frac{F_0}{k}}_{\text{Static Displacement (constant)}}\underbrace{\left[1 - e^{-\zeta\omega_n t}\left(cos\ \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\, sin\ \omega_d t\right)\right]}_{\text{DLF (We have to find the maximum of this part)}}$$

To find the u_{max}, we have to solve $\left(\frac{du}{dt} = 0\right)$ and find t, and substitute it in above equation.

$$\frac{du}{dt} = \zeta\omega_n \underbrace{e^{-\zeta\omega_n t}}_{\neq 0}\left(cos\ \omega_d t + \frac{\zeta}{\underbrace{\sqrt{1-\zeta^2}}_{\text{For } \zeta \ll 1,\ \approx 1}}\, sin\ \omega_d t\right) - \underbrace{e^{-\zeta\omega_n t}}_{\neq 0}\left(-\omega_d\, sin\ \omega_d t + \frac{\zeta\omega_d}{\underbrace{\sqrt{1-\zeta^2}}_{\text{For } \zeta \ll 1,\ \approx 1}}\, cos\ \omega_d t\right) = 0$$

$$\frac{du}{dt} = \cancel{\zeta\omega_n cos\ \omega_d t} + \zeta^2\omega_n\, sin\ \omega_d t + \omega_d\, sin\ \omega_d t - \cancel{\zeta\omega_d cos\ \omega_d t} = 0$$

$$\frac{du}{dt} = \underbrace{(\zeta^2\omega_n + \omega_d)}_{\text{Constant and it can't be zero}}\, sin\ \omega_d t = 0$$

$$\sin \omega_d t = 0 \longrightarrow \omega_d t = \pi \longrightarrow \boxed{t = \frac{\pi}{\omega_d}}$$

For $\zeta \ll 1, \approx 1$

$$DLF = 1 + e^{-\zeta\omega_n \frac{\pi}{\omega_d}} = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 1 + e^{-\zeta\pi}$$

Note: $(\omega_d = \sqrt{1-\zeta^2}\,\omega_n)$

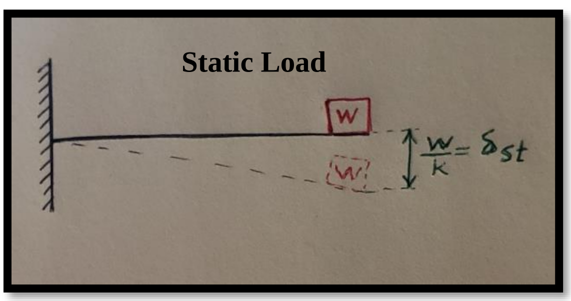
$$\boxed{DLF = 1 + e^{-\zeta\pi}}$$

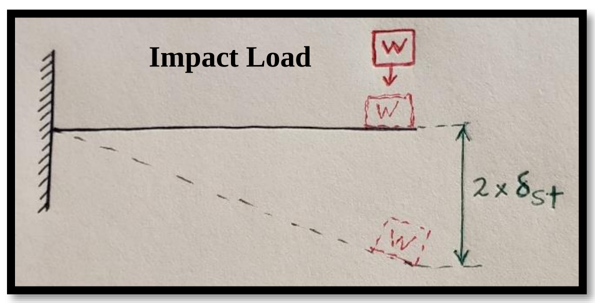
Even in case of damped system, as long as we have a small damping ratio (ζ less than 10%), DLF would be very close to DLF for undamped system ($DLF \cong 2$). So, for the case of impact loading, it would be a fair assumption to consider $DLF = 2$ in the design.

$\zeta = 0.01$ $\qquad DLF = 1 + e^{-\zeta\pi} = 1.97 \cong 2$

$\zeta = 0.02$ $\qquad DLF = 1 + e^{-\zeta\pi} = 1.94 \cong 2$

Here is comparison between displacement caused by a static load and an impact load:





Applications for Duhamel's Integral:

First Application: How to evaluate the maximum displacement u_{max} vs static displacement $(\frac{F_0}{k})$, if we have general forcing function.

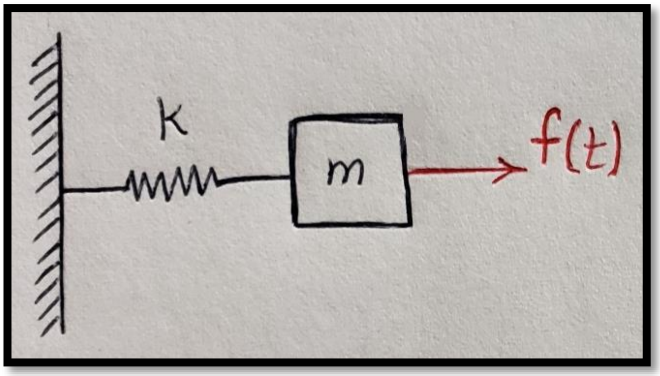
Second Application: Shock Spectrum

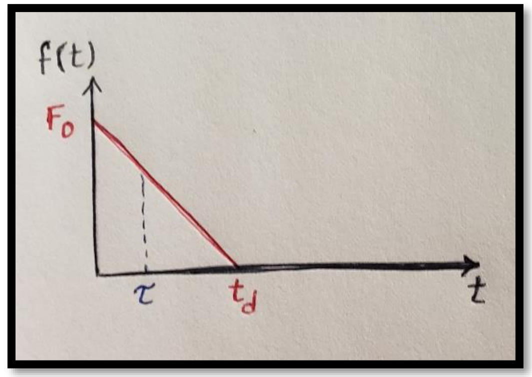
First Application:

We already study the simple differential equations and saw how to solve, but in the case of Duhamel's Integral, the integration is the most important part. In the forced vibration, we studied the harmonic forces (and periodic forces) to drive displacement function $u\ (t)$. But the ultimate goal is finding the maximum displacement u_{max} to find the maximum stresses σ_{max} for the design purposes. In the case of harmonic forces, independent to details of force function (amplitude, frequency, etc.), the maximum displacement was equal to multiplication of static displacement and frequency response function $u_{max} = \frac{F_0}{k} \times |H|$. However, in general force function, we have to divide the function to different regions and check each of them for maximum displacement and compare with each other to find where the maximum displacement will be happened.

C) General Forcing Function

Example 36: There is a ramp load apply to an undamped system as it is shown in following graph. Time τ represent any arbitrary time from zero to time t_d. Find the maximum displacement u_{max} vs static displacement $(\frac{F_0}{k})$ of this system.





Wrong assumption: There is no load after time t_d on this system so we can assume there is no load applied on this system at all and system has not any vibration!

This is wrong assumption because before time t_d, the force causes some initial condition (initial velocity) on system and after time t_d, we can assume system as a free vibration case!

According to the graph, forcing function would be:

$$f(t) = \begin{cases} \left(1 - \frac{\tau}{t_d}\right) F_0 & 0 \le t < t_d \\ 0 & t > t_d \end{cases}$$

In this case, we can divide the function to two regions: 1) $0 \le t < t_d$ 2) $t > t_d$

1) $0 \le t < t_d$

Displacement function for an Undamped system($\zeta = 0$), would be:

Note: we are taking integral from 0 to t!

$$u(t) = \frac{1}{m\omega_n} \int_0^t f(\tau)\ \sin \omega_n (t - \tau)\, d\tau$$

$$u(t) = \frac{F_0}{m\omega_n} \int_0^t \left(1 - \frac{\tau}{t_d}\right) \sin \omega_n (t - \tau)\, d\tau$$

$$\sin \omega_n (t - \tau) = \boxed{\sin \omega_n t} \cos \omega_n \tau - \boxed{\cos \omega_n t} \sin \omega_n \tau$$

Constant

$$u(t) = \frac{F_0}{m\omega_n}\int_0^t \left(1-\frac{\tau}{t_d}\right) sin\ \omega_n t\ cos\ \omega_n\tau\ d\tau - \frac{F_0}{m\omega_n}\int_0^t \left(1-\frac{\tau}{t_d}\right)\ cos\ \omega_n t\ sin\ \omega_n\tau\ d\tau$$

$$u(t) = \frac{F_0}{m\omega_n}\left\{sin\ \omega_n t\left[\underbrace{\boxed{\int_0^t cos\ \omega_n\tau\ d\tau}}_{1} - \frac{1}{t_d}\underbrace{\boxed{\int_0^t \tau\ cos\ \omega_n\tau\ d\tau}}_{2}\right] - cos\ \omega_n t\left[\underbrace{\boxed{\int_0^t sin\ \omega_n\tau\ d\tau}}_{3} - \frac{1}{t_d}\underbrace{\boxed{\int_0^t \tau\ sin\ \omega_n\tau\ d\tau}}_{4}\right]\right\}$$

We have 4 integrals that need to be solved:

1) $\int_0^t cos\ \omega_n\tau\ d\tau = \frac{1}{\omega_n}\ sin\ \omega_n\tau\Big|_0^t = \boxed{\frac{1}{\omega_n}\ sin\ \omega_n t}$

3) $\int_0^t sin\ \omega_n\tau\ d\tau = -\frac{1}{\omega_n}\ cos\ \omega_n\tau\Big|_0^t = \boxed{\frac{1}{\omega_n} - \frac{1}{\omega_n}\ cos\ \omega_n t}$

For integration number 2 & 4, we need to use the integration by parts:

$$\int_a^b u\ dv = uv\Big|_a^b - \int_a^b v\ du$$

2) $\int_0^t \tau\ cos\ \omega_n\tau\ d\tau$

$u = \tau \quad , \quad du = d\tau \quad , \quad dv = cos\ \omega_n\tau\ d\tau \quad , \quad v = \frac{1}{\omega_n}\ sin\ \omega_n\tau$

$$\int_0^t \tau\ cos\ \omega_n\tau\ d\tau = \frac{\tau}{\omega_n}\ sin\ \omega_n\tau\Big|_0^t - \int_0^t \frac{1}{\omega_n}\ sin\ \omega_n\tau\ d\tau = \frac{t}{\omega_n}\ sin\ \omega_n t - \frac{1}{\omega_n}\left[-\frac{1}{\omega_n} cos\ \omega_n\tau\right]\Big|_0^t$$

$$\boxed{\int_0^t \tau\ cos\ \omega_n\tau\ d\tau = \frac{t}{\omega_n}\ sin\ \omega_n t + \frac{1}{{\omega_n}^2}(cos\ \omega_n t - 1)}$$

4) $\int_0^t \tau\ sin\ \omega_n\tau\ d\tau$

$u = \tau \quad , \quad du = d\tau \quad , \quad dv = sin\ \omega_n\tau\ d\tau \quad , \quad v = -\frac{1}{\omega_n}\ cos\ \omega_n\tau$

$$\int_0^t \tau\ sin\ \omega_n\tau\ d\tau = -\frac{\tau}{\omega_n}\ cos\ \omega_n\tau\Big|_0^t + \int_0^t \frac{1}{\omega_n}\ cos\ \omega_n\tau\ d\tau = -\frac{t}{\omega_n}\ cos\ \omega_n t + \frac{1}{\omega_n}\left[\frac{1}{\omega_n} sin\ \omega_n\tau\right]\Big|_0^t$$

$$\boxed{\int_0^t \tau\ sin\ \omega_n\tau\ d\tau = -\frac{t}{\omega_n}\ cos\ \omega_n t + \frac{1}{{\omega_n}^2}\ sin\ \omega_n t}$$

Total response of system, $u(t)$ for $\mathbf{(0 < t \le t_d)}$:

$$\boxed{\begin{aligned} u(t) = \frac{F_0}{m\omega_n}\Bigg\{ & sin\ \omega_n t\left[\frac{1}{\omega_n}\ sin\ \omega_n t - \frac{t}{\omega_n t_d}\ sin\ \omega_n t - \frac{1}{{\omega_n}^2 t_d} cos\ \omega_n t + \frac{1}{{\omega_n}^2 t_d}\right] \\ & - cos\ \omega_n t\left[\frac{1}{\omega_n} - \frac{1}{\omega_n}\ cos\ \omega_n t + \frac{t}{\omega_n t_d}\ cos\ \omega_n t - \frac{1}{{\omega_n}^2 t_d}\ sin\ \omega_n t\right]\Bigg\} \end{aligned}}$$

Simplify version of $u(t)$:

$$\boxed{u(t) = \frac{F_0}{k}\left\{1 - \frac{t}{t_d} + \frac{1}{\omega_n t_d} sin\ \omega_n t - cos\ \omega_n t\right\}}$$

Now for finding the maximum displacement (u_{max}), we have to take derivative of $u(t)$ and set it equal zero and solve it for t, put this t value back to the $u(t)$ to find the u_{max}.

