Introduction

Concepts that you need for this course

1) Vector Calculus: Trigonometry
   - How to write a vector
   - How to calculate the length of a vector
   - How to compute the angle between two vectors
   - Dot product & cross product

2) Linear Algebra: Matrices, multiplication of vector and matrix

3) Reference frame

4) Derivatives: In addition, to study the position, we will work with velocity, acceleration, and higher derivatives in this course

5) Rigid bodies

6) Statics: Free body diagrams, Force, Moments, etc.

What we will cover in this course

- In this course, we **just** have a focus on **planar linkages** while in industry and advanced research, there are many spatial linkages.
- Sawing machine is an example of a planar motion device (two planar motions parallel to each other)
- We will define the components of the linkage like joints, links, different types of joints, etc.
- We will learn how to calculate the degree of freedom (D.O.F) for a linkage. In other words, we can find if a linkage can able to move, and if yes, how many actuators are required to control its motion?
- We will study the force, displacement, velocity, and acceleration of the linkages.
- There are many different designs (linkages) for doing a task, but we are looking for the most efficient one!
- Machine (Mechanical): A machine is a human-made device that uses power to apply forces and control movement to perform an action.
- We will study the quick return mechanism (in one direction going slow when in the other direction moving fast)
- We will study input/output angular motion relation (car steering system). For instance, in a 4-wheel drive vehicle, we will have different angles for different wheels.

- We will study the 4bar linkages in detail.
- We will study the slider-crank linkages (rotation to translation motion like car engine (cylinder-piston) or for the cases that we need lots of force)
- We will study the cam linkages (Angular to linear motion with creating a profile)
- We will see how to design a proper cam to generate the desired motion.
- We will study gears, chains, and belts. These are used for changing the velocity or torque from input to output with a constant ratio.

Definitions

- Kinematics: (Only) Study of motion (don't care about forces)
- Dynamics: Forces of systems in Motion
- Statics: Forces on unmoved systems
- Machine: A set of elements linked by joints that allow relative motion between them, to transmit motion and/or forces.

There are two possible study processes for any machine:

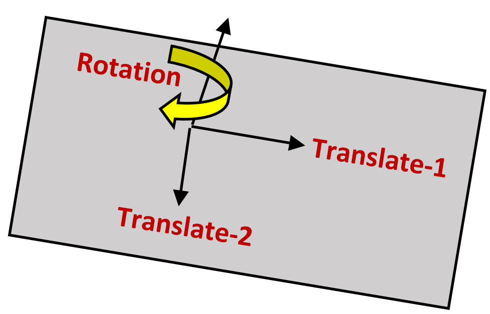
a) Analysis: Given a machine, calculate its motion and forces.
b) Synthesis: (Design process) – Given a motion/forces, create a machine able to do the task.

Hypotheses in this course

- We are dealing with rigid bodies (shapes are not changing, no deformation, no break)
- Most of the time, we will assume the conservation of energy for problems (Only for force study sometimes, if in the problem mentioned, we will care about friction)

Power in = Power out

- We will work with planar motion

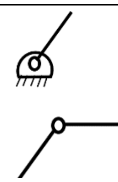
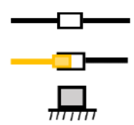
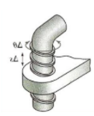


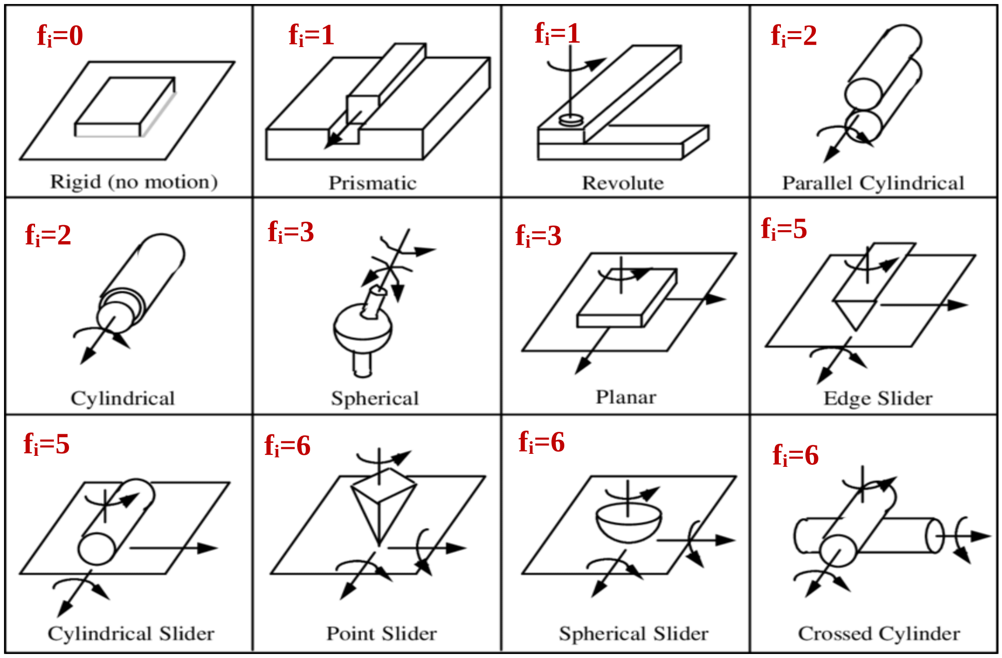
Types of Joints and Degree of Freedom

Machine → Links + Joints

- Links: Rigid Bodies (It can be in any shape but there is no deformation during motion)
- Joints: The connection between two surfaces. The shape of the surfaces determines the motion.

Various Joints

Name	Symbol	Example	Kinematic Sketch	Type of motion	DOF
Revolute (Hinge, Pin)	R			Rotation (only) about an axis	1
Prismatic (Slider)	P			Translation (only) along a direction	1
Cam	×		×	Translation + Rotation about a point	1
Helical (Screw)	H		×	Translation along + Rotation about a point	1



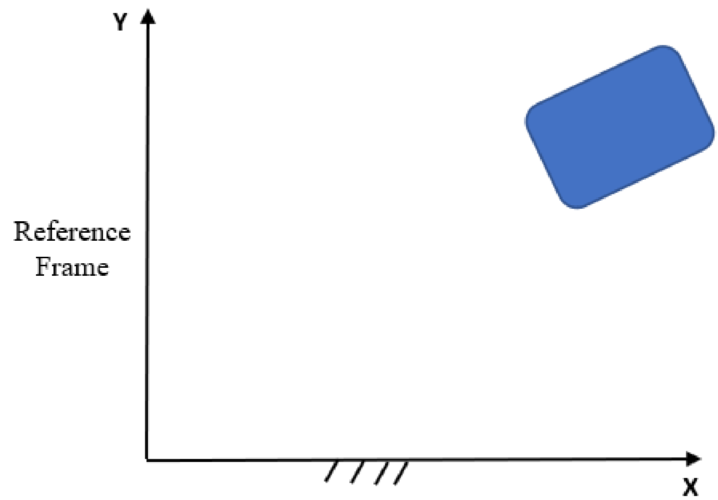
- f_i = Degree of Freedom of joints

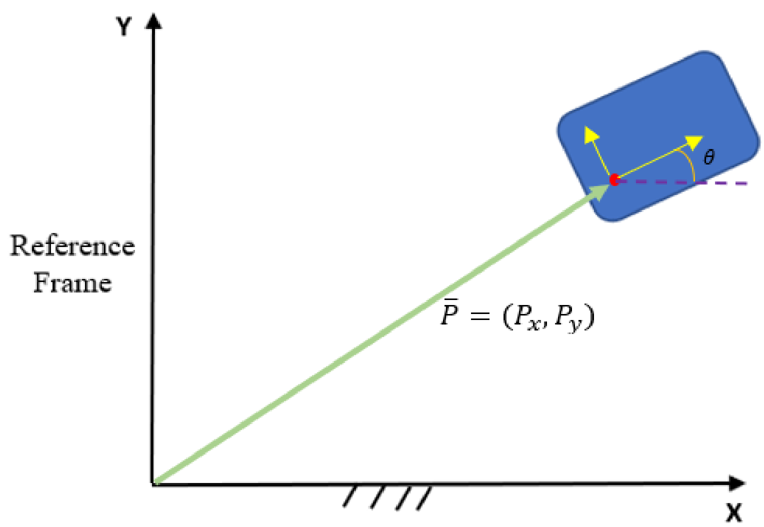
Degree of Freedom (D.O.F)

- Degree of Freedom (D.O.F): The number of **independent** parameters needs to uniquely define the position of a body at a given instant of time and relative to a **reference frame**.

Example: How many parameters do we need to define the position of the following body?

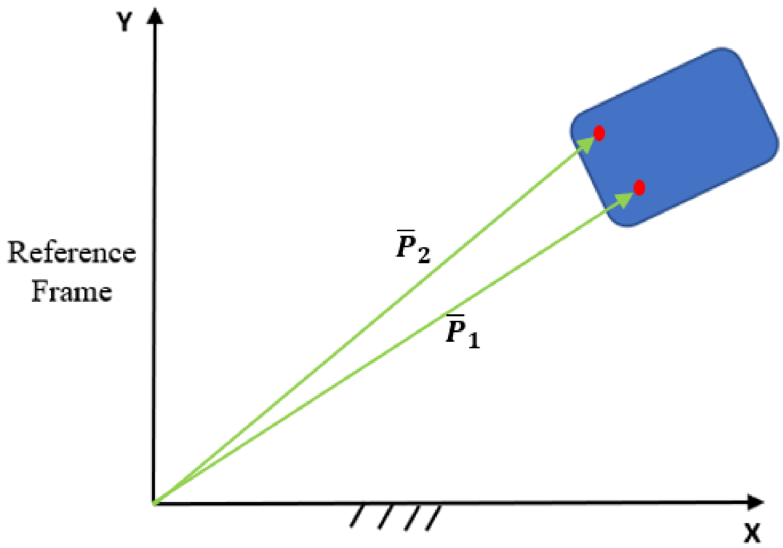
A) 1
B) 2
C) 3
D) 4





Rigid body in the plane: 3 D.O.F

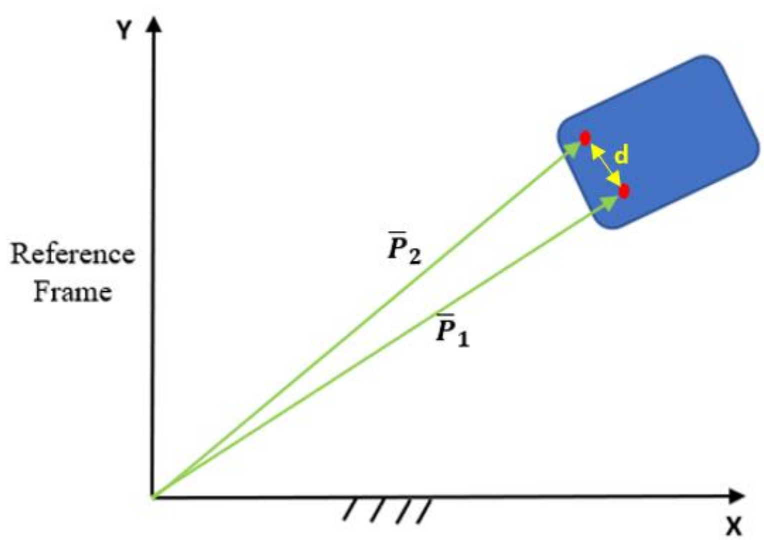
Another way is defining 2 points on the body to be fully defined.



$$\overline{P}_1 = (P_{1x}, P_{1y})$$

$$\overline{P}_2 = (P_{2x}, P_{2y})$$

What is the extra parameter?