2) $t > t_d$

For this part, we can use two different methods:

A) As it can be seen after time t_d, there is no force acting on the system and the system is undamped. So, we have a free vibration of an undamped system that having some initial conditions which can be find from previous part ($u(t_d)$ & $\dot{u}(t_d)$)

B) We can write the Duhamel's integral for part 1 (integral from $0\ to\ t_d$) plus part 2 (integral from $t_d\ to\ t$) for any $t > t_d$

$$u(t) = \frac{F_0}{m\omega_n}\int_0^{t_d}\left(1 - \frac{\tau}{t_d}\right)\ sin\ \omega_n(t-\tau)\ d\tau + \underbrace{\cancelto{0}{\frac{1}{m\omega_n}\int_{t_d}^{t} f(\tau)\ sin\ \omega_n(t-\tau)\ d\tau}}_{f(\tau) = 0 \text{ for } t > t_d}$$

If we solve it, we will get the same result as previous section, but this time, integral from 0 to t_d. So, the result for second part of the graph would be:

$$u(t) = \frac{F_0}{m{\omega_n}^2}\left\{sin\ \omega_n t\left[-\frac{1}{\omega_n t_d} cos\ \omega_n t_d + \frac{1}{\omega_n t_d}\right] - cos\ \omega_n t\left[1 - \frac{1}{\omega_n t_d}\ sin\ \omega_n t_d\right]\right\}$$

$$\boxed{u(t) = \frac{F_0}{k}\left\{sin\ \omega_n t\left[-\frac{1}{\omega_n t_d} cos\ \omega_n t_d + \frac{1}{\omega_n t_d}\right] - cos\ \omega_n t\left[1 - \frac{1}{\omega_n t_d}\ sin\ \omega_n t_d\right]\right\}}$$

Note: The results from both methods would be same!

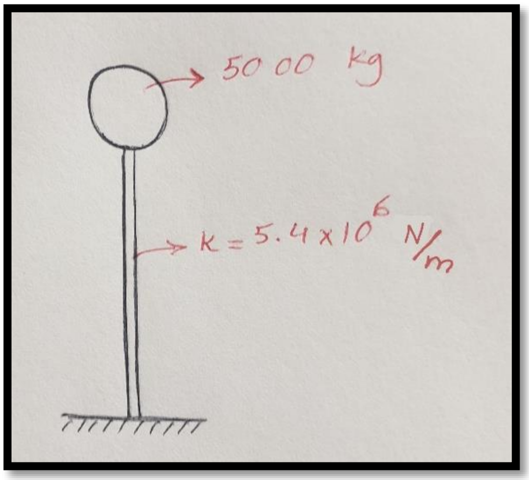
Then, we can find the maximum displacement (u_{max}), for this part same as the first part of the graph and compare them to find which of them is the true u_{max} for the system.

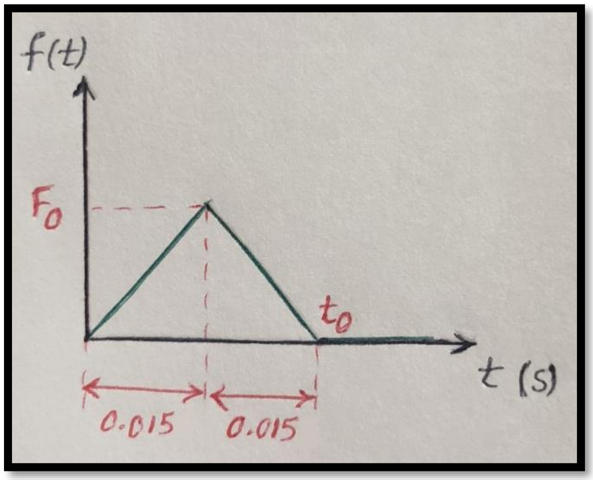
In general, a "General Force Function", based on the shape of the function, can be divide to several sections/parts (any changes in the forcing function will be consider as a new section). So, we have to evaluate a response function section by section, find the u_{max} for each of them, and compare to find the true u_{max} for the system.

- Do you think a blast loading or a static load has more severe effect on a structure? For answering to this question, look at the following problem.

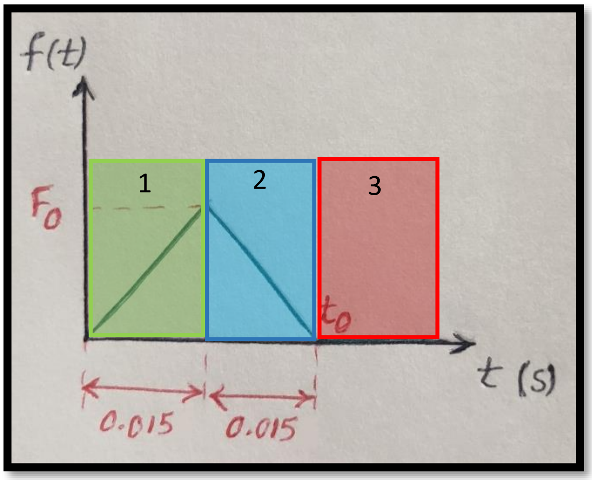
C) General Forcing Function

Example 37: The following figure showing a water tower carries 5,000 kg of water. The column has a stiffness of $k = 5.4 \times 10^6\ N/m$. This water tower is subjected to a blast loading (very short duration of loading) as shown in the following graph. Find the maximum displacement u_{max} of this water tower. Assume this structure as an undamped system.





In this case, we have 3 regions:



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.4\times10^6}{5000}} = 32.86\ \frac{rad}{s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{32.86} = 0.191\ sec$$

Undamped $\Longrightarrow \zeta = 0$

Now, to evaluate the response of system, we can use the Duhamel's integral for an undamped case..

$$u(t) = \frac{1}{m\omega_n}\int_0^t f(\tau)\ \sin\omega_n(t-\tau)\,d\tau$$

In this case, we have three regions for $f(\tau)$:

$$f(\tau) = \begin{cases} \dfrac{2F_0}{t_0}\tau & 0 < t < \dfrac{t_0}{2} \\ 2F_0\left(1-\dfrac{\tau}{t_0}\right) & \dfrac{t_0}{2} < t < t_0 \\ 0 & t_0 < t \end{cases}$$

Region 1: $0 < t < \frac{t_0}{2}$

$$u(t) = \frac{1}{m\omega_n}\int_0^t \frac{2F_0}{t_0}\tau\ \sin\omega_n(t-\tau)\,d\tau$$

Note: We are taking integral from $0\ to\ t$ **Not** from $0\ to \frac{t_0}{2}$! Actually, we are interested to drive an expression for the response of system caused by the force in region 1, for any moment (not only in region 1).

If we solve the integral for region 1, we will have:

Static response

$$\boxed{u(t) = \frac{2F_0}{m{\omega_n}^2 t_0}\left\{t + \frac{1}{\omega_n}[\sin\omega_n(t-\tau)]\Big|_0^t\right\} = \frac{2F_0}{kt_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right) = \left(\frac{F_0}{k}\right)\frac{2}{t_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)}$$

Note: Always the response to general loading depends on two important quantities: 1) natural frequency of the system (ω_n) 2) Duration of the forcing function (t_0). We will study later how can reach to maximum response as a function of these two quantities (Shock Spectrum)

Region 2: $\frac{t_0}{2} < t < t_0$

For this case, we have to add the response of system from $0\ to \frac{t_0}{2}$ to the response of system after time $\frac{t_0}{2}$.

$$u(t) = \frac{1}{m\omega_n}\int_0^{\frac{t_0}{2}} \frac{2F_0}{t_0}\tau\ sin\ \omega_n(t-\tau)\ d\tau + \frac{1}{m\omega_n}\int_{\frac{t_0}{2}}^{t} 2F_0\left(1-\frac{\tau}{t_0}\right)\ sin\ \omega_n(t-\tau)\ d\tau$$

$$\boxed{u(t) = \frac{2F_0}{kt_0}\left(t_0 - t + \frac{1}{\omega_n}\left[2\ sin\ \omega_n\left(t-\frac{t_0}{2}\right) - sin\ \omega_n t\right]\right)}$$

Region 3: $t_0 < t$

In this case, we have two options: After time t_0, we can look at the system as an undamped free vibration with some initial conditions ($u(t_0)$ & $\dot{u}(t_0)$) or we can rewrite the Duhamel's integral for all three regions.

$$u(t) = \frac{1}{m\omega_n}\int_0^{\frac{t_0}{2}} \frac{2F_0}{t_0}\tau\ sin\ \omega_n(t-\tau)\ d\tau + \frac{1}{m\omega_n}\int_{\frac{t_0}{2}}^{t_0} 2F_0\left(1-\frac{\tau}{t_0}\right)\ sin\ \omega_n(t-\tau)\ d\tau$$
$$+ \underbrace{\cancel{\frac{1}{m\omega_n}\int_{t_0}^{t} f(t)\ sin\ \omega_n(t-\tau)\ d\tau}}_{0}$$

$$u(t) = \frac{1}{m\omega_n}\int_0^{\frac{t_0}{2}} \frac{2F_0}{t_0}\tau\ sin\ \omega_n(t-\tau)\ d\tau + \frac{1}{m\omega_n}\int_{\frac{t_0}{2}}^{t_0} 2F_0\left(1-\frac{\tau}{t_0}\right)\ sin\ \omega_n(t-\tau)\ d\tau$$

$$\boxed{u(t) = \frac{2F_0}{kt_0\omega_n}\left[2\ sin\ \omega_n\left(t-\frac{t_0}{2}\right) - sin\ \omega_n t - sin\ \omega_n(t-t_0)\right]}$$

For finding the maximum displacement of the system, we have to take derivative of all three region, set them equal zero, find the value for t, put it back to equations and find the maximum displacement for each region (if there is any) and compare them to find the maximum displacement for the system.

1) Finding the maximum displacement for the first region

$$u(t) = \frac{2F_0}{kt_0}\left(t - \frac{1}{\omega_n} sin\ \omega_n t\right)$$

$$\frac{du}{dt} = \frac{2F_0}{kt_0}(1 - cos\ \omega_n t)$$

For region (1), $\frac{2F_0}{kt_0} > 0$ and maximum value for t is $0.015\ sec$, so $1 - \cos\omega_n t > 0$. Therefore, always $\frac{du}{dt} > 0$ for whole period of $0 < t < \frac{t_0}{2}$. That means $\frac{du}{dt}$ never become zero for any value of t in the region (1) and there is no maximum displacement for the region (1).

2) Finding the maximum displacement for the second region

$$u(t) = \frac{2F_0}{kt_0}\left(t_0 - t + \frac{1}{\omega_n}\left[2\sin\omega_n\left(t - \frac{t_0}{2}\right) - \sin\omega_n t\right]\right)$$

$$\frac{du}{dt} = \frac{2F_0}{kt_0}\left(-1 + 2\cos\omega_n\left(t - \frac{t_0}{2}\right) - \cos\omega_n t\right)$$

Let's first check for start and end point of the region (2).

For $t = \frac{t_0}{2}$

$$\frac{du}{dt} = \frac{2F_0}{kt_0}\left(-1 + \cancelto{2}{2\cos\omega_n\left(\frac{t_0}{2} - \frac{t_0}{2}\right)} - \cos\omega_n\frac{t_0}{2}\right) > 0$$

For $t = t_0$

$$\frac{du}{dt} = \frac{2F_0}{kt_0}\left(-1 + 2\cos\omega_n\left(t_0 - \frac{t_0}{2}\right) - \cos\omega_n t_0\right) = \frac{2F_0}{kt_0}\left(-1 + 2\cos(0.493) - \cos(0.986)\right) > 0$$

For region (2), $\frac{2F_0}{kt_0} > 0$ and for any value of t $\left(\frac{t_0}{2} < t < t_0\right)$, $\frac{du}{dt} > 0$. That means $\frac{du}{dt}$ never become zero for any value of t in the region (2) and there is no maximum displacement for the region (2). So, if there is any maximum displacement for the system, it has to be happen in region (3).

3) Finding the maximum displacement for the third region

First we can simplify the $u(t)$ for this region.

$$u(t) = \frac{2F_0}{kt_0\omega_n}\left[\underbrace{2\sin\omega_n\left(t - \frac{t_0}{2}\right)} - \sin\omega_n t - \underbrace{\sin\omega_n(t - t_0)}\right]$$

We can expand these parts

$$u(t) = \frac{2F_0}{kt_0\omega_n}\Big[2\, sin\, \omega_n t\, cos\ \ \omega_n \frac{t_0}{2} - 2\, cos\, \omega_n t\, sin\, \omega_n \frac{t_0}{2} - sin\, \omega_n t$$
$$- sin\, \omega_n t\, cos\, \omega_n t_0 + cos\, \omega_n t\, sin\, \omega_n t_0\Big]$$

$$u(t) = \frac{2F_0}{kt_0\omega_n}[sin\, \omega_n t[2\, cos\, \overbrace{(0.493)}^{\omega_n \frac{t_0}{2}\ (rad)} - 1 - cos\, \overbrace{(0.986)}^{\omega_n t_0\ (rad)}] + cos\, \omega_n t[-2\, sin\, (0.493)$$
$$+ sin\, (0.986)]]$$

$$\boxed{u(t) = \frac{2F_0}{kt_0\omega_n}[\overset{A}{0.208}\, sin\, \omega_n t - \overset{B}{0.113}\, cos\, \omega_n t]}$$

Harmonic Function

$$u(t) = \frac{2F_0}{kt_0\omega_n}\sqrt{A^2 + B^2}\, sin\, (\omega_n t + \emptyset)$$

Instead of take the derivative of this equation and equate to zero to find the u_{max}, we can see the response is a harmonic function, so the maximum value would be:

$$u_{max} = \frac{2F_0}{kt_0\omega_n}\sqrt{A^2 + B^2} = \frac{2 \times 5000 \times 9.81}{5.4 \times 10^6 \times 0.03 \times 32.86}\sqrt{0.208^2 + 0.113^2} = \boxed{0.004\, m}$$

$$\delta_{static} = \frac{F_0}{k} = \frac{5000 \times 9.81}{5.4 \times 10^6} = \boxed{0.009\, m}$$

As you can see, the maximum displacement made by a blast load is even less than a half of a static load! Therefore, it is not necessary the maximum response of a blast load (or any other short duration dynamic loads) be higher than corresponding static displacement of system.

C) General Forcing Function

General Approach-Numerical Integration:

In order to find the response under a general loading, using the analytical integral is hard and have to be solve for each regions. A more general approach to find the response of system under a general loading is using numerical integration.

$$u(t) = \frac{1}{m\omega_n} \int_0^t f(\tau)\ sin\ \omega_n(t-\tau)\ d\tau$$

$$u(t) = \frac{1}{m\omega_n} \int_0^t f(\tau)\ sin\ \omega_n t\ cos\ \omega_n \tau - cos\ \omega_n t\ sin\ \omega_n \tau\ d\tau$$

$\omega_n t = constant$

$$u(t) = \frac{1}{m\omega_n} sin\ \omega_n t \underbrace{\boxed{\int_0^t f(\tau)\ cos\ \omega_n \tau\ d\tau}}_{A} - \frac{1}{m\omega_n} cos\ \omega_n t \underbrace{\boxed{\int_0^t f(\tau)\ sin\ \omega_n \tau\ d\tau}}_{B}$$

$$u(t) = \frac{1}{m\omega_n}\{A\ sin\ \omega_n t - B\ cos\ \omega_n t\} = \frac{1}{m\omega_n}\sqrt{A^2+B^2}\ sin\ (\omega_n t + \emptyset)$$

(A & B are time dependent)

You can see, always $u(t)$ can be represent as a harmonic function! So, we only need to find the *A* and *B*. The procedure to find *A & B* is called numerical integration. The most common types of numerical integration methods: 1) Central Difference method 2) Newmark-β numerical integration 3) Wilson-θ scheme numerical integration.

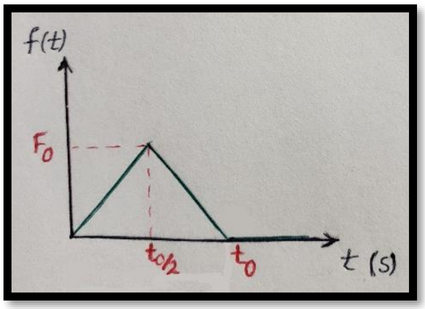
We are not cover any of these methods in these course and you can know about them in more advance level courses.

Second Application: Shock Spectrum/Response Spectrum

The maximum displacement under any kind of short duration loading which we call that shock loading is a function of two variables: time duration of the load (t_0) and natural frequency of the system (ω_n). Depending on these two variables, you may have a response with maximum displacement higher than corresponding static displacement or lower than that.

Note: We only know for impact loading the factor of safety should be equal “2”, and for other types of loading you have to evaluate the maximum value.

In harmonic loading the maximum displacement was multiplication of static displacement by maximum value of the frequency response function ($u_{max} = \frac{F_0}{k}|H|_{max}$). Shock Spectrum/Response Spectrum is sort of equivalent of that concept for maximum displacement in case of a general loading. To make it clear, let's look at the previous example again.

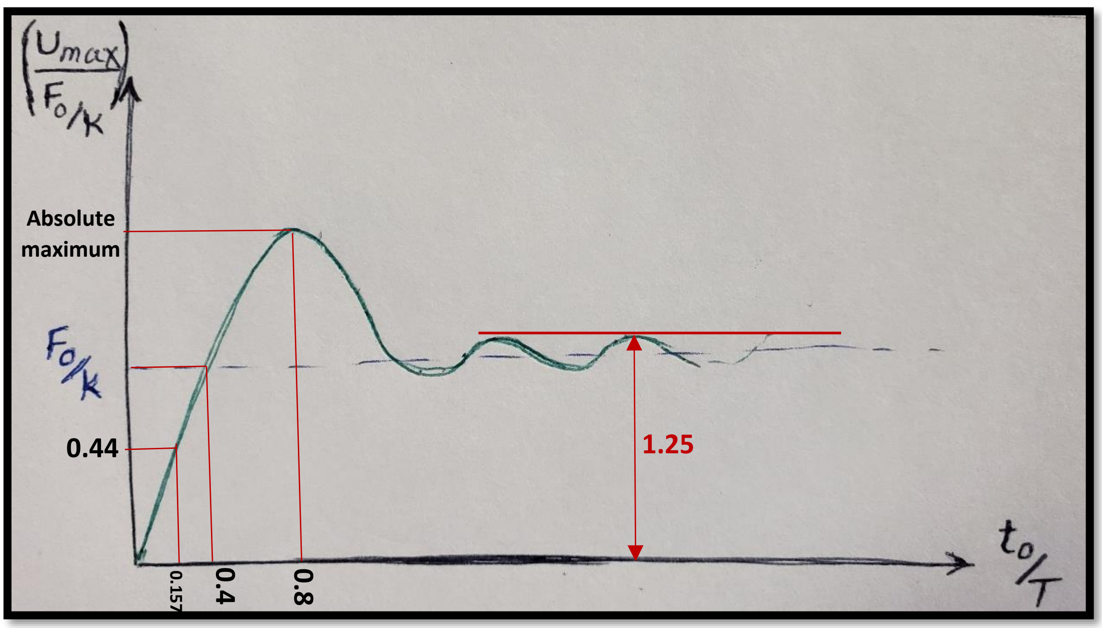


We found the u_{max} like this:

$$u(t) = \frac{2\frac{F_0}{k}}{t_0\omega_n}[0.208\,sin\,\omega_n t - 0.113\,cos\,\omega_n t]$$

$$u_{max} = \frac{2\frac{F_0}{k}}{t_0\omega_n} = 0.004m \quad , \quad \delta_{static} = \frac{F_0}{k} = 0.009\,m$$

If we plot the $\left(\frac{u_{max}}{\frac{F_0}{k}}\right)$ respect to $\frac{t_0}{T}$ (t_0= time duration of general force, T = Natural period of the system, $T = \frac{2\pi}{\omega_n}$), we will have:

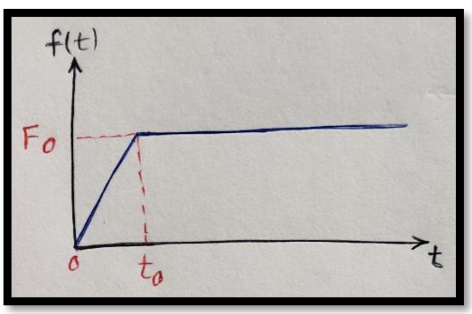


This plot showing the maximum response respect to normalized force time duration. In this problem, we had

$T = 0.191\ sec$
$t_0 = 0.03\ sec$ $\Rightarrow$ $\boxed{\dfrac{t_0}{T} = 0.157}$

The response spectrum is a plot that can give you quickly the maximum dynamic response for any values of $\left(\frac{t_0}{T}\right)$. This plot give you lots of information. From this plot, the $\left(\frac{u_{max}}{\frac{F_0}{k}}\right)$ would be equal 0.44 for $\frac{t_0}{T} = 0.157$. For this type of loading which was discussed in this example, for $\left(\frac{t_0}{T}\right) = 0.4$, the maximum response of system would be equal to static displacement. In the other words, for any shock loading with the shape look like this example, as long as $\left(\frac{t_0}{T}\right) \leq 0.4$, the dynamic response would never exceed the corresponding static displacement! Also, from this graph, we can see for $\left(\frac{t_0}{T}\right) = 0.8$, we will have the absolute maximum dynamic response! Form this graph, we can quickly find what would be the critical time duration for force (t_0). Also, for $\left(\frac{t_0}{T}\right) > 0.8$, the dynamic response would be oscillate between 1-1.25 of static displacement. The shock spectrum is available for a wide range of short duration loadings and these plots can be used for all kinds of practical research purposes.

Example 38: Set up Shock Spectrum/Response Spectrum for following forcing function.



$$u(t) = \frac{1}{m\omega_n}\int_0^t f(\tau)\ sin\ \omega_n(t-\tau)\ d\tau$$

$$f(\tau) = \begin{cases} \dfrac{F_0}{t_0}\tau & 0 < t < t_0 \\ F_0 & t_0 < t \end{cases}$$

If we solve it for both regions, we will find the responses equal to:

Region 1:

$$u(t) = \frac{F_0}{k}\left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0}\right)$$

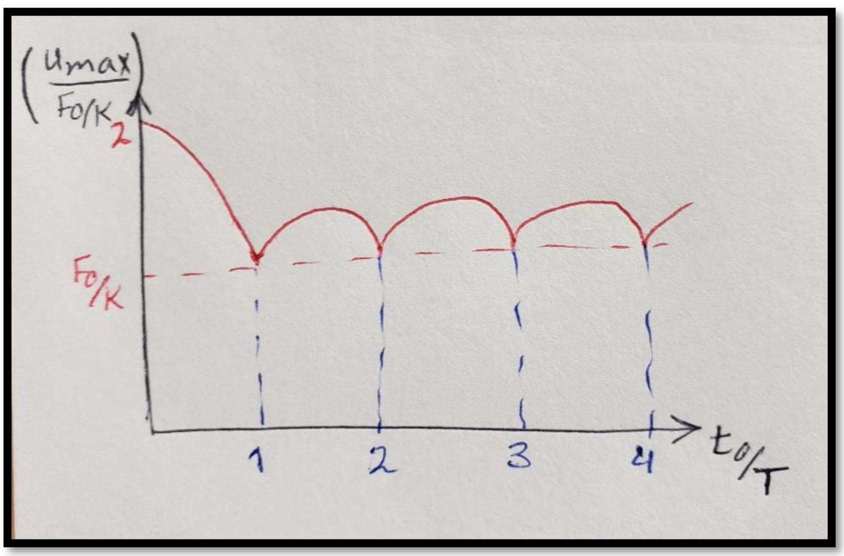
Region 2:

$$u(t) = \frac{F_0}{k}\left(1 + \frac{\sin \omega_n (t - t_0)}{\omega_n t_0} - \frac{\sin \omega_n t}{\omega_n t_0}\right)$$

Now, we want to plot the u_{max} vs $\left(\frac{t_0}{T}\right)$. We can find the $u_{max}$by taking derivative of previous equations, equate to zero, find t values, substitute these values back in $u(t)$ to find the u_{max}. After solving for both regions, the u_{max} would be:

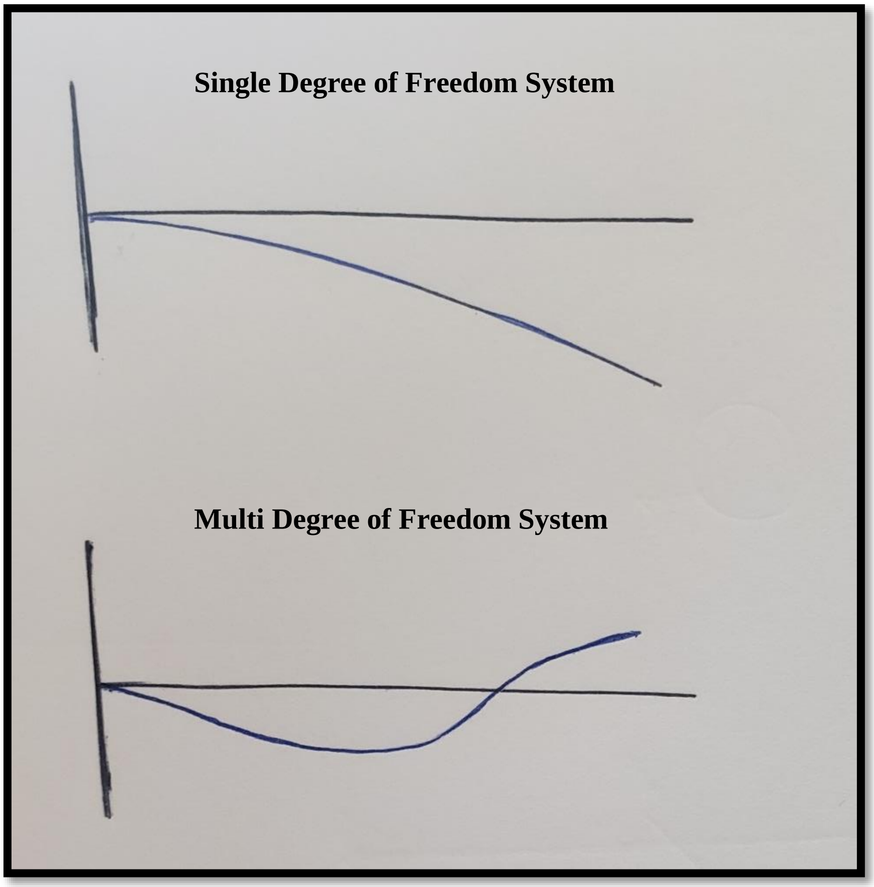
$$\boxed{u_{max} = \frac{F_0}{k}\left[1 + \frac{\sqrt{2(1 - \cos \omega_n t_0)}}{\omega_n t_0}\right]}$$

Then, based on the u_{max}, we can plot the shock spectrum for this type of force and use it for other experiments (e.g. earthquake, blast loads, etc).