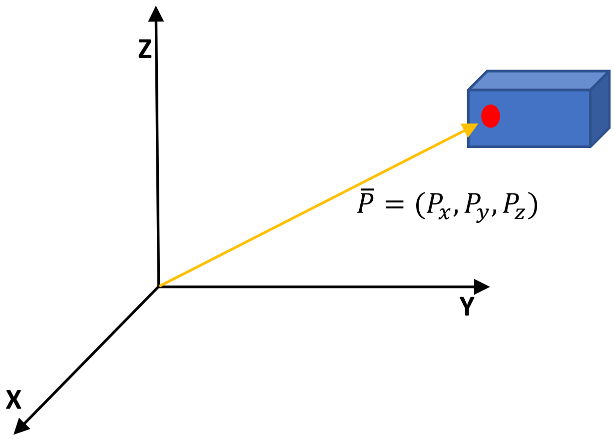


The parameters are not independent because: $(\overline{P}_2 - \overline{P}_1).(\overline{P}_2 - \overline{P}_1) = d^2$

So, D.O.F is always: 3

Spatial motion (3D):

- What is the D.O.F of the following body?



3 to locate origin and 3 to define orientation: 6 D.O.F

Mobility (D.O.F) for mechanisms

- Mobility and D.O.F are equivalent concepts for us.
- Mobility: Parameters are needed to specify the position of all links of a mechanism.

Example: For a 4-bar linkage in a planar case, we have 12 parameters (3 for each link) but some of them are dependent because they are connected by joints!

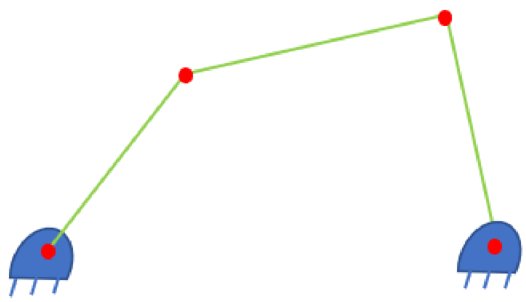
Formula for D.O.F. (planar case):

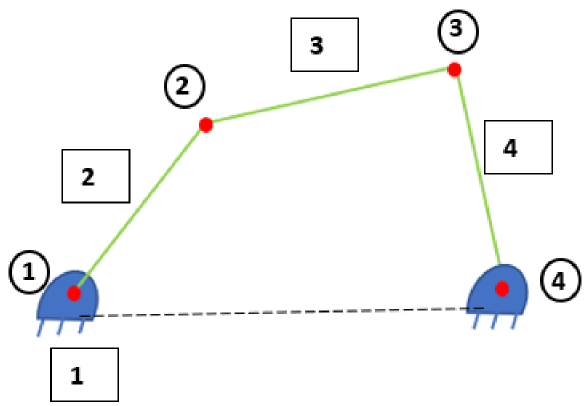
1) For each link: 3 parameters
2) If we have **n** links: 3n parameters
3) Subtract the ground link: 3(n-1) parameters
4) Each joint has f degree of freedom ➡ restricts (3-f) degree of freedom for a link

For a linkage with **n** links (counting the ground) and **j** joints, having f_i D.O.F each:

$$M = \underbrace{3(n-1)}_{\text{D.O.F of all Links}} - \underbrace{\sum_{i=1}^{j}(3 - f_i)}_{\text{Constraints imposed by joints}}$$

Example: Find the mobility of the following mechanism.





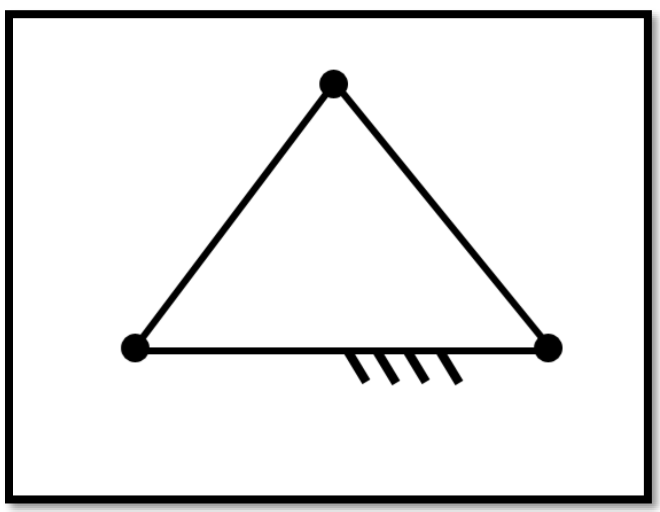
$n = 4$, $j = 4$, $f_i = 1$

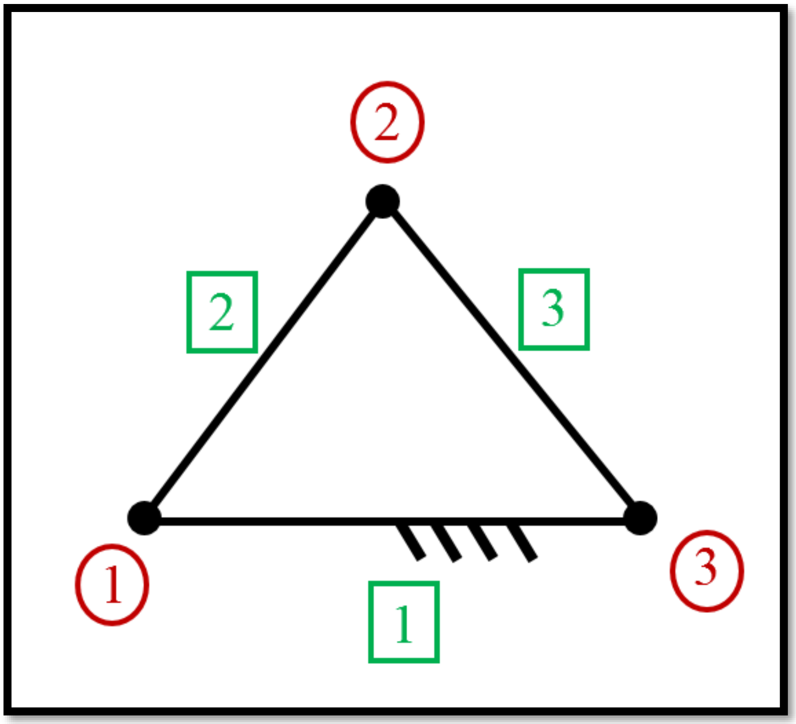
$$M = 3\times(4-1) - 4\times(3-1) = 9 - 8 = 1$$

Joint	D.O.F. (f_i) (Motion allowed by the joint)	Restriction 2D ($3 - f_i$)	Restriction 3D ($6 - f_i$)
Revolute (R)	1	3-1=2	6-1=5
Prismatic (P)	1	2	5
Cam	2	1	4
Spherical (S)	3	×	3
Cylindrical (C)	2	× (is not defined in the plane)	4

Examples:

1)

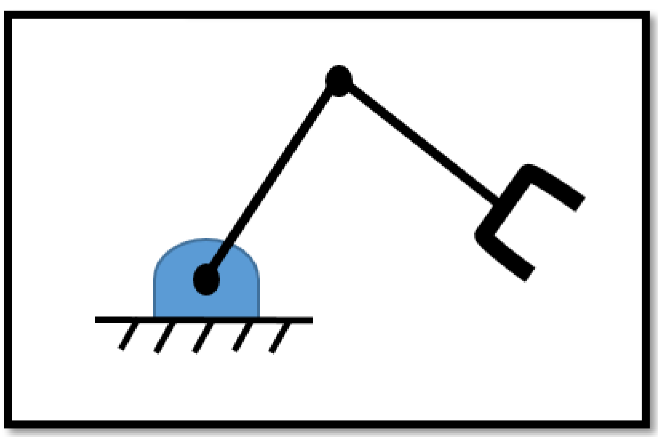


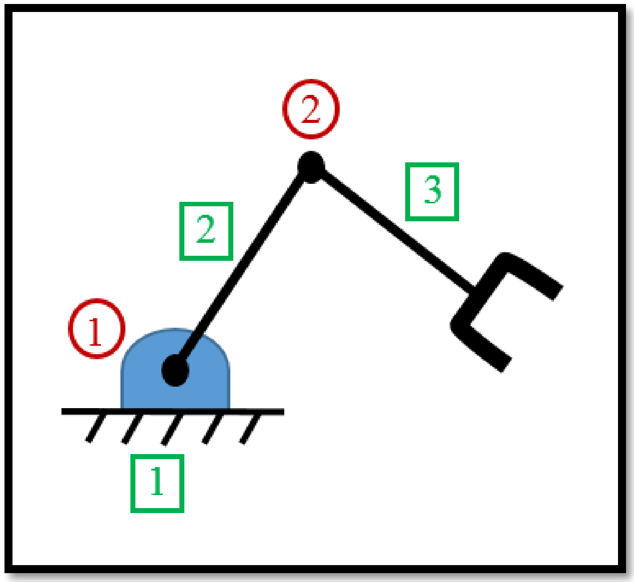


n = 3 , j = 3 , $f_i = 1$

$$M = 3(n-1) - \sum_{i=1}^{j}(3-f_i) = 3 \times (3-1) - 3 \times (3-1) = 6-6 = 0$$