Two Degree of Freedom Systems

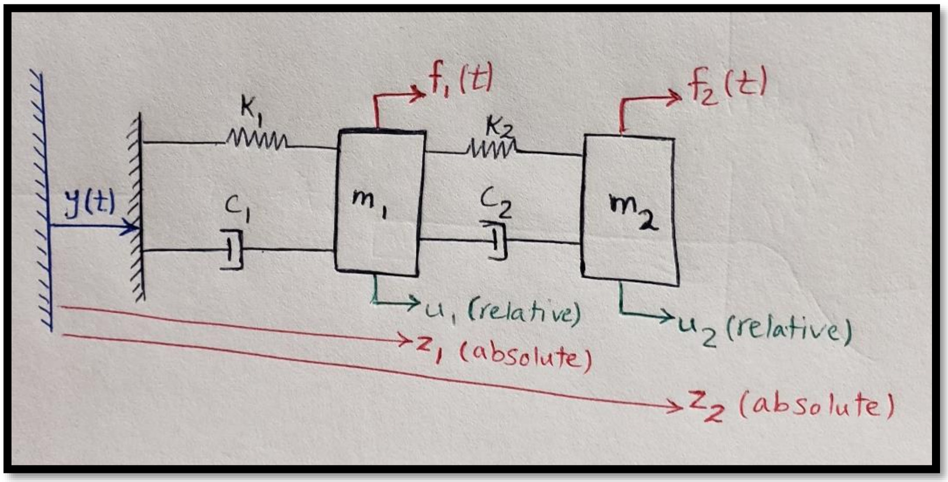
For single degree of freedom systems we assumed that the overall deformation of structure can be demonstrate by looking at a single location in the structure. In general this assumption is not true! Especially if you have vibration that goes to higher frequency of the excitation (e.g. a long beam which is moving very fast).



In fact, any structure made of infinite number of D.O.F. To solve this problem, one way is considering structure as continues system and study the vibration of continues systems (it is a little difficult to study). The alternative option would be making an assumption that the structure made of finite degrees of freedom (finite number of lumped masses) which each one move in different directions (multi degree of freedom system). The M.D.O.F. give you more accurate and better understanding about the way structure deform. The study of M.D.O.F. is required using matrix analysis. To make it easier and more understandable, before talking about M.D.O.F. systems, let's start with two D.O.F. systems.

Study of two D.O.F. systems is easy way to introduce the fundamental concepts of M.D.O.F. systems. For instance "Modal Coordinates", "Orthogonality of Modes", "Modal transformation", "Mode Shapes", etc.

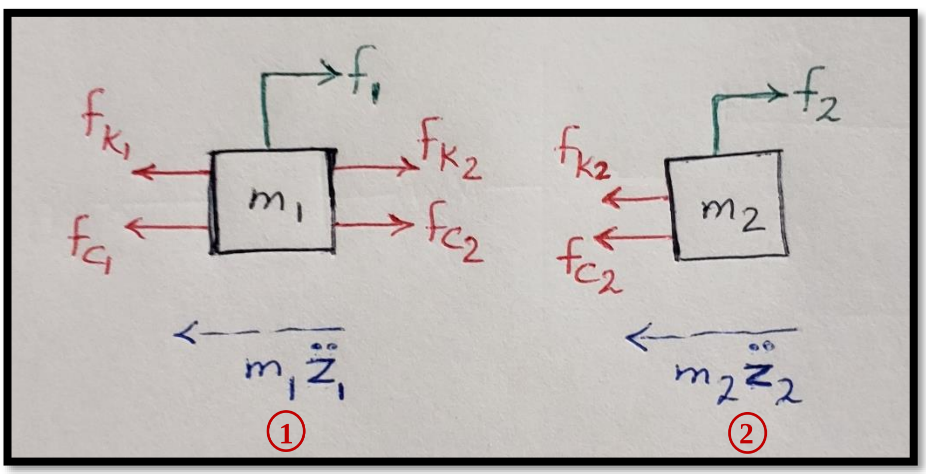
The following picture shows the most general case for a 2 D.O.F system. This system includes two masses (m_1 & m_2) and each of these masses subjected to separate forcing function $f_1(t)$ & $f_2(t)$. Because of these forcing functions, each mass is experiencing a displacement respect to a base (u_1 & u_2). Also, this base is moving respect to a reference frame. The u_1 & u_2 are relative displacement for masses (respect to the base) and we can show the absolute displacement of masses with (z_1 & z_2) (respect to the reference frame).



In order to drive the equation of motion for this 2 D.O.F. system, there are two approaches: 1) Newton's law 2) Lagrange's Formula

1) Newton's law (Dynamic equilibrium, F.B.D, and D'Alembert force, etc.)

In this case we have two free body diagrams.



For each of these free body diagrams, we can write the equation of equilibrium.

1) $m_1\ddot{z}_1 = f_1 + f_{k_2} + f_{c_2} - f_{k_1} - f_{c_1}$

2) $m_2\ddot{z}_2 = f_2 - f_{k_2} - f_{c_2}$

$f_{k_1} = k_1 u_1$, $f_{k_2} = k_2(u_2 - u_1)$, $f_{c_1} = c_1\dot{u}_1$, $f_{c_2} = c_2(\dot{u}_2 - \dot{u}_1)$

$z_1 = u_1 + y$ & $z_2 = u_2 + y$

$$\boxed{\begin{gathered} m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + (k_1 + k_2)u_1 - k_2u_2 = f_1 - m_1\ddot{y} \\ m_2\ddot{u}_2 - c_2\dot{u}_1 + c_2\dot{u}_2 - k_2u_1 + k_2u_2 = f_2 - m_2\ddot{y} \end{gathered}}$$

Equations of motion in terms of relative displacements (u_1&u_2)

1. $\ddot{y}$ is input acceleration

Practice: Write the equation of motion in term of absolute displacement.

$$\boxed{\begin{gathered} m_1\ddot{z}_1 + (c_1 + c_2)\dot{z}_1 - c_2\dot{z}_2 + (k_1 + k_2)z_1 - k_2z_2 = f_1 + k_1y + c_1\dot{y} \\ m_2\ddot{z}_2 - c_2\dot{z}_1 + c_2\dot{z}_2 - k_2z_1 + k_2z_2 = f_2 \end{gathered}}$$

Equations of motion in terms of absolute displacements (z_1&z_2)

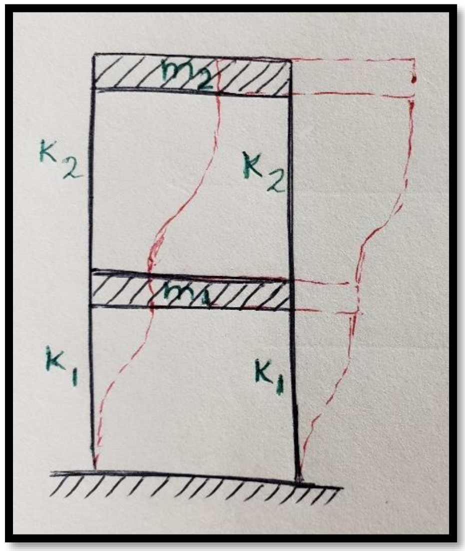
In terms of applied forces

For S.D.O.F system, we had one equation in terms of one variable, so we use homogenous deferential equation of second order and solve it directly. But in this case, we have two equations and two variables (u_1 & u_2) and in each of them we have both variables (they are coupled). Therefore, we cannot solve these equations independently/directly.

In this case, it is possible to transform these two equations to another coordinate system where these equations become decoupled (each equation only in term of one variable) and solve the equations there and return them back to original coordinate system.

Note: In the S.D.O.F. we only have a single natural frequency for system. In multi degree of freedom systems, system doesn't have just one fundamental frequency but can vibrate under the action of load which excite the system in multi frequencies (the first frequency called fundamental frequency, and higher frequency called second mode, third mode,…).

Example 39: The following figure showing a two story building. The columns are rigidly connected to the masses m_1 & m_2. The columns for each level of this building have same stiffness. Find the equation of motion for this building.



This is an **undamped** two degree of freedom system ($c = 0$). First of all, let's find the equivalent stiffness for each level of this building. Columns are parallel to each other, so:

For level 1: $k_1 = \frac{12E_1I_1}{{l_1}^3} \Longrightarrow \boxed{k_{eq1} = \frac{24E_1I_1}{{l_1}^3}}$

For level 2: $k_2 = \frac{12E_2I_2}{{l_2}^3} \Longrightarrow \boxed{k_{eq2} = \frac{24E_2I_2}{{l_2}^3}}$

In this case, we only look at the **free vibration** of the system. Also, we **don't** have a **base motion**. So, based on these information and from the general formulation for two degree of freedom systems, the equations of motion for this system would be:

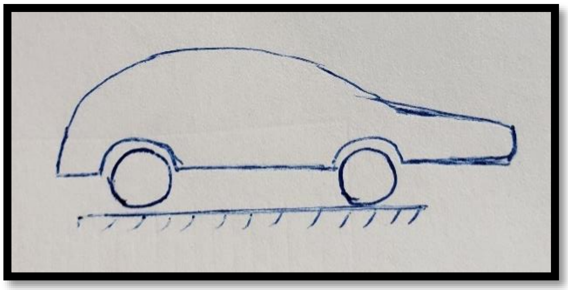
$$m_1\ddot{u}_1 + \cancelto{0}{(c_1 + c_2)}\dot{u}_1 - \cancelto{0}{c_2\dot{u}_2} + (k_1 + k_2)u_1 - k_2u_2 = \cancelto{0}{f_1} - \cancelto{0}{m_1\ddot{y}}$$
$$m_2\ddot{u}_2 - \cancelto{0}{c_2\dot{u}_1} + \cancelto{0}{c_2\dot{u}_2} - k_2u_1 + k_2u_2 = \cancelto{0}{f_2} - \cancelto{0}{m_2\ddot{y}}$$

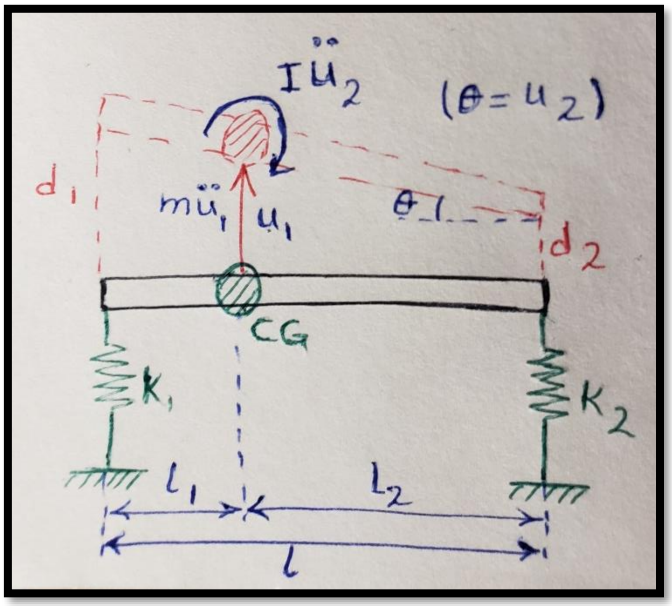
$$\boxed{\begin{aligned} m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0 \\ m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = 0 \end{aligned}} \Longrightarrow \text{Equation of motion}$$

However, in this case, we had translational motion for both movements, what happened if we have one translational and one rotational motion?

Two Degree of Freedom Systems

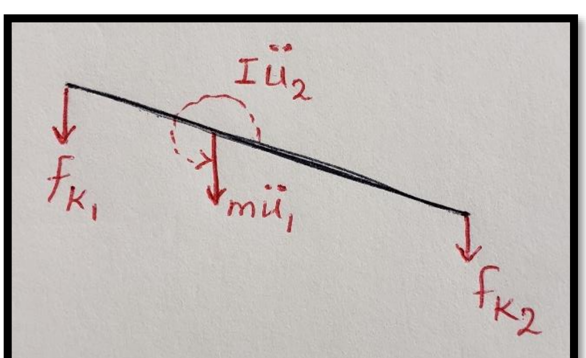
Example 40: The following figure showing the model of a car. Suspension system of this car showing with two springs with stiffness of k_1& k_2. This car experience both translational and rotational motions. Find the equation of motion for this car.





(CG: Center of Gravity of the car)

Free Body Diagram:



To set up the equation of motion, we will have:

$\sum F_y = 0$ 1

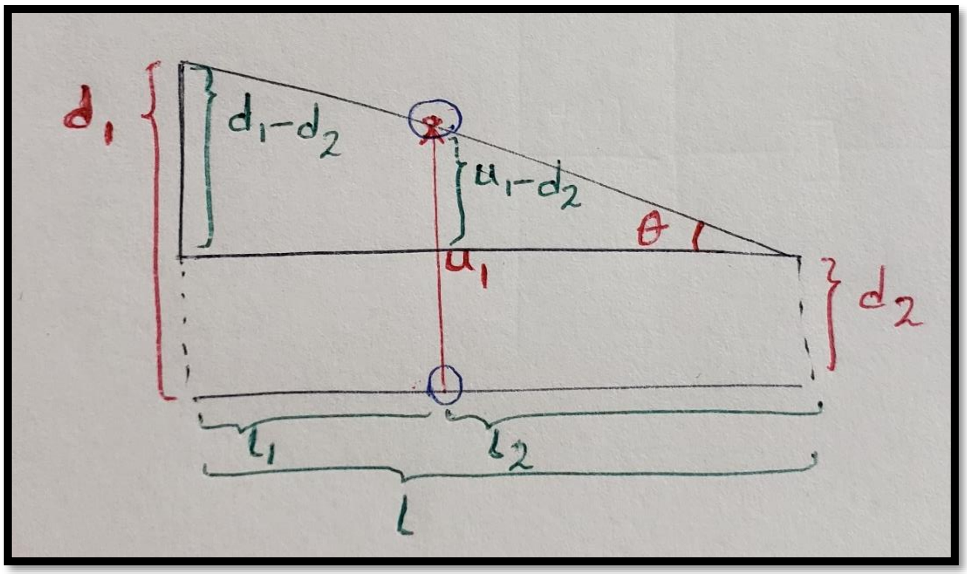
$\sum M_{CG} = 0$ 2

$f_{k_1} = k_1 d_1$ & $f_{k_2} = k_2 d_2$

$m_1 \ddot{u}_1 + f_{k_1} + f_{k_2} = 0$ 1

$I \ddot{u}_2 + f_{k_1}.l_1 - f_{k_2}.l_2 = 0$ 2

However, we want to express the above equations in terms of u_1& u_2.