2)

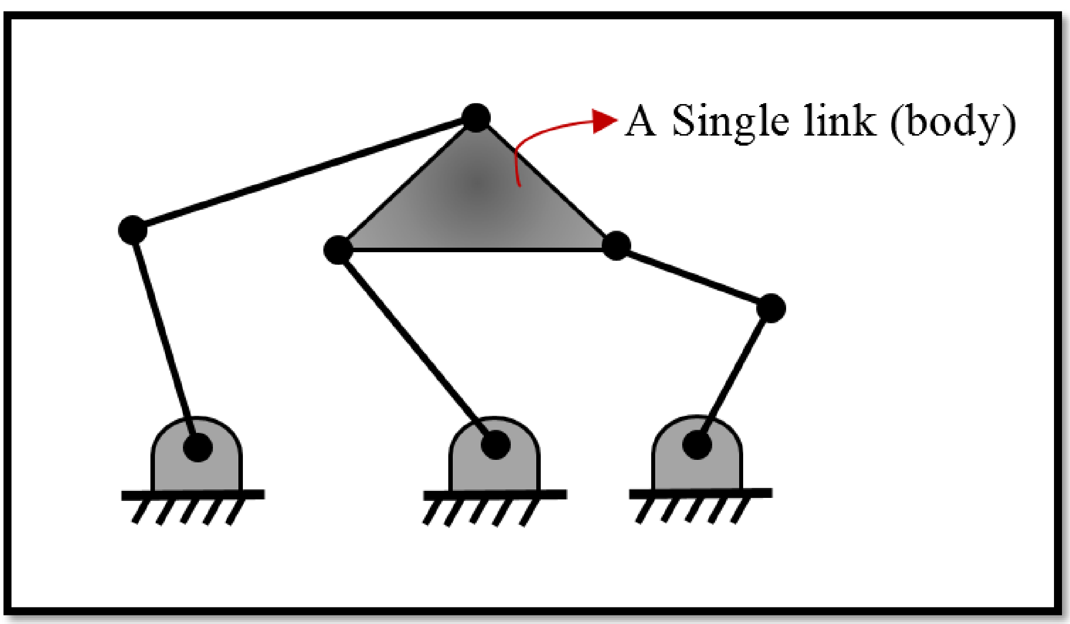


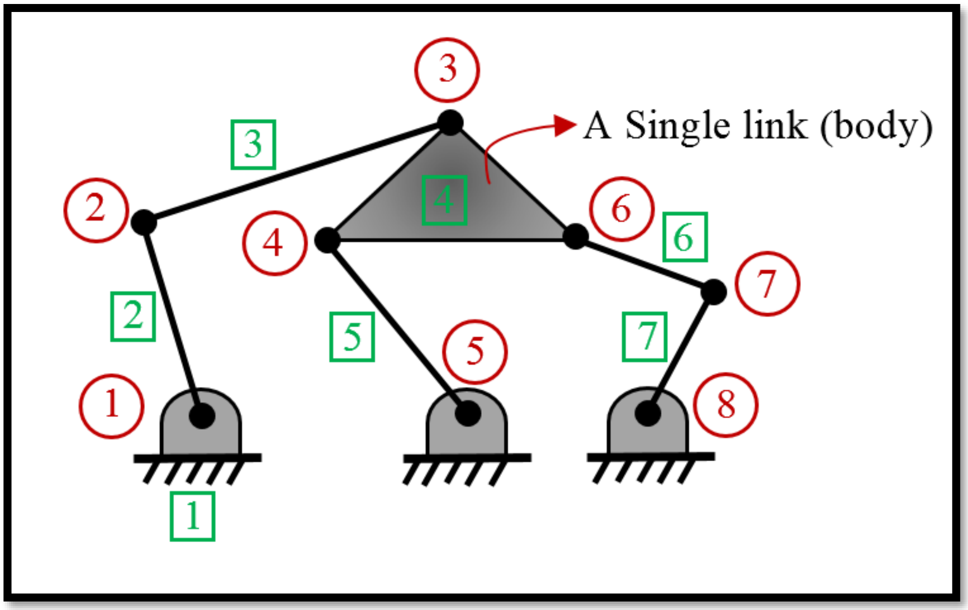


$n = 3 \quad , \quad j = 2 \quad , \quad f_i = 1$

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (3-1) - 2 \times (3-1) = 6 - 4 = 2$$

3)



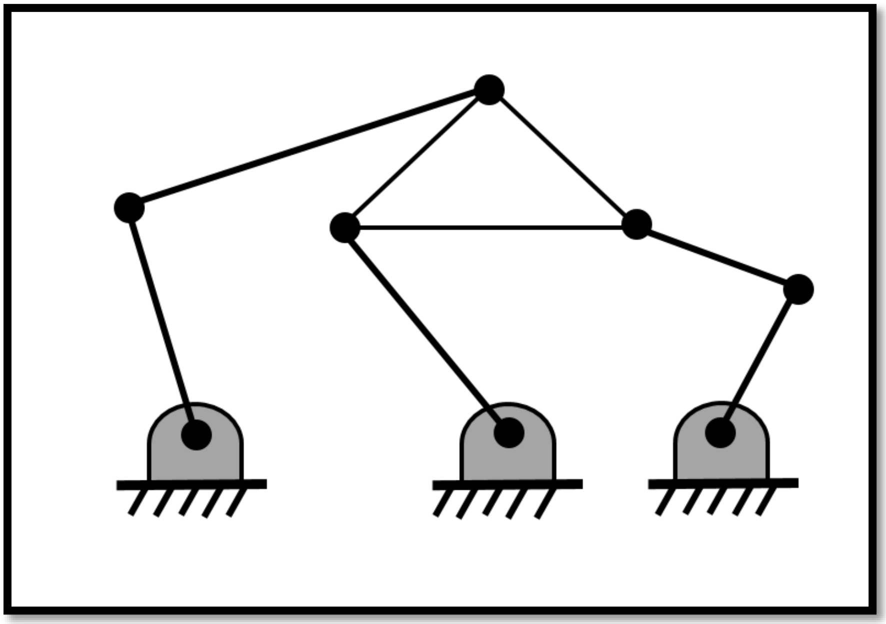


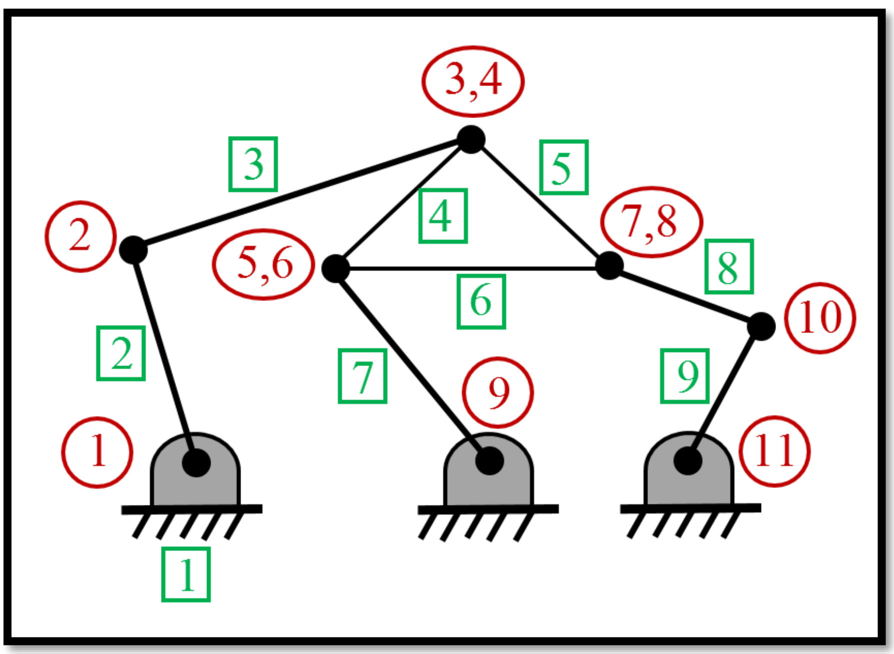
n = 7 , j = 8 , $f_i = 1$

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (7-1) - 8 \times (3-1) = 18 - 16 = 2$$

Degree of Freedom (Continue)

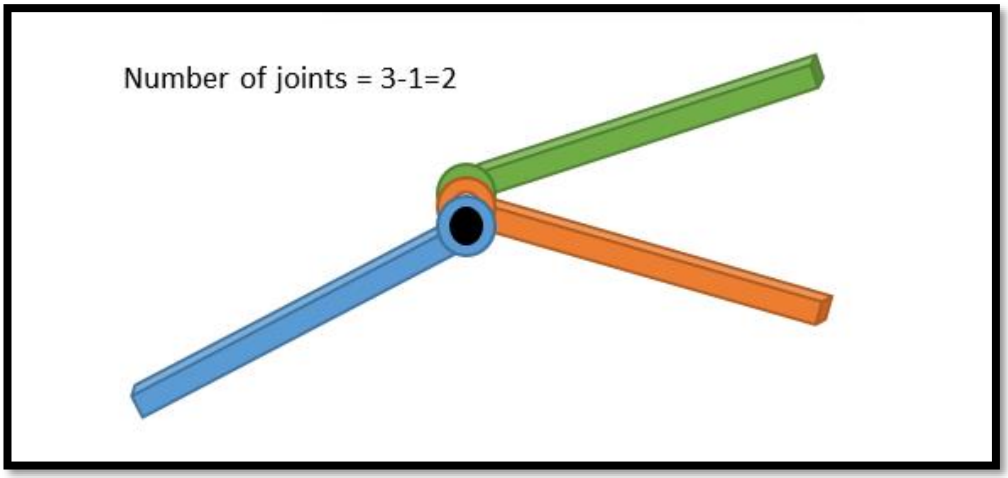
4)





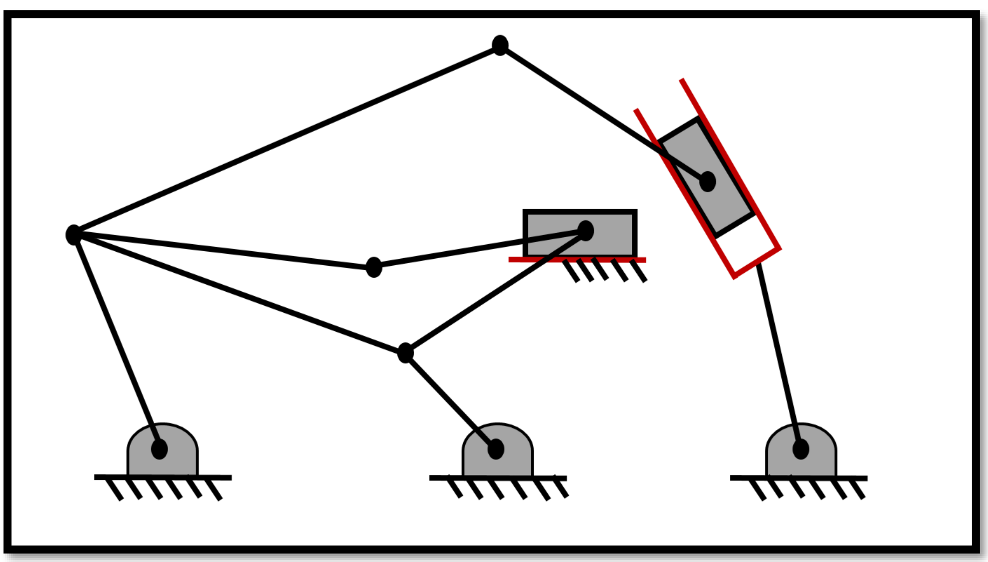
n = 9 , j = 11 , $f_i = 1$

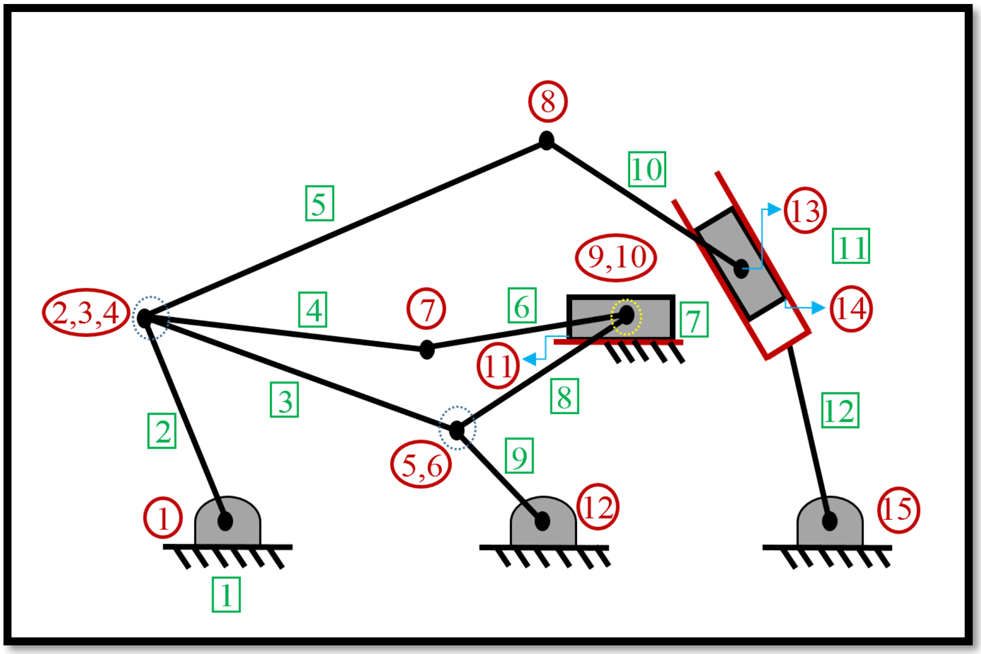
$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (9-1) - 11 \times (3-1) = 24 - 22 = 2$$



Number of joints = Number of links connected to a single joint -1

5)



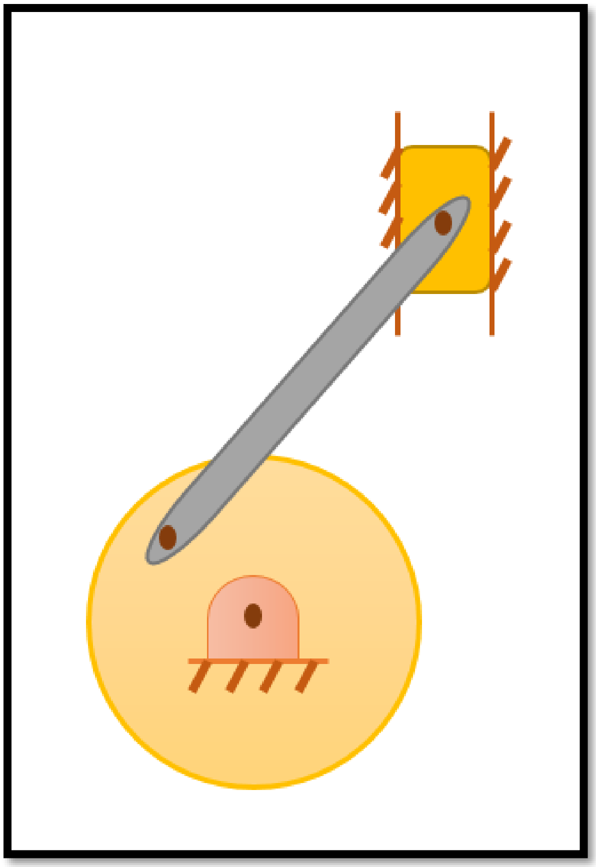


n = 12 , j = 15 , $f_i = 1$ (for both prismatic and revolute joints)

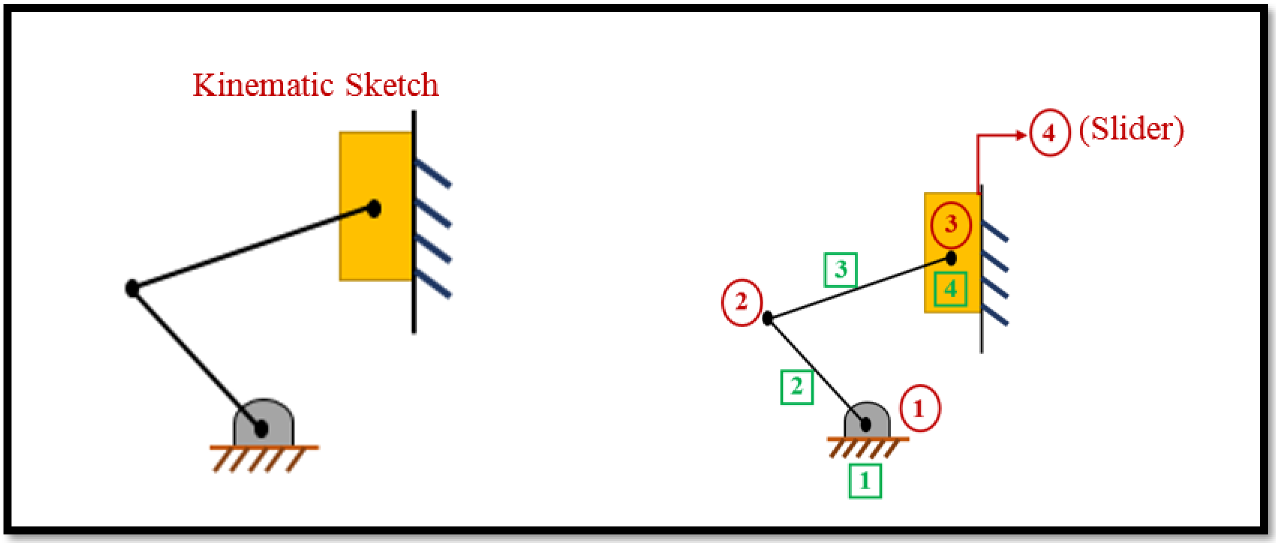
$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (12-1) - 15 \times (3-1) = 33 - 30 = 3$$

So, in this example, if you know 3 information (such as the angle between two links or the length of the links) then you can find all other information about this mechanism!

6)



For this example and similar cases, first, you have to draw the kinematic sketch!

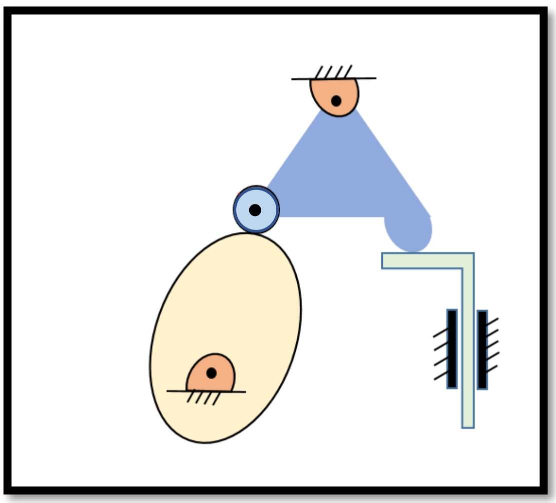


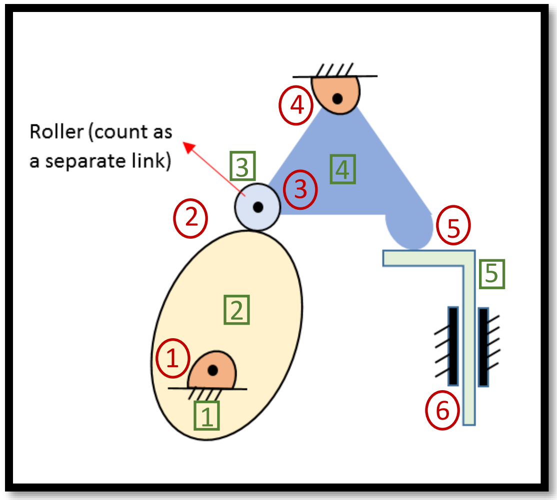
n = 4 , j = 4 , $f_i = 1$ (for both prismatic and revolute joints)