$\theta = u_2 = \frac{d_1 - d_2}{l}$ & $\frac{u_1 - d_2}{d_1 - d_2} = \frac{l_2}{l}$

We can use these two relations to write the equation of motion in terms of u_1& u_2.

$d_1 = u_1 - u_2 l_2 + u_2 l \Longrightarrow \boxed{d_1 = u_1 + u_2 l_1}$

$\boxed{d_2 = u_1 - u_2 l_2}$

$f_{k_1} = k_1 d_1 = k_1(u_1 + u_2 l_1)$ & $f_{k_2} = k_2(u_1 - u_2 l_2)$

$$\boxed{\begin{aligned} &m_1 \ddot{u}_1 + (k_1 + k_2)u_1 + (k_1 l_1 - k_2 l_2)u_2 = 0 \\ &I \ddot{u}_2 + (k_1 l_1 - k_2 l_2)u_1 + (k_1 {l_1}^2 + k_2 {l_2}^2)u_2 = 0 \end{aligned}} \Longrightarrow \text{Equation of motion}$$

1) Lagrange's Equation
   This method is very general and you don't need a F.B.D. and doesn't matter if we are working with flexible structure or rigid body. Lagrange's dynamics, itself is an advance course and here we will briefly talk about this approach (we will not go through all details, steps, energy methods, etc.).

   If you have a system with q_i number of degrees of freedom (generalized coordinates), if you use this approach, you can drive equation of motion for all degrees of freedom. Lagrange stablish by "L" which is equal to difference between kinetic energy and potential energy.

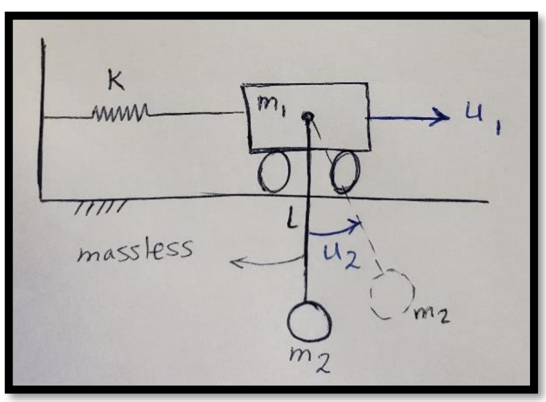
$$L = ke - pe$$

   Following formula which is derived based on energy method can be used directly to drive the equation of motion.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

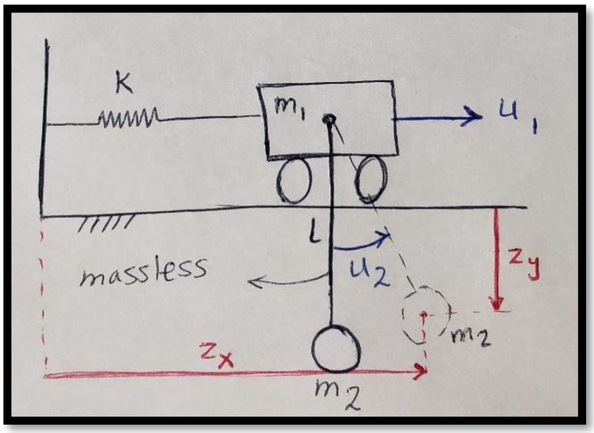
   Note: for two degrees of freedom system $i = 1,2$, and "q" is equivalent "u" (displacement in this case)
   Note: This approach same as all other energy methods is only applicable to conservative systems (**no damping or damping is negligible**).

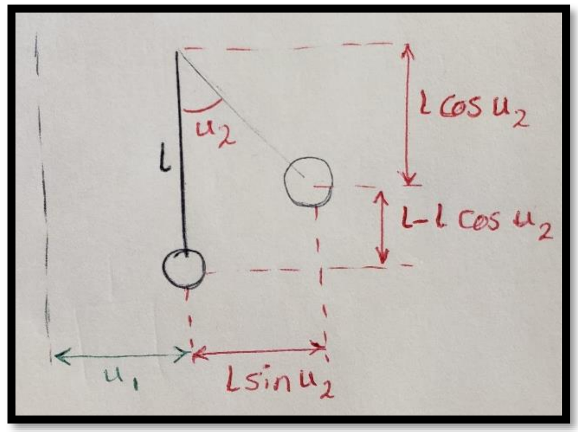
   Example 41: The following figure shows a vehicle (m_1) connected to a wall with a spring with the stiffness of k and moving along direction u_1. Mass (m_2) with a massless rod with length of l is attached to this vehicle. When the vehicle is moving, m_2 is also moving (u_2). In this case we have two degrees of freedom (two generalized coordinates). Find two equations of motion for this system.



First, we have to set up kinetic and potential energies.

In this case, we have both translational and rotational motions. To write the kinetic & potential energies, we will find the translational coordinates of rotational motion of m_2 (z_x & z_y)





$$ke = \frac{1}{2} m v^2 = \frac{1}{2} m_1 \dot{u}_1{}^2 + \frac{1}{2} m_2 (\dot{z}_x{}^2 + \dot{z}_y{}^2)$$

$$z_x = u_1 + l \sin u_2 \Longrightarrow \dot{z}_x = \dot{u}_1 + l\, \dot{u}_2 \cos u_2$$
$$z_y = l \cos u_2 \Longrightarrow \dot{z}_y = -l\, \dot{u}_2 \sin u_2$$

$$ke = \frac{1}{2} m_1 \dot{u}_1{}^2 + \frac{1}{2} m_2 [(\dot{u}_1 + l\, \dot{u}_2 \cos u_2)^2 + (-l\, \dot{u}_2 \sin u_2)^2]$$

$$\boxed{ke = \frac{1}{2} m_1 \dot{u}_1{}^2 + \frac{1}{2} m_2 (\dot{u}_1{}^2 + l^2 \dot{u}_2{}^2 + 2 l \dot{u}_1 \dot{u}_2 \cos u_2)}$$

$$h = l - l \cos u_2 = l(1 - \cos u_2)$$

$$\boxed{pe = \frac{1}{2} k x^2 + m g h = \frac{1}{2} k u_1{}^2 + m_2 g l (1 - \cos u_2)}$$

$$L = ke - pe$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$i = 1$:

$$\frac{\partial L}{\partial \dot{u}_1} = m_1 \dot{u}_1 + m_2 \dot{u}_1 + m_2 l \dot{u}_2 \cos u_2 \qquad \text{(There is no } \dot{u}_1 \text{ in } pe\text{)}$$

If u_2 be a small angle $\Longrightarrow \frac{\partial L}{\partial \dot{u}_1} = m_1 \dot{u}_1 + m_2 \dot{u}_1 + m_2 l \dot{u}_2 \cancelto{1}{\cos u_2} \cong (m_1 + m_2) \dot{u}_1 + m_2 l \dot{u}_2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) = (m_1 + m_2) \ddot{u}_1 + m_2 l \ddot{u}_2$$

$\frac{\partial L}{\partial u_1} = -ku_1$ (There is no u_1 in ke, "$-$" sign because of $L = ke - pe$)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) - \frac{\partial L}{\partial u_1} = (m_1 + m_2)\ddot{u}_1 + m_2 l\ddot{u}_2 + ku_1 = 0}$$

$i = 2$:

$\frac{\partial L}{\partial \dot{u}_2} = m_2 l^2 \dot{u}_2 + m_2 l\, \dot{u}_1 \cos u_2$ (There is no $\dot{u}_2$ in pe)

If u_2 be a small angle $\Longrightarrow$ $\frac{\partial L}{\partial \dot{u}_2} = m_2\, l^2 \dot{u}_2 + m_2 l\, \dot{u}_1 \cancelto{1}{\cos u_2} \cong m_2\, l^2 \dot{u}_2 + m_2 l\, \dot{u}_1$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right) = m_2 l^2 \ddot{u}_2 + m_2 l \ddot{u}_1$

$\frac{\partial L}{\partial u_2} = -m_2 l \dot{u}_1 \dot{u}_2 \,sin\, u_2 - m_2 gl \,sin\, u_2$

If u_2 be a small angle $\Longrightarrow$ $\frac{\partial L}{\partial u_2} = -m_2 l \cancelto{0}{\dot{u}_1 \dot{u}_2}\, \cancelto{u_2}{sin\, u_2} - m_2 gl\, \cancelto{u_2}{sin\, u_2} \cong -m_2 g l u_2$

Multiplication of two small numbers

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right) - \frac{\partial L}{\partial u_2} = m_2 l^2 \ddot{u}_2 + m_2 l \ddot{u}_1 + m_2 g l u_2 = 0}$$

In this case, if we used Newton's law, it will be so much harder to solve!

Two Degree of Freedom Systems

Matrix form representation of the equation of motion:

The Equations of motion in terms of relative displacements ($u_1 \& u_2$) can be write as a matrix form:

$$m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + (k_1 + k_2)u_1 - k_2u_2 = f_1 - m_1\ddot{y}$$

$$m_2\ddot{u}_2 - c_2\dot{u}_1 + c_2\dot{u}_2 - k_2u_1 + k_2u_2 = f_2 - m_2\ddot{y}$$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\substack{\textbf{Mass}\\ \textbf{Matrix}\\ ([M] \text{ or } \underset{\sim}{M})\\ \text{(Diagonal matrix)}}} \underbrace{\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}}_{\substack{\textbf{Acceleration}\\ \textbf{Vector}\\ \underset{\sim}{\ddot{u}}}} + \underbrace{\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}}_{\substack{\textbf{Damping}\\ \textbf{Matrix}\\ ([C] \text{ or } \underset{\sim}{C})}} \underbrace{\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}}_{\substack{\textbf{Velocity}\\ \textbf{Vector}\\ \underset{\sim}{\dot{u}}}} + \underbrace{\begin{bmatrix} -k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\substack{\textbf{Stiffness}\\ \textbf{Matrix}\\ ([K] \text{ or } \underset{\sim}{k})}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{\substack{\textbf{Displacement}\\ \textbf{Vector}\\ \underset{\sim}{u}}} = \underbrace{\begin{Bmatrix} f_1 - m_1\ddot{y} \\ f_2 - m_2\ddot{y} \end{Bmatrix}}_{\substack{\textbf{Forcing}\\ \textbf{Function}\\ \textbf{Vector}\\ \underset{\sim}{f}}}$$

So, we can write the equation of motion for M.D.O.F systems like the scalar form that we had for S.D.O.F system. However, you have to remember, it is a matrix equation and it is a coupled system of differential equation and you have transform it to the normal coordinate to decoupled and then solve it.

$$\boxed{\underset{\sim}{M}.\underset{\sim}{\ddot{u}} + \underset{\sim}{C}\underset{\sim}{\dot{u}} + \underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{f}}$$

State-Space formulation: The second order differential equation is not easy to solve, so we will do a variable substitution and convert it to the first order differential equation.

Just imagine we have:

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad , \quad \dot{u} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \quad , \quad \ddot{u} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

If we define Z vector as:

$$Z = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad , \quad z_1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad , \quad z_2 = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \Rightarrow \dot{z}_2 = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

Note: From matrix algebra multiplication of a matrix to its invers would be equal to an identity matrix.

$$A.A^{-1} = [I] = \begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1\end{bmatrix}$$

Based on above information, if we multiply the equation of motion by M^{-1}:

$$\underset{\sim}{\ddot{u}} + \underset{\sim}{M}^{-1}\underset{\sim}{C}\underset{\sim}{\dot{u}} + \underset{\sim}{M}^{-1}\underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{M}^{-1}\underset{\sim}{f}$$

$$\boxed{\underset{\sim}{\ddot{u}} = -\underset{\sim}{M}^{-1}\underset{\sim}{C}\underset{\sim}{\dot{u}} - \underset{\sim}{M}^{-1}\underset{\sim}{K}\underset{\sim}{u} + \underset{\sim}{M}^{-1}\underset{\sim}{f}}$$

Now, if we use State-Space formulation, we can convert it to the first order differential equation. Then, just we need to take one integration to find z_1& z_2.