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (4-1) - 4 \times (3-1) = 9 - 8 = 1$$

- To see the animation for various mechanisms there are many websites. One of the famous ones is: https://www.mekanizmalar.com/

7)

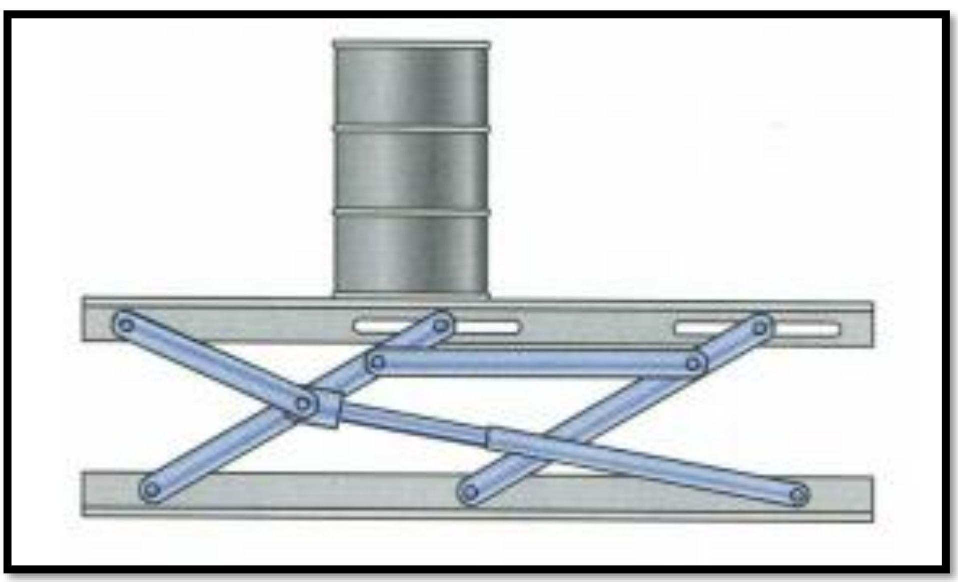


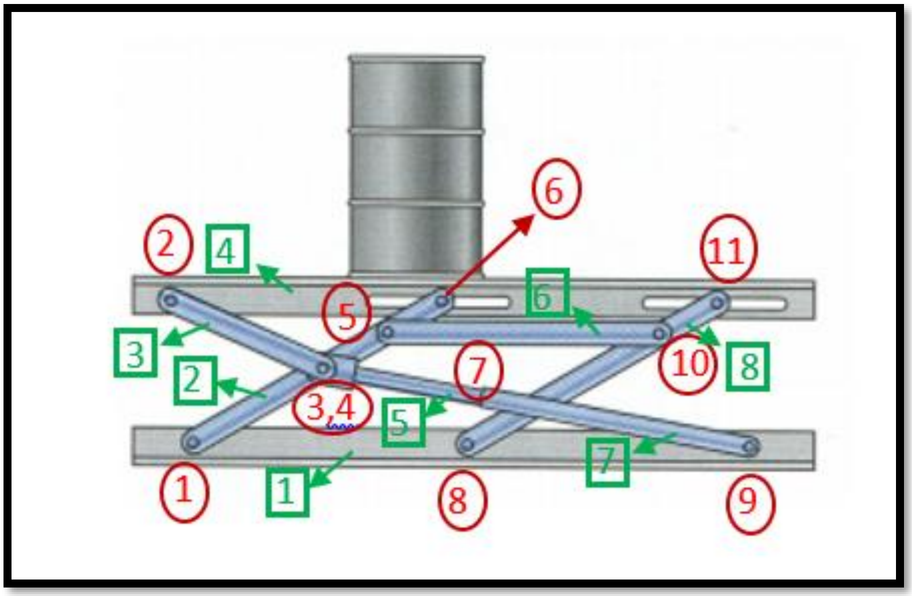


n = 5 , j = 6 , $f_i = 1$ (for all joints except joint #2), $f_i = 2$ (for joint #2)

$M = 3(n-1) - \sum_{i=1}^{j}(3-f_i) = 3 \times (5-1) - 5 \times (3-1) - 1 \times (3-2) = 12 - 10 - 1 = 1$

8)



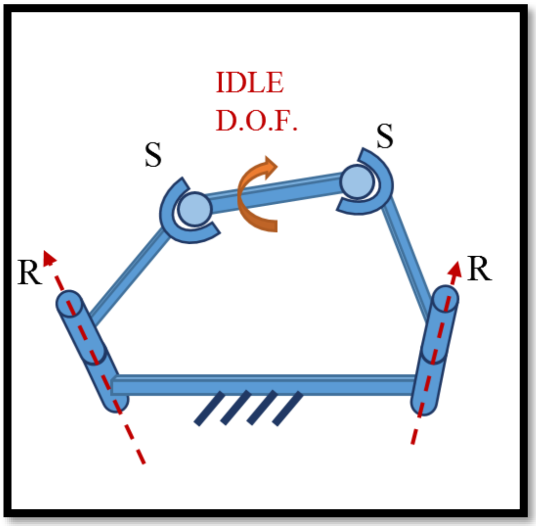


n = 8 , j = 11 , $f_i = 1$ (for all joints except joint #6 & # 11), $f_6 = f_{11} = 2$

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (8-1) - 9 \times (3-1) - 2 \times (3-2) = 21 - 18 - 2 = 1$$

Remarks: Sometimes the D.O.F. formula doesn't work!

1) Idle degree of freedom: An internal D.O.F. that does not affect the input/output relation of the linkage. For instance, RSSR linkage. From the formula, you will find D.O.F. equals 2 while we just need only one actuator to control the mechanism! (the second one is internal D.O.F.)



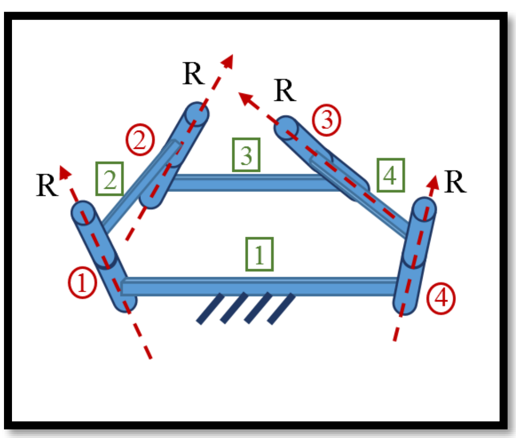
2) Over-constrained mechanisms: They have negative mobility!

Example: Find the mobility of a 4-bar linkage in Space.

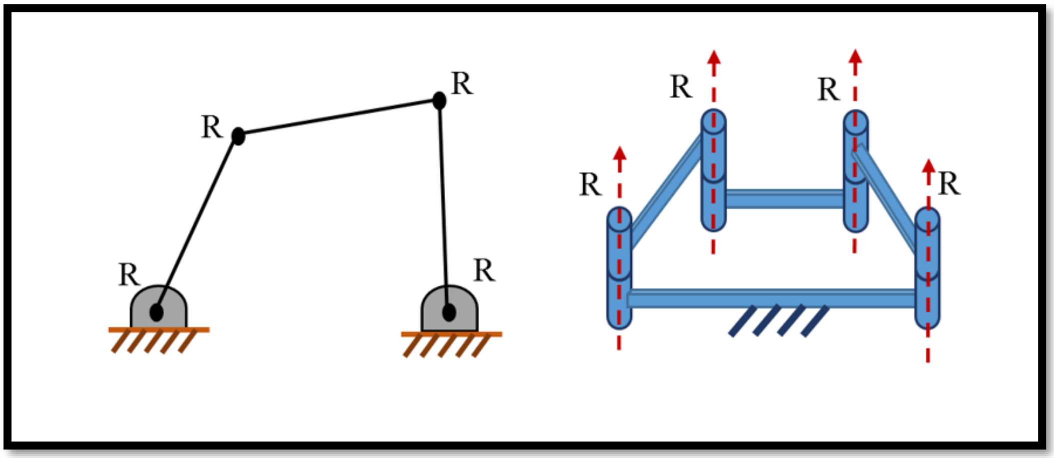
In general:

$$n = 4 \quad , \quad j = 4 \quad , \quad f_i = 1$$

$$M = 6(n-1) - \sum_{i=1}^{j}(6 - f_i) = 6\times(4\text{-}1)\text{-}4\times(6\text{-}1) = 18\text{-}20 = \text{-}2 \qquad \text{(Over-constrained)}$$



If the axes of revolute joints are parallel with each other (like a planar case)



n = 4 , j = 4 , $f_i = 1$

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3\times(4\text{-}1)\text{-}4\times(3\text{-}1) = 9\text{-}8 = 1$$

- Some over-constrained linkages are movable when the axes satisfy certain geometric constraints (such as parallel)

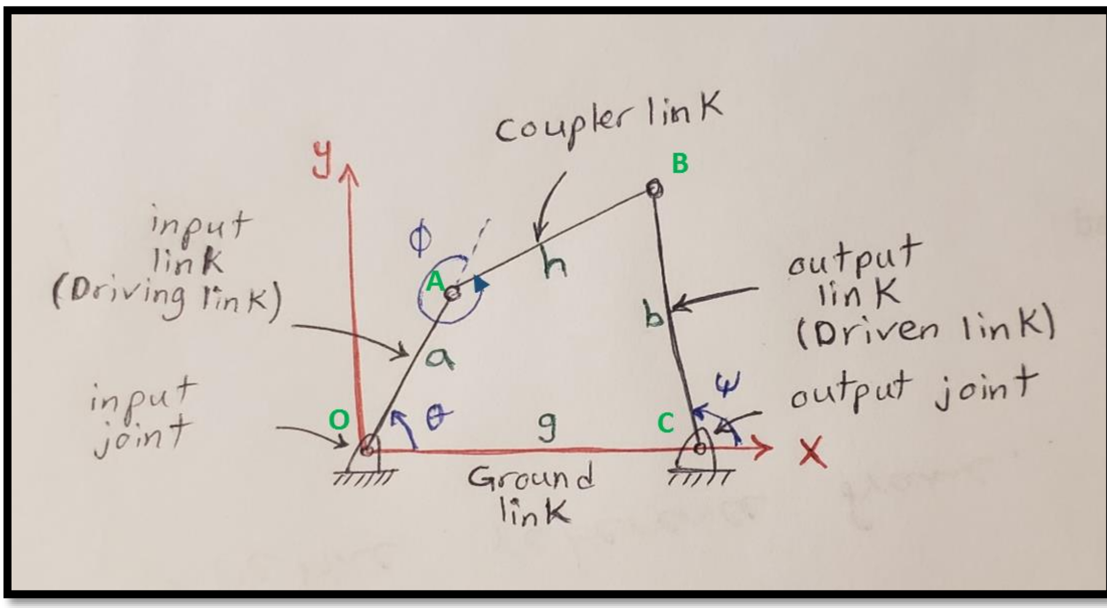
For more examples about link/joint/mobility, you can take a look at Ch.1 of any kinematic book.