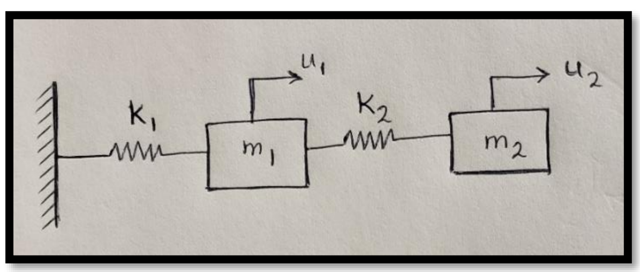
$$\underset{\sim}{\dot{z}_2} = -\underset{\sim}{M}^{-1}\underset{\sim}{C}\underset{\sim}{z_2} - \underset{\sim}{M}^{-1}\underset{\sim}{K}\underset{\sim}{z_1} + \underset{\sim}{M}^{-1}\underset{\sim}{f}$$

In general form we can write it:

$$\underset{\sim}{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \quad , \qquad \underset{\sim}{\dot{z}} = \underset{\sim}{A}\underset{\sim}{z} + \underset{\sim}{B}\underset{\sim}{f} \;\; , \;\; \underset{\sim}{A} = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right] \;\; , \;\; \underset{\sim}{B} = \left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right]$$

MDOF: In this part we will see all tools to cover the concepts and formulation for two degree of freedom system which would be same formulation for multi degree of freedom system (just matrix has larger dimension). We have Undamped, damped, free vibration and forced vibration same as S.D.O.F. system but in this course we just cover Undamped free vibration case.

Undamped Free Vibration:



From before, the equation of motions would be:

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0$$

$$m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = 0$$

These are two simultaneously linear homogeneous differential equations. In S.D.O.F. system, we assumed the solution would be a harmonic function, took first and second derivative of that and put it back in the equation and try to find constant value “A” and “ω”. We will do exactly same thing for two D.O.F. system, with only difference we have two harmonic functions.

$u_1 = A_1 \sin \omega t$ & $u_2 = A_2 \sin \omega t$

If we take first and second derivatives and put them back in the equations of motion, we will have a homogenous set of equations.

$$m_1(-\omega^2 A_1)\cancel{\sin \omega t} + (k_1 + k_2)A_1 \cancel{\sin \omega t} - k_2 A_2 \cancel{\sin \omega t} = 0$$
$$m_2(-\omega^2 A_2)\cancel{\sin \omega t} - k_2 A_1 \cancel{\sin \omega t} + k_2 A_2 \cancel{\sin \omega t} = 0$$

$\sin \omega t \neq 0$

$$\Longrightarrow \begin{cases} (-\omega^2 m_1 + k_1 + k_2)A_1 - k_2 A_2 = 0 \\ -k_2 A_1 + (-\omega^2 m_2 + k_2)A_2 = 0 \end{cases}$$

From matrix algebra there is **no unique solution** for homogeneous set of simultaneous equations. In other words, you cannot solve for two variables A_1& A_2 because they are not independent. Only we can do is finding the ratios of those two variables!

There are many different ways that we can solve system of homogeneous, simultaneous equation. One way of solving a set of homogeneous, simultaneous equation is Cramer's rule.

Example:

$$\begin{cases} a_1 x_1 + a_2 x_2 = 0 \\ b_1 x_1 + b_2 x_2 = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}}_{A} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Longrightarrow x_i = \frac{det|A_i|}{det|A|}$$

For instance: $x_2 = \dfrac{\cancelto{0}{det\begin{vmatrix} 0 & a_1 \\ 0 & b_1 \end{vmatrix}}}{det\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ $\Longrightarrow$

We will have trivial solution, for x_1 & $x_2 = 0$ or **non-trivial** solution for: $det\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

$$x_2 = \frac{\cancelto{0}{det\begin{vmatrix} 0 & a_1 \\ 0 & b_1 \end{vmatrix}}}{\cancelto{0}{det\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}} = \frac{0}{0}$$

$\Longrightarrow$ That means $x_2 \neq 0$

For our case:

$$det\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

This determinant consists of some system parameters (m_1, m_2, k_1, k_2), and it is the function of frequency (ω). We show this equation with $D(\omega) = 0$ and it is named: *frequency equation*. This would be a quadratic equation in terms of (ω^2) and it will have two solutions for frequencies corresponding to this two degree of freedom system.

$$(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2) - {k_2}^2 = 0$$

.

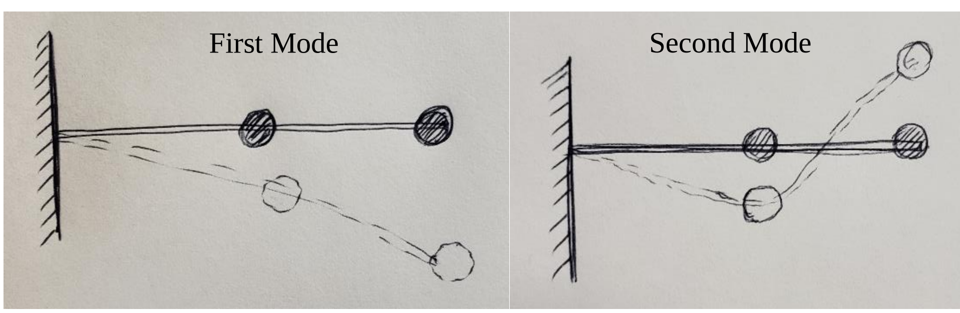
.

.

$$\boxed{{\omega_{1,2}}^2 = \frac{1}{2}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right) \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}}$$

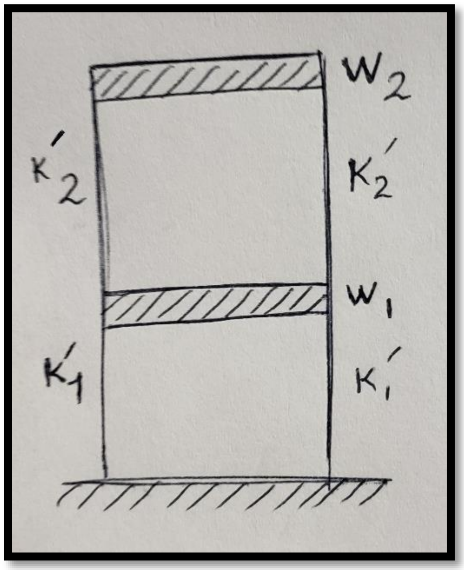
These roots are the frequency associated with those two degrees of freedom. Now to solve for A_1 & A_2, we need to substitute these ω values to the main equations. However, when we have a homogenous set of equation, the equations are not independent! So, you can use only one of those equations and divide it by one of the variables (A_1 or A_2) and we just able to find the ratio of those two variables (relative value of those variables not the actual value ($\frac{A_1}{A_2}$ or $\frac{A_2}{A_1}$)). For instance for ω_1 you will find a value for $\frac{A_2}{A_1}$ and for ω_2 you will find another value for $\frac{A_2}{A_1}$. Usually, we normalized $\frac{A_2}{A_1}$ with putting $A_1 = 1$. Each frequency shows how the structure deformed based on that frequency. The displacement based on ω_1 and its corresponding ratio of $\frac{A_2}{A_1}$ is named "first mode" and displacement related to ω_2 and its corresponding ratio of $\frac{A_2}{A_1}$ is named "second mode". In two degree of freedom we have two frequencies which are the natural frequencies of the system. The first frequency is called "fundamental frequency" and higher frequencies correspond to higher modes.

Example: If we have a beam with two lump mass the first and second modes would be:

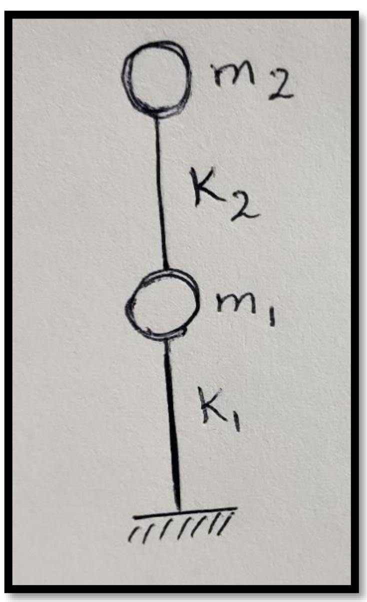


Two Degree of Freedom Systems

Example 42: There is a two story building with the weight of W_1& W_2 for first and second floors respectively. The columns in each floor has same stiffness. Assume $W_1 = W_2 = W$, $I_1 = 2I_2 = I$ (second moment of area for the columns), and $l_1 = l_2 = l$.



Free Body Diagram:



$k_1 = 2\left(\frac{12EI}{l^3}\right)$

$k_2 = \frac{k_1}{2} = \left(\frac{12EI}{l^3}\right)$

From the equation of motion that we have for undamped-free vibration:

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0$$

$$m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = 0$$

$$\Longrightarrow \begin{cases} \ddot{u}_1 + 3Bu_1 - Bu_2 = 0 \\ \ddot{u}_2 + Bu_2 - Bu_1 = 0 \end{cases} \quad , \qquad B = \left(\frac{12EIg}{Wl^3}\right)$$

$$u_1 = A_1 sin\,\omega t$$

$$u_2 = A_2 sin\,\omega t$$

$$\Longrightarrow \begin{cases} -A_1\omega^2 + 3BA_1 - BA_2 = 0 \\ -A_2\omega^2 + BA_2 - BA_1 = 0 \end{cases}$$

If we substitute u_1& u_2 in equations of motion and find the determinant, we will have:

$$det\begin{vmatrix} -\omega^2 + 3B & -B \\ -B & -\omega^2 + B \end{vmatrix} = 0$$

We need to solve for the roots of this determinant to find the frequencies of system.

$$\omega^4 - 4B\omega^2 + 2B^2 = 0$$

$$\boxed{\omega^2 = \begin{cases} 0.586\,B \\ 3.414\,B \end{cases}}$$

* As a numerical example, let's assume $\frac{EIg}{Wl^3} = 2$, then we will have:

$$\boxed{\begin{aligned} \omega_1 &= 3.75\,rad/s \\ \omega_2 &= 9.05\,rad/s \end{aligned}}$$

Now, for finding A_1 & A_2, we have to substitute these ω values in **one** of the equation of motions (two equations are dependent) and divide it by one of the variables (A_1).

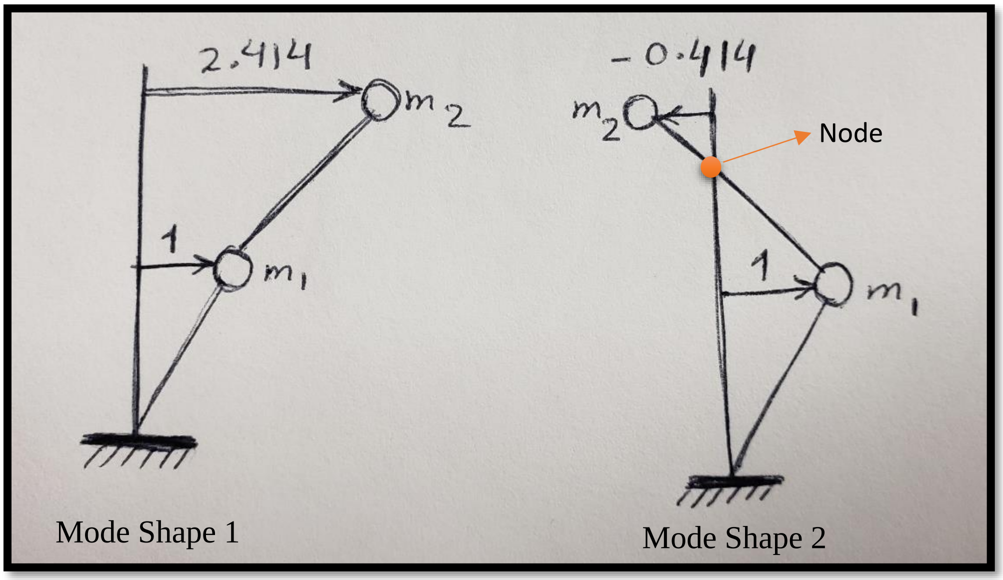
$$-A_1\omega^2 + 3BA_1 - BA_2 = 0 \Longrightarrow -\omega^2 + 3B - B\left(\frac{A_2}{A_1}\right) = 0$$

$$A_2 = \frac{-\omega^2+3B}{B}A_1 \xrightarrow{\text{If we take } A_1 = 1} \boxed{A_2 = \frac{-\omega^2+3B}{B}}$$

$${\omega_1}^2 = 0.586\,B \quad \begin{cases} A_1 = 1 \\ A_2 = 2.414 \end{cases}$$

$${\omega_2}^2 = 3.414\,B \quad \begin{cases} A_1 = 1 \\ A_2 = -0.414 \end{cases}$$

ω_1 and ω_2 are the natural frequencies and A_1& A_2 are the shapes corresponding to each natural frequencies which we call them mode shapes.



Node: There is point along the structure that doesn't move (stay stationary). In other words, node is a point on the structure which is experiences zero deformation (is not subjected to any dynamic force, no displacement, no stresses, etc). The number of nodal points is depends on what modes of vibration we are consider, higher modes having more nodal points.

Summary of Steps:

1) Set up the equation of motion
2) Consider a solution of type $u_i = A_i sin\, \omega t$ (for two D.O.F. $i = 1,2$) and substitute it in equation of motion
3) Consider the determinant of coefficient for part (2) equal zero ($D(\omega) = 0$, frequency equation)
4) Find the roots of frequency equation (frequency of the system)
5) Substitute these roots in **one** of the equations in part (2) and for each value of (ω) find the corresponding relative deformed shapes $\left(\text{i.e.} \frac{A_2}{A_1}\right)$
6) Plot A_1 & A_2 $(A_1 = 1)$ for each ω (mode shapes)

Mode Shape

For a free vibration of an undamped system, if we follow the steps:

1) $$\underset{\sim}{M}.\underset{\sim}{\ddot{u}} + \underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{0}$$

$\underset{\sim}{M}$: Mass matrix

$\underset{\sim}{K}$: Stiffness matrix

$\underset{\sim}{u}$: Vector of displacement

2) $$u_i = \emptyset_i \, sin \, \omega t \quad \text{(We use } \emptyset_i \text{ rather than } A_i\text{)}$$

3) Substitute (2) in to (1) and set the eigenvalue problem

$$(\underset{\sim}{K} - \omega^2 . \underset{\sim}{m}).\{\Phi_i\} = \{0\}$$

Note: From the matrix algebra the following equation is named eigenvalue equation:

$$(\underset{\sim}{A} - \lambda . \underset{\sim}{I}).\{x\} = \{0\}$$

We can rewrite the equation from (3) by pre-multiply by m^{-1}:

Eigenvalues

$$(\underset{\sim}{m}^{-1}\underset{\sim}{K}).\{\Phi_i\} = \omega^2.\{\Phi_i\}$$

$$\{\Phi_i\} = \begin{Bmatrix} \emptyset_{1i} \\ \emptyset_{2i} \end{Bmatrix}$$

D.O.F. Frequency

Eigenvectors
(Mode shapes)

For instance $\emptyset_{11}$ is shape for D.O.F. "1" for mode frequency "1". In other words, for two degree of freedom system, we will have $\emptyset_{11}$ & $\emptyset_{21}$ for ω_1 and $\emptyset_{21}$ & $\emptyset_{22}$ for ω_2 as we discussed before. The $\omega's$ (frequencies) are "Eigenvalues" and $\emptyset's$ (Mode Shapes) are "Eigenvectors".

$$\omega_1 : \begin{Bmatrix} \emptyset_{11} \\ \emptyset_{21} \end{Bmatrix}$$

$$\omega_2 : \begin{Bmatrix} \emptyset_{21} \\ \emptyset_{22} \end{Bmatrix}$$

$$\{\Phi\} = [\{\Phi_1\}, \{\Phi_2\}, \dots, \{\Phi_n\}]$$

Mode shape matrix

4) From $(coefficients) = 0$, find the $\omega's$ (frequencies)

5) Substitute the $\omega's$ in equation (3), and find corresponding mode shape ($\emptyset's$)

$$\{\Phi_i\} = \begin{Bmatrix} \emptyset_{1i} \\ \emptyset_{2i} \end{Bmatrix}$$

Two Degree of Freedom Systems

Properties of Mode Shapes:

1. Mode Shapes are invariant with respect to a constant. That means if you have a mode shape (modal Metrix) Φ_i, it would be same as $c \times \Phi_i$.

$\{\Phi_i\} = \begin{Bmatrix} \emptyset_{1i} \\ \emptyset_{2i} \end{Bmatrix}$ $\{\Phi_i\} = \begin{Bmatrix} c\ \emptyset_{1i} \\ c\ \emptyset_{2i} \end{Bmatrix}$

2.

2.1. Mode shapes are orthogonal with respect to each other relative to mass or stiffness matrices. Based on definition of orthogonality, we will have:

$$\{\Phi_1\}^T[m]\{\Phi_2\} = \underset{\sim}{0}$$
$$\{\Phi_1\}^T[k]\{\Phi_2\} = \underset{\sim}{0}$$

And in general:

$$\{\Phi_i\}^T[m]\{\Phi_j\} = \underset{\sim}{0}$$
$$\{\Phi_i\}^T[k]\{\Phi_j\} = \underset{\sim}{0} \qquad (\boldsymbol{i \neq j})$$

2.2. Mode shapes are not orthogonal with respect to themselves & relative to mass or stiffness matrices. However, in relation to mass matrix that would be equal to a constant matrix (a diagonal matrix) and for stiffness would be equal to multiplication of that constant matrix to ω^2.

In general:

$$\{\Phi_i\}^T[m]\{\Phi_i\} = constant = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_i \end{bmatrix} \qquad (\boldsymbol{i \neq j})$$

$$\{\Phi_i\}^T[k]\{\Phi_i\} = \omega^2 \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_i \end{bmatrix}$$

$$\begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_i \end{bmatrix} = Generalized\ Mass\ Matrix\ (modal\ Mass\ Matrix)$$

Note: The property # 2 is very important and in order to verify if the mode shape and frequency that we calculated for the system are correct the relationships in the property number #2 must hold.

Normalization of Mode Shapes

Mode shape represents the quality shape of the deformation. There are several approaches to normalize the mode shapes.

1. Set $\emptyset_1 = 1$ & find other relative displacement with respect to that.

$$\frac{x_1}{x_2} \Longrightarrow x_1 = 1 \Longrightarrow x_2 = ?$$

2. Consider the maximum value of each mode & set that equal to 1.

3. Set modal mass corresponding to each mode equal to 1.

1

$$\{\Phi_i\}^T[k]\{\Phi_i\} = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_i \end{bmatrix}$$

Find the relation for the rest of them

Note: Between these three approaches, last one is most commonly used especially in finite element code because need less computation.

Example 43: Consider the following two degree of freedom system. This system is consist of a rigid bar attached to a spring at one end and pivot at the other end. A mass (m) with a spring is connected to the bar in the point with a distance of l_1 from the pivot. The bar has a mass of m_{bar} and total length of l_2. Whole system start oscillating. Find the mode shapes, corresponding frequencies, and check if the mode shape are orthogonal with respect to each other. Assume $y < l_1\theta$ (the spring is in compression).

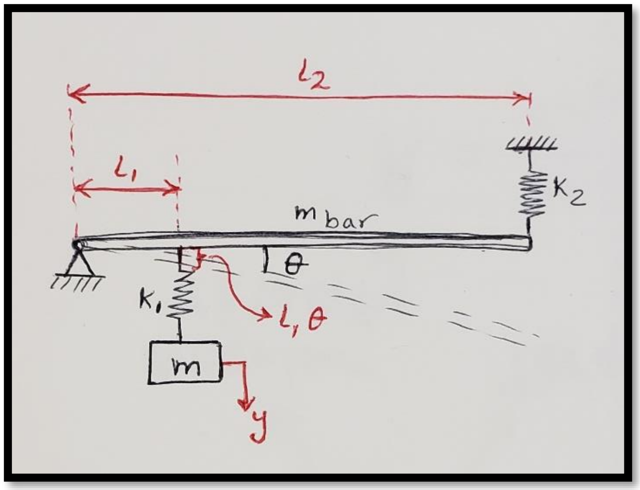
There are some relations between parameters in this problem (a, b, c are constant values):

$l_1 = al_2 = l$

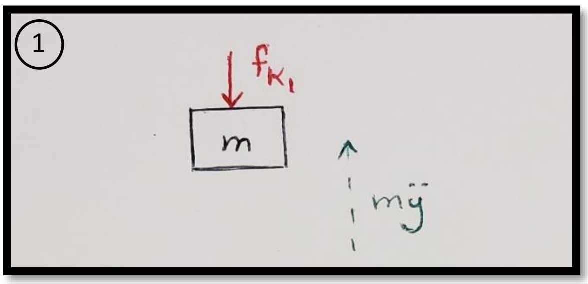
$k_2 = bk_1$

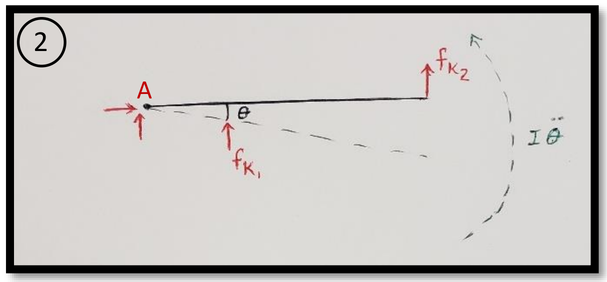
$m_{bar} = (3m).c$

$\Omega^2 = \frac{k_1}{m}$



Free body diagram:





Step 1: Equation of Motions

1)

$$m\ddot{y} - f_{k_1} = 0$$

$$f_{k_1} = k_1(l_1\theta - y)$$

$$\boxed{m\ddot{y} + k_1 y - k_1 l_1 \theta = 0}$$

1st Equation of motion

2)

$$\sum M_A = 0$$

$$I\ddot{\theta} + f_{k_1}(l_1) + f_{k_2}(l_2) = 0$$

$$f_{k_2} = k_2(l_2\theta) \qquad \& \qquad I = \frac{1}{3} m_{bar} {l_2}^2$$

$$I\ddot{\theta} + k_1(l_1\theta - y)l_1 + k_2(l_2\theta)l_2 = 0$$

$$\boxed{I\ddot{\theta} + \left(k_1 {l_1}^2 + k_2 {l_2}^2\right)\theta - k_1 l_1 y = 0}$$

2nd Equation of motion

Step 2: Find the mode shapes

$$\underset{\sim}{m}.\underset{\sim}{\ddot{u}} + \underset{\sim}{k}\underset{\sim}{u} = \underset{\sim}{0}$$

$$\underset{\sim}{m} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}$$

$$\underset{\sim}{k} = \begin{bmatrix} k_1 & -k_1 l_1 \\ -k_1 l_1 & k_1 {l_1}^2 + k_2 {l_2}^2 \end{bmatrix}$$

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} y \\ \theta \end{Bmatrix}$$

Based on the given relation between parameters, equations of motion would be:

$$m\ddot{y} + k_1 y - k_1 l_1 \theta = 0$$

$$\ddot{y} + \Omega^2 y - \Omega^2 l\theta = 0$$

$$I\ddot{\theta} + \left(k_1 {l_1}^2 + k_2 {l_2}^2\right)\theta - k_1 l_1 y = 0$$

After simplification $\Rightarrow$ $\ddot{\theta} + \Omega^2 \left(\frac{a^2+b}{c}\right)\theta - \Omega^2 \left(\frac{a^2}{cl}\right) y = 0$

$$\boxed{\begin{aligned} & l_1 = a l_2 = l \\ & k_2 = b k_1 \\ & m_{bar} = (3m).c \\ & \Omega^2 = \frac{k_1}{m} \\ & I = \frac{1}{3} m_{bar} {l_2}^2 \end{aligned}}$$

$$\begin{cases} \ddot{y} + (\Omega^2) y - (\Omega^2 l)\theta = 0 \\ \ddot{\theta} - \Omega^2 \left(\dfrac{a^2}{cl}\right) y + \Omega^2 \left(\dfrac{a^2+b}{c}\right)\theta = 0 \end{cases}$$

$$\begin{cases} \ddot{u}_1 + (\Omega^2) u_1 - (\Omega^2 l) u_2 = 0 \\ \ddot{u}_2 - \Omega^2 \left(\dfrac{a^2}{cl}\right) u_1 + \Omega^2 \left(\dfrac{a^2+b}{c}\right) u_2 = 0 \end{cases}$$

In the matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underset{\sim}{m}} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \Omega^2 \underbrace{\begin{bmatrix} 1 & -l \\ -\dfrac{a^2}{cl} & \dfrac{a^2+b}{c} \end{bmatrix}}_{\underset{\sim}{k}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underset{\sim}{0}$$

Step 3: Find the corresponding frequencies

$$det(\underset{\sim}{K} - \omega^2 . \underset{\sim}{m}) = 0 \quad \Longrightarrow \quad \text{Find } \omega's$$

$$det\left(\Omega^2 \begin{bmatrix} 1 & -l \\ -\frac{a^2}{cl} & \frac{a^2+b}{c} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = det\left(\Omega^2 \begin{bmatrix} 1-\omega^2 & -l \\ -\frac{a^2}{cl} & \frac{a^2+b}{c} - \omega^2 \end{bmatrix}\right) = 0$$

$\Omega^2 \neq 0 \quad \Longrightarrow \quad det\left(\begin{bmatrix} 1-\omega^2 & -l \\ -\frac{a^2}{cl} & \frac{a^2+b}{c} - \omega^2 \end{bmatrix}\right) = 0$

After simplification:

$$\omega^4 - \omega^2\left(\frac{a^2+b+c}{c}\right) + \frac{b}{c} = 0$$

To be more realistic, let's give some values to the parameters:

$l_1 = l = 12.5\ cm = 0.125\ m$

$l_2 = 50\ cm = 0.5\ m$

$m = 1\ kg$

$m_{bar} = 0.3\ kg$

$k_1 = 34\ N/m$

$k_2 = 17\ N/m$

$I = 0.025\ m^4$

$\Omega^2 = \frac{k_1}{m} = 34$

$a = \frac{l}{l_2} = 0.25$

$b = 0.5$

$c = \frac{m_{bar}}{3} = 0.1$

$\frac{a^2+b+c}{c} = 6.625$

$\omega^4 - \omega^2\left(\frac{a^2+b+c}{c}\right) + \frac{b}{c} = 0 \quad \Longrightarrow \quad \omega^4 - 6.625\ \omega^2 + 5 = 0 \quad \Longrightarrow \quad \boxed{\omega^2 = 0.869 \quad , \quad 5.756}$

$\omega_1 = 0.932 \qquad \& \qquad \omega_2 = 2.4$

Step 3: Find the mode shape

$$\left(\begin{bmatrix} 1-\omega^2 & -l \\ -\dfrac{a^2}{cl} & \dfrac{a^2+b}{c}-\omega^2 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

y

θ

$\omega_1 = 0.932 \longrightarrow \quad y = 1 \quad , \quad \theta = 1.048 \longrightarrow \quad mode\ shape\ 1$