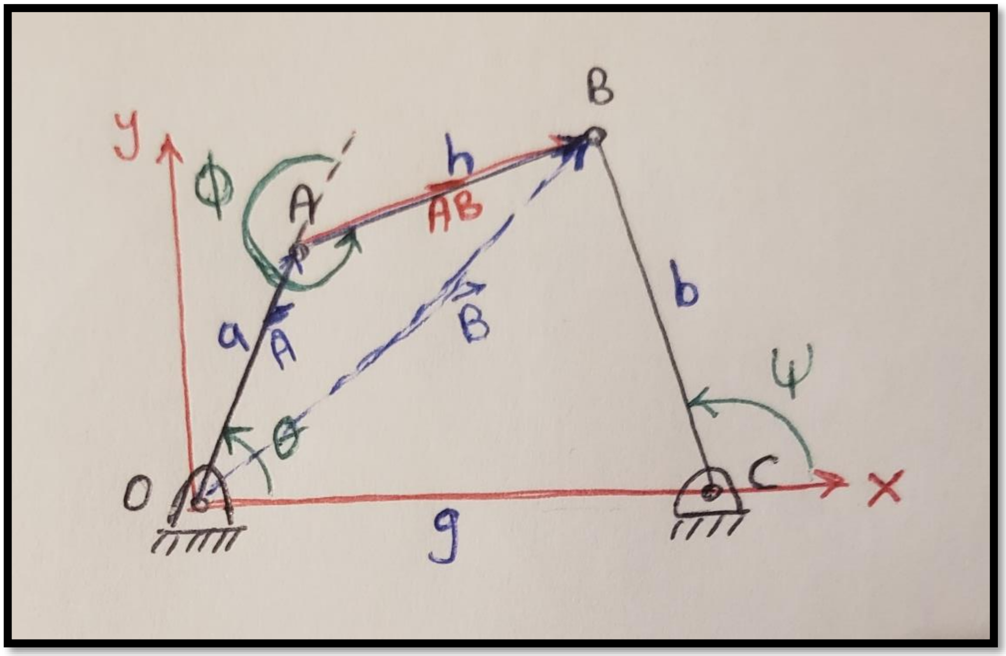
Linkage Analysis

1. Kinematic sketch
2. Mobility
3. Position analysis: Determine the position of any point in the linkage.

Steps:

1. First, we have to define the reference frame.
2. Define variables and parameters (give the name to them)
3. Link length: a, b, h, g
4. Angular/joint variables (measure from the previous link in CCW direction): $\theta\ (input\ angle), \emptyset\ (coupler\ angle), \Psi\ (output\ angle)$
5. Pivots: O, A, B, C
6. Vector coordinates of points



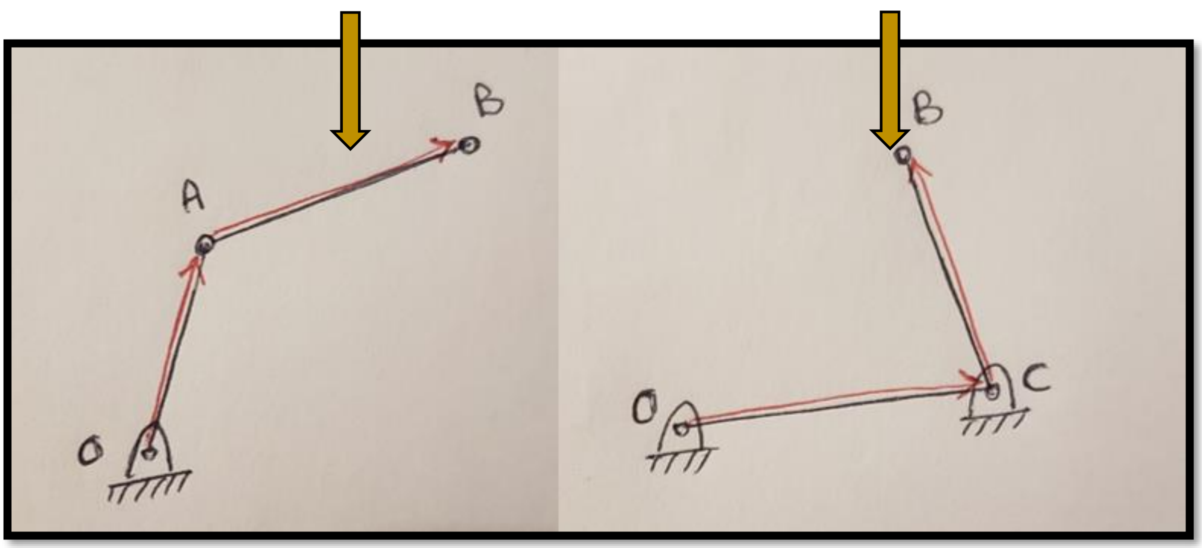


$$\bar{O} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

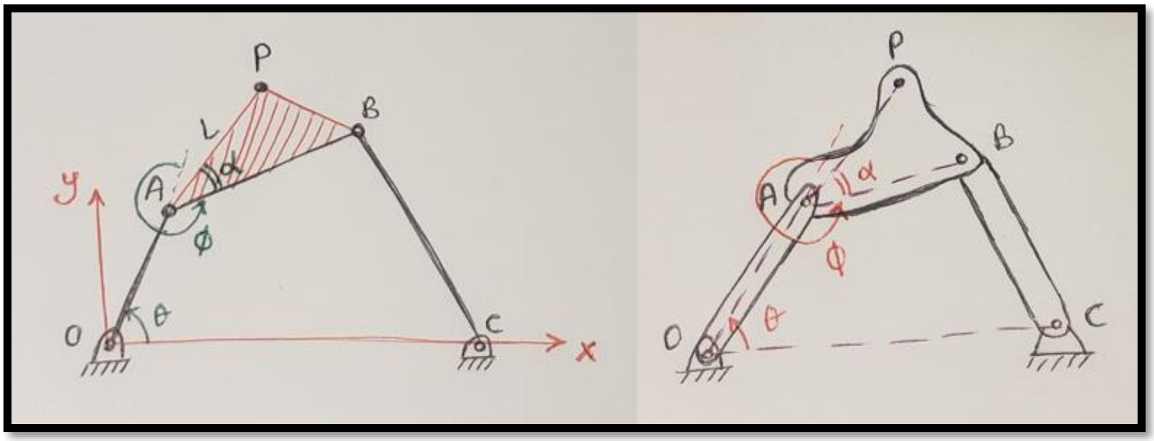
$$\bar{A} = \begin{Bmatrix} a\ cos\theta \\ a\ sin\theta \end{Bmatrix}$$

$$\bar{C} = \begin{Bmatrix} g \\ 0 \end{Bmatrix}$$

$$\bar{B} = \overline{OA} + \overline{AB} = \begin{Bmatrix} a\ cos\theta \\ a\ sin\theta \end{Bmatrix} + \begin{Bmatrix} h\cos(\theta + \emptyset) \\ h\sin(\theta + \emptyset) \end{Bmatrix} \quad \textbf{or} \quad \bar{B} = \overline{OC} + \overline{CB} = \begin{Bmatrix} g \\ 0 \end{Bmatrix} + \begin{Bmatrix} b\ \cos(\Psi) \\ b\ \sin(\Psi) \end{Bmatrix}$$



What if we want any point on the coupler links?



$$\bar{P} = \begin{Bmatrix} a\cos\theta + l\cos(\theta + \emptyset + \alpha) \\ a\sin\theta + l\sin(\theta + \emptyset + \alpha) \end{Bmatrix}$$

Watch the video for 4-bar linkage and motion of point "p".
(https://www.mekanizmalar.com/four-bar-infinity-curve.html)

$a,\ l,\ \alpha$: Fixed parameters (constant values)

θ : Input variable (we control that)
$\emptyset$: Variable which is related to θ in a nonlinear way and we need to find $\emptyset\ (\theta)$
Ψ : Variable which is related to θ in a nonlinear way and we need to find $\Psi\ (\theta)$

- How do we know $\emptyset$ & Ψ are dependent variables?

Because the mobility equals "1" (M=1), that means we just need to control one variable for the 4-bar linkage!

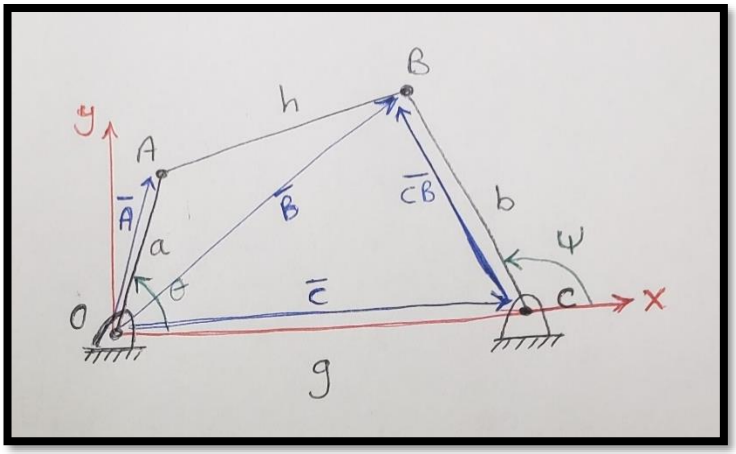
Two Ways of Finding Angular Relations:

1) Distance Constraints: The distance between two points in a link is fixed.

$$\overline{AB} = \bar{B} - \bar{A}$$

$$(\bar{B} - \bar{A}) \cdot (\bar{B} - \bar{A}) = h^2 \quad \text{(Length square)}$$

Dot product (yields a **scalar**)



$$\Bigg(\underbrace{\begin{Bmatrix} g + b\cos\Psi \\ b\sin\Psi \end{Bmatrix}}_{\bar{\boldsymbol{B}}} - \underbrace{\begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix}}_{\bar{\boldsymbol{A}}}\Bigg) \cdot \Bigg(\begin{Bmatrix} g + b\cos\Psi \\ b\sin\Psi \end{Bmatrix} - \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix}\Bigg) = h^2$$

$$(g + b\cos\Psi - a\cos\theta)^2 + (b\sin\Psi - a\sin\theta)^2 = h^2$$

Except for Ψ, everything else in the above equation is known, so we can find the relation between Ψ *and* θ (it is complicated). You can solve it numerically (e.g. Matlab) or with computer algebra (e.g. Maple/ Mathematica) or solve it by hand.

$g^2 + b^2\cos^2\Psi + a^2\cos^2\theta + 2gb\cos\Psi - 2ga\cos\theta - 2ab\cos\Psi\cos\theta +$

$b^2\sin^2\Psi + a^2\sin^2\theta - 2ab\sin\Psi\sin\theta = h^2$

After simplification and collection by Ψ:

$$\underbrace{\big(2b(g - a\cos\theta)\big)}_{\boldsymbol{A(\theta)}}\cos\Psi + \underbrace{(-2ab\sin\theta)}_{\boldsymbol{B(\theta)}}\sin\Psi = \underbrace{2ag\cos\theta + h^2 - g^2 - b^2 - a^2}_{\boldsymbol{C(\theta)}}$$

$$A(\theta)\cos\Psi + B(\theta)\sin\Psi = C(\theta)$$

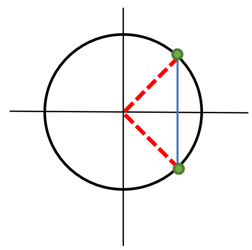
Divide by $\sqrt{A^2(\theta) + B^2(\theta)}$:

$$\underbrace{\frac{A(\theta)}{\sqrt{A^2(\theta) + B^2(\theta)}}}_{\boldsymbol{\cos\delta}}\cos\Psi + \underbrace{\frac{B(\theta)}{\sqrt{A^2(\theta) + B^2(\theta)}}}_{\boldsymbol{\sin\delta}}\sin\Psi = \frac{C(\theta)}{\sqrt{A^2(\theta) + B^2(\theta)}}$$

$$\cos\delta\ \cos\Psi + \sin\delta\ \sin\Psi = \frac{C(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}$$

$$\cos(\Psi-\delta) = \frac{C(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}$$

$$(\Psi-\delta) = \pm \arccos\left(\frac{C(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}\right)$$ (**arccos** has two solutions!)



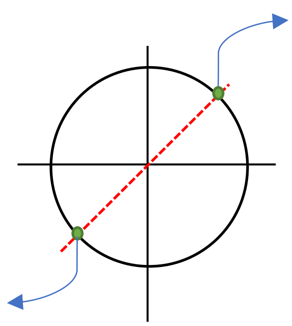
$$\tan\delta = \frac{\dfrac{B(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}}{\dfrac{A(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}} = \frac{B(\theta)}{A(\theta)}$$

$$\delta = \arctan\left(\frac{B(\theta)}{A(\theta)}\right)$$

$$\boxed{\boldsymbol{\Psi = \arctan\left(\frac{B(\theta)}{A(\theta)}\right) \pm \arccos\left(\frac{C(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}\right)}}$$

Note: For the arctan, use the double-value function or keep track of the signs.

$$\arctan\left(\frac{+B(\theta)}{+A(\theta)}\right) = \arctan\left(\frac{B(\theta)}{A(\theta)}\right)$$

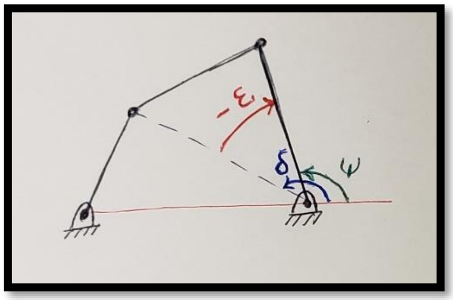


$$\arctan\left(\frac{-B(\theta)}{-A(\theta)}\right) = \arctan\left(\frac{B(\theta)}{A(\theta)}\right)$$

Or we can write it: $\Psi = \delta \pm \varepsilon$, $\varepsilon = arccos\left(\frac{C(\theta)}{\sqrt{A^2(\theta)+B^2(\theta)}}\right)$

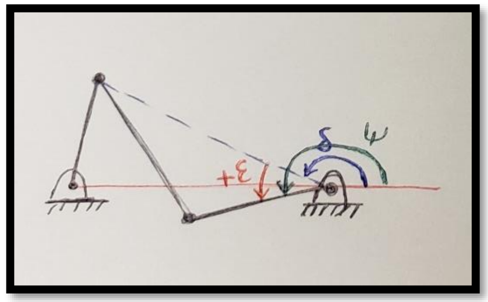
In the above equation, we express $\boldsymbol{\Psi}$ as a summation of two angles. What are these two angles? There are two solutions for Ψ for every value of θ.

$-\boldsymbol{\varepsilon}$: because CW



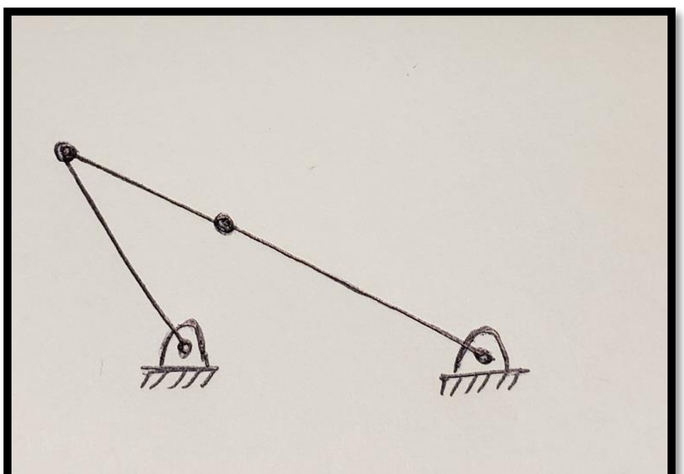
Solution 1

$+\boldsymbol{\varepsilon}$: because CCW



Solution 2

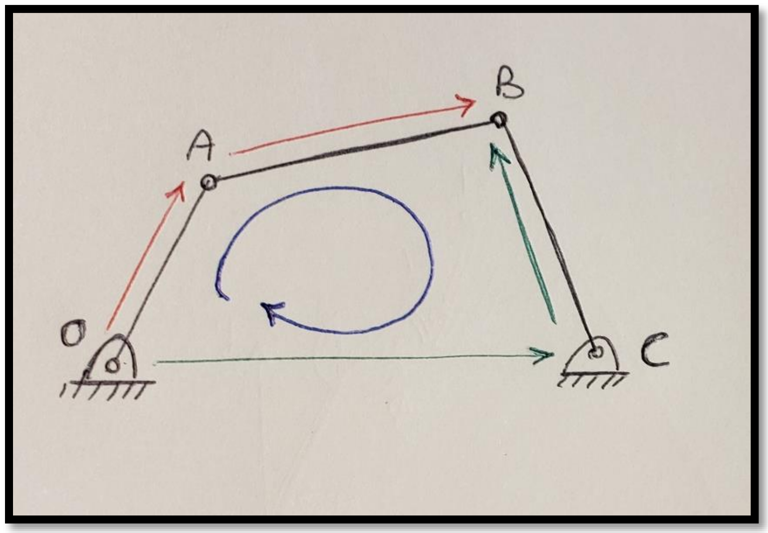
- Just in the following case (ε=0), when the coupler link and output link are aligned, we will have a single solution. This solution is not desired because in this case mechanism will be locked and not move anymore. In kinematic, this situation is named singularity.



- We have now $\boldsymbol{\Psi}(\boldsymbol{\theta})$ (2 Solutions)
- We need $\emptyset(\boldsymbol{\theta})$

Linkage Analysis (Continue)

2) **Loop Equations:** We can express the vector coordinates of points using different paths along the linkage (we can create closed loops)



$$\bar{B} = \overline{OA} + \overline{AB}$$
$$\bar{B} = \overline{OC} + \overline{CB}$$

Loop:

Zero Vector

$$\overline{OA} + \overline{AB} + \overline{BC} + \overline{CO} = \bar{\emptyset}$$

$$\overline{OA} + \overline{AB} = \overline{OC} + \overline{CB} \qquad \text{Loop Equation}$$

$$\begin{Bmatrix} a\cos\theta + h\,cos\,(\theta + \emptyset) \\ a\sin\theta + h\,sin\,(\theta + \emptyset) \end{Bmatrix} = \begin{Bmatrix} g + b\,cos\,(\Psi) \\ b\,sin\,(\Psi) \end{Bmatrix}$$

We know the Ψ from previous calculations and θ is the input (known). Now, we solve for $\emptyset$!