$\omega_2 = 2.4 \longrightarrow \quad y = 1 \quad , \quad \theta = -38.048 \longrightarrow \quad mode\ shape\ 2$

Step 4: Check if the mode shapes are orthogonal with respect to each other

$$\{\Phi_1\} = \begin{Bmatrix} \emptyset_{11} \\ \emptyset_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.048 \end{Bmatrix}$$

$$\{\Phi_2\} = \begin{Bmatrix} \emptyset_{21} \\ \emptyset_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -38.048 \end{Bmatrix}$$

$$\{\underset{\sim}{\Phi}\}^T[\underset{\sim}{m}]\{\underset{\sim}{\Phi}\}$$

$$\begin{bmatrix} 1 & 1.048 \\ 1 & -38.048 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.025 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1.048 & -38.048 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.027 & 0 \\ 0 & 40 \end{bmatrix}}_{Generalized\ Mass\ Matrix}$$

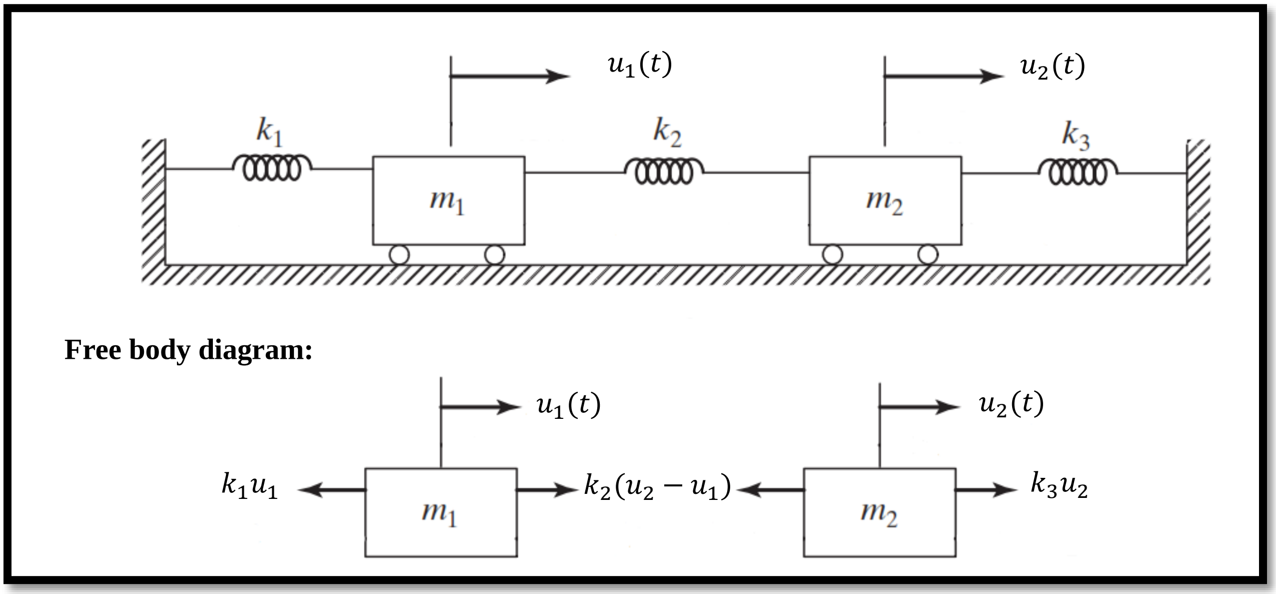
$$\{\Phi_1\}^T[m]\{\Phi_2\}$$

$$\begin{bmatrix} 1 & 1.048 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.025 \end{bmatrix}\begin{bmatrix} 1 \\ -38.048 \end{bmatrix} = [0]$$

Both conditions are satisfied and the mode shape are orthogonal with respect to each other.

Finding the Solution for Free Vibration 2nd DOF of an Undamped System

Look at following system.



Let's go over all steps for this problem.

For this system, the equations of motion will be:

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0$$

$$m_2\ddot{u}_2 - k_2u_1 + (k_2 + k_3)u_2 = 0$$

If m_1 & m_2 can oscillate harmonically with same frequency and phase but different amplitude then we can say:

$$u_1 = \Phi_1 cos\,(\omega t + \theta)$$

$$u_2 = \Phi_2 cos\,(\omega t + \theta)$$

Φ_1 & Φ_2: The maximum amplitudes of u_1& u_2

With replacing these equations in the equations of motion and simplifying, we will reach:

$$[-m_1\omega^2 + (k_1 + k_2)]\Phi_1 - k_2\Phi_2 = 0$$

$$-k_2\Phi_1 + [-m_2\omega^2 + (k_2 + k_3)]\Phi_2 = 0$$

As you seen before, for finding Φ_1 & Φ_2, we have to write these equations and make determinant of coefficients equal zero.

$$det\begin{bmatrix}[-m_1\omega^2 + (k_1 + k_2)] & -k_2 \\ -k_2 & [-m_2\omega^2 + (k_2 + k_3)]\end{bmatrix} = 0$$

From that determinant, we can find two natural frequencies (ω) for system:

$$\omega_1{}^2, \omega_2{}^2 = \frac{1}{2}\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right] \pm \frac{1}{2}\sqrt{\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right]^2 - 4\left[\frac{(k_1+k_2)(k_2+k_3) - k_2{}^2}{m_1 m_2}\right]}$$

For each of these natural frequencies, we will have different mode shapes (Φ_1 & Φ_2).

$$\omega_1 \Longrightarrow \{\Phi_1\} = \begin{Bmatrix} \emptyset_{11} \\ \emptyset_{12} \end{Bmatrix}$$

$$\omega_2 \Longrightarrow \{\Phi_2\} = \begin{Bmatrix} \emptyset_{21} \\ \emptyset_{22} \end{Bmatrix}$$

However, two equations of motions are homogenous set of equation and they are not independent! So, we are only able to find the ratio of $r_1 = \frac{\emptyset_{12}}{\emptyset_{11}}$ & $r_2 = \frac{\emptyset_{22}}{\emptyset_{21}}$.

$$r_1 = \frac{\emptyset_{12}}{\emptyset_{11}} = \frac{[-m_1\omega_1{}^2 + (k_1+k_2)]}{k_2} = \frac{k_2}{[-m_2\omega_1{}^2 + (k_2+k_3)]}$$

$$r_2 = \frac{\emptyset_{22}}{\emptyset_{21}} = \frac{[-m_1\omega_2{}^2 + (k_1+k_2)]}{k_2} = \frac{k_2}{[-m_2\omega_2{}^2 + (k_2+k_3)]}$$

So, the normal modes of vibration corresponding ω_1 & ω_2 will be:

$$\{\Phi_1\} = \begin{Bmatrix} \emptyset_{11} \\ \emptyset_{12} \end{Bmatrix} = \begin{Bmatrix} \emptyset_{11} \\ r_1\emptyset_{11} \end{Bmatrix}$$

$$\{\Phi_2\} = \begin{Bmatrix} \emptyset_{21} \\ \emptyset_{22} \end{Bmatrix} = \begin{Bmatrix} \emptyset_{21} \\ r_2\emptyset_{21} \end{Bmatrix}$$

$\Longrightarrow$ Modal vectors

Then, solution for each mode will be:

$$u_1 = \begin{Bmatrix} \emptyset_{11} cos\,(\omega_1 t + \theta_1) \\ r_1\emptyset_{11} cos\,(\omega_1 t + \theta_1) \end{Bmatrix}$$ 1st mode

$$u_2 = \begin{Bmatrix} \emptyset_{21} cos\,(\omega_2 t + \theta_2) \\ r_2\emptyset_{21} cos\,(\omega_2 t + \theta_2) \end{Bmatrix}$$ 2nd mode

The general solution can be obtained by a linear superposition of the two normal modes:

$u = c_1 u_1 + c_2 u_2$

c_1 & c_2 are constant and can be combine with $\emptyset_{11}$ & $\emptyset_{21}$values (constants).

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \emptyset_{11} cos\,(\omega_1 t + \theta_1) + \emptyset_{21} cos\,(\omega_2 t + \theta_2) \\ r_1 \emptyset_{11} cos\,(\omega_1 t + \theta_1) + r_2 \emptyset_{21} cos\,(\omega_2 t + \theta_2) \end{Bmatrix}$$

However, we still need to find the value for $\emptyset_{11}$, $\emptyset_{21}$, θ_1, and θ_2. To obtain these values we have to use two initial conditions for each mass (the rest of steps is in the book). The final results will be:

$$\boxed{\begin{gathered} \emptyset_{11} = \frac{1}{(r_2 - r_1)} \sqrt{\{r_2 u_1(0) - u_2(0)\}^2 + \frac{\{-r_2 \dot{u}_1(0) + \dot{u}_2(0)\}^2}{{\omega_1}^2}} \\ \emptyset_{21} = \frac{1}{(r_2 - r_1)} \sqrt{\{-r_1 u_1(0) + u_2(0)\}^2 + \frac{\{r_1 \dot{u}_1(0) - \dot{u}_2(0)\}^2}{{\omega_2}^2}} \\ \theta_1 = tan^{-1}\left[\frac{-r_2 \dot{u}_1(0) + \dot{u}_2(0)}{\omega_1[r_2 u_1(0) - u_2(0)]}\right] \\ \theta_2 = tan^{-1}\left[\frac{r_1 \dot{u}_1(0) - \dot{u}_2(0)}{\omega_2[-r_1 u_1(0) + u_2(0)]}\right] \end{gathered}}$$

If in this example we have $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, $m_1 = 10$, $m_2 = 1$, and initial conditions of $u_1(0) = 1$, $u_2(0) = \dot{u}_1(0) = \dot{u}_2(0) = 0$,

$$\begin{bmatrix} [-m_1\omega^2 + (k_1 + k_2)] & -k_2 \\ -k_2 & [-m_2\omega^2 + (k_2 + k_3)] \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$det \begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} = 0$$

$$10\omega^4 - 85\omega^2 + 150 = 0 \Rightarrow {\omega_1}^2 = 2.5 \quad , \quad {\omega_2}^2 = 6 \Rightarrow \boxed{\omega_1 = 1.5811 \quad , \quad \omega_2 = 2.4495}$$

$$r_1 = \frac{\emptyset_{12}}{\emptyset_{11}} = \frac{[-m_1{\omega_1}^2 + (k_1 + k_2)]}{k_2} = \frac{[-10 \times 2.5 + (35)]}{5} = 2$$

$$r_2 = \frac{\emptyset_{22}}{\emptyset_{21}} = \frac{[-m_1\omega_2{}^2 + (k_1 + k_2)]}{k_2} = \frac{[-10 \times 6 + (35)]}{5} = -5$$

Normal modes will be:

$$\{\Phi_1\} = \begin{Bmatrix} \emptyset_{11} \\ \emptyset_{12} \end{Bmatrix} = \begin{Bmatrix} \emptyset_{11} \\ r_1\emptyset_{11} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \emptyset_{11}$$

$$\{\Phi_2\} = \begin{Bmatrix} \emptyset_{21} \\ \emptyset_{22} \end{Bmatrix} = \begin{Bmatrix} \emptyset_{21} \\ r_2\emptyset_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} \emptyset_{21}$$

Then, solution for each mode will be:

$$u_1 = \begin{Bmatrix} \emptyset_{11} cos\ (1.5811t + \theta_1) + \emptyset_{12} cos\ (2.4495t + \theta_2) \\ r_1\emptyset_{11} cos\ (1.5811t + \theta_1) \end{Bmatrix} \qquad \text{1st mode}$$

$$u_2 = \begin{Bmatrix} \emptyset_{21} cos\ (\omega_2 t + \theta_2) \\ r_2\emptyset_{21} cos\ (\omega_2 t + \theta_2) \end{Bmatrix} \qquad \text{2nd mode}$$

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \emptyset_{11} cos\ (1.5811t + \theta_1) + \emptyset_{21} cos\ (2.4495t + \theta_2) \\ 2\emptyset_{11} cos\ (1.5811t + \theta_1) - 5\emptyset_{21} cos\ (2.4495t + \theta_2) \end{Bmatrix}$$

$$u_1(0) = 1 = \emptyset_{11} cos\ \theta_1 + \emptyset_{21} cos\ \theta_2$$

$$u_2(0) = 0 = 2\emptyset_{11} cos\ \theta_1 - 5\emptyset_{21} cos\ \theta_2$$

$$\dot{u}_1(0) = 0 = -1.5811\emptyset_{11}\ sin\ \theta_1 - 2.4495\emptyset_{21}\ sin\ \theta_2$$

$$\dot{u}_2(0) = 0 = -3.1622\emptyset_{11}\ sin\ \theta_1 - 12.2475\emptyset_{21}\ sin\ \theta_2$$

$\emptyset_{11} cos\ \theta_1 = \frac{5}{7}$, $\emptyset_{21} cos\ \theta_2 = \frac{2}{7}$, $\emptyset_{11}\ sin\ \theta_1 = 0$, $\emptyset_{21}\ sin\ \theta_2 = 0$

$\emptyset_{11} = \frac{5}{7}$, $\emptyset_{21} = \frac{2}{7}$, $\theta_1 = 0$, $\theta_2 = 0$

$$\boxed{\begin{aligned} u_1(t) &= \frac{5}{7}\ cos\ 1.5811t + \frac{2}{7} cos\ 2.4495t \\ u_2(t) &= \frac{10}{7} cos\ 1.5811t - \frac{10}{7} cos\ 2.4495t \end{aligned}}$$