Isolate $cos\,(\theta + \emptyset)$, $sin\,(\theta + \emptyset)$ in the equation and divide to create $tan\,(\theta + \emptyset)$:

$$\frac{\cancel{h}\,sin\,(\theta + \emptyset)}{\cancel{h}\,cos\,(\theta + \emptyset)} = \frac{b\,sin\,(\Psi) - a\sin\theta}{g + b\,cos\,(\Psi) - a\cos\theta}$$

- h is the length of the link, so it is always positive.

$$tan\,(\theta + \emptyset) = \frac{b\ sin\,(\Psi) - a\sin\theta}{g + b\ cos\,(\Psi) - a\cos\theta}$$

$$\boxed{\emptyset = arctan\left(\frac{b\ sin\,(\Psi) - a\sin\theta}{g + b\ cos\,(\Psi) - a\cos\theta}\right) - \theta}$$

- Note 1: Double-value arctan in this equation
- Note 2: Use the same units for arctan and θ (degrees or radians for both of them)

If we know:

- Input value θ
- Links lengths
- $\Psi(\theta)$ and $\emptyset(\theta)$

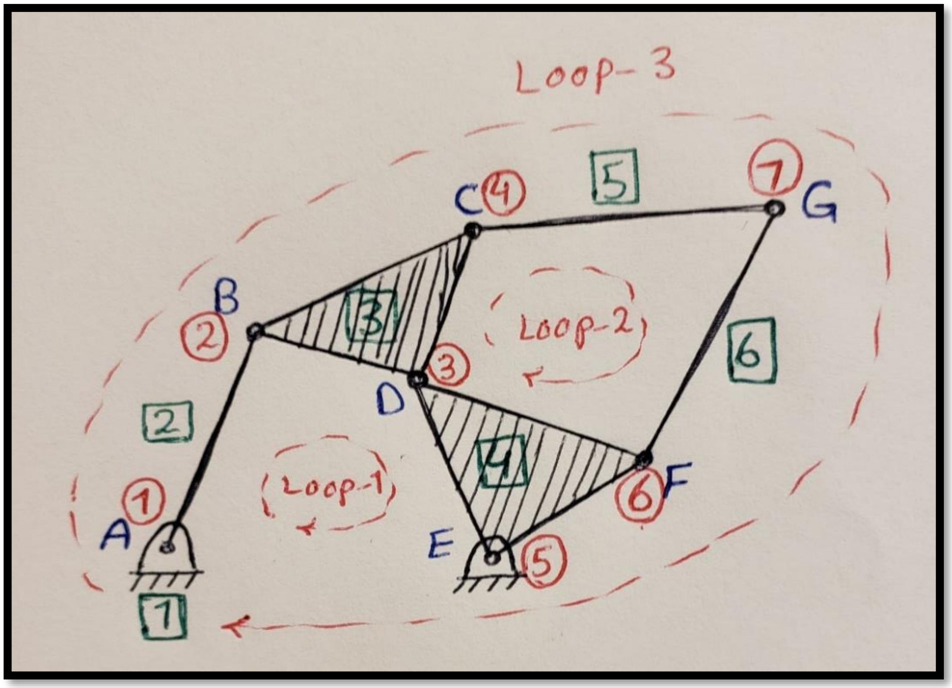
We can fully determine the position of any point in the 4-bar linkage.

- Independent loops: For most of the planar kinematic chains the independent loops can be determined by visual inspection and are arbitrarily selected. Each independent loop must contain at least one link or joint that other independent loops do not contain and each link and/or joint must appear in at least one of the independent loops.

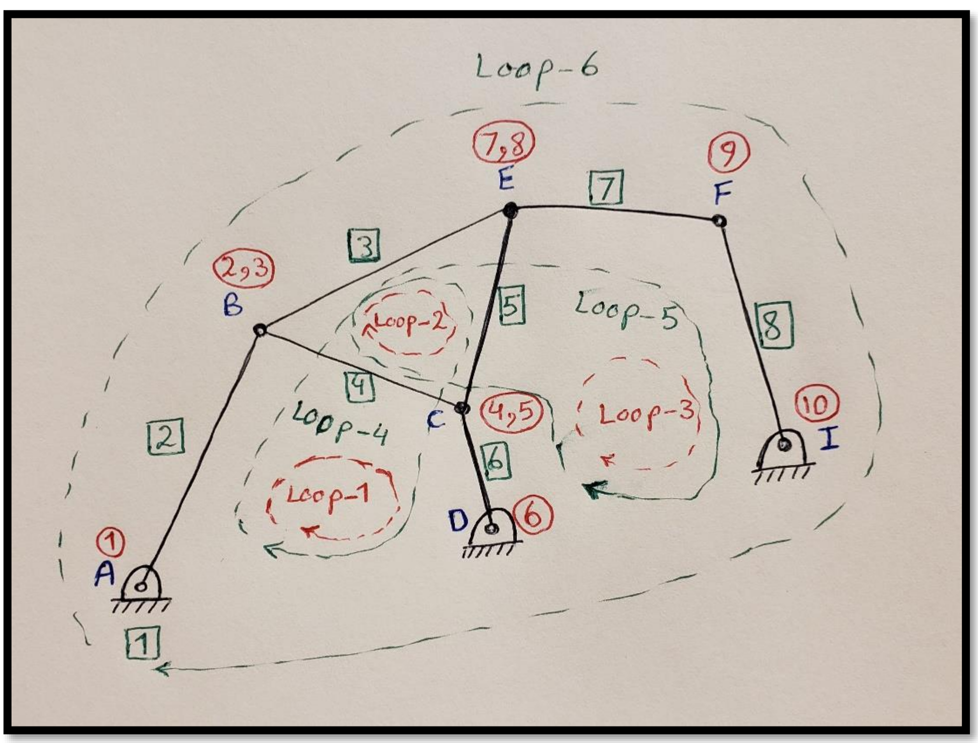
- However, we can use the following equation to check if we found the correct number of independent loops or not!

$$L = j - n + 1$$

Example: Find the number of independent loops for the following mechanisms.



$n = 6\ ,\ j = 7 \qquad L = j - n + 1 = 7 - 6 + 1 = 2$ Independent loops



$n = 8\ ,\ j = 10 \qquad L = j - n + 1 = 10 - 8 + 1 = 3$ Independent loops

Classification of 4-bar Linkages

(Based on whether the link fully rotates or rock (back and forth motion))

Usually, for designing a mechanism, for instance, a 4-bar linkage, we want to know about the amount of rotation of each link. That is important for us because usually, we want the input link can able to fully rotate (because of attaching motor) while the output link can be designed based on the task to do a different range of rotation.

Let's take a look at the following video and see how a pump jack (4-bar linkage) works.
https://www.youtube.com/watch?v=Gny3hqxBqCk

Grashof Classification

- For Grashof classification, we need to know the length of the links.
- s: Smallest link
- l: Largest link
- p and q: They are two middle-size links

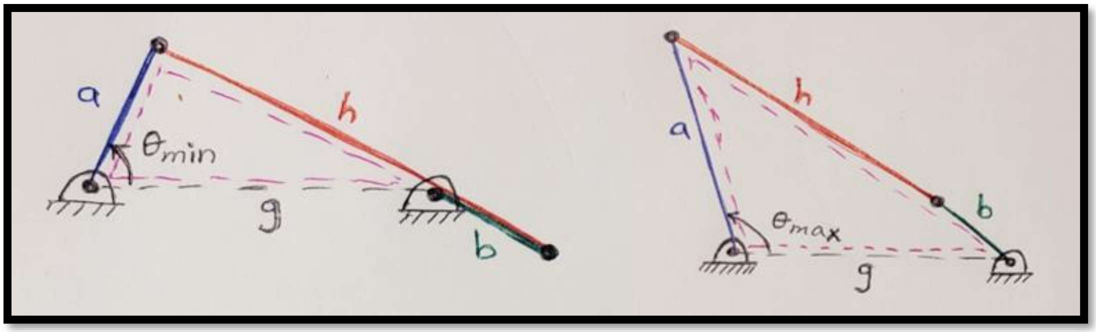
1) One link can fully rotate (Grashof or Grashof-1 linkage)	$s + l < p + q$
2) No link can fully rotate (Non-Grashof or Grashof-2 linkage)	$s + l > p + q$
3) Foldable linkage	$s + l = p + q$

- In case #1, the shortest link (s) will fully rotate!

Limit Angles

After knowing about one link can fully rotate or not, still we need to know more information about the limit angle (how much the other one rocks or can fully rotate).

Graphically: Limits for θ (input angle)



We can use the cos law for the above triangles:

$$(h-b)^2 = a^2 + g^2 - 2ag\,cos\theta_{min}$$

$$\boxed{cos\theta_{min} = \frac{-(h-b)^2 + a^2 + g^2}{2ag}}$$

$$(h+b)^2 = a^2 + g^2 - 2ag\,cos\theta_{max}$$

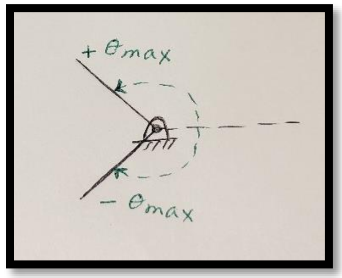
$$\boxed{cos\theta_{max} = \frac{a^2 + g^2 - (h+b)^2}{2ag}}$$

Linkage Analysis (Continue)

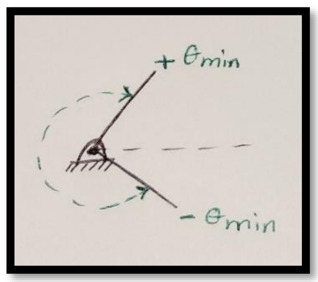
Limit Angles

Four different cases based on $\boldsymbol{\theta_{min}}$ & $\boldsymbol{\theta_{max}}$:

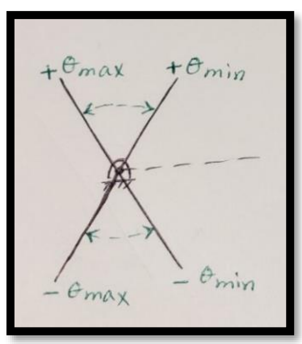
1) If θ_{min} is a complex number ($cos\theta_{min} > 1 \; or \; cos\theta_{min} < -1$), that means there is no θ_{min}! (Input link will go through $\theta = 0$)



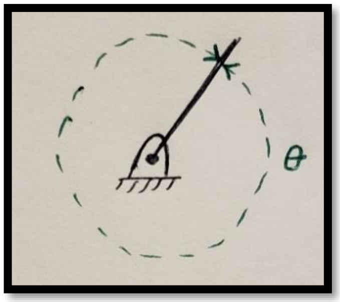
2) If θ_{max} is a complex number ($cos\theta_{max} > 1 \; or \; cos\theta_{max} < -1$), that means there is no θ_{max}! (Input link will go through $\theta = \pi$)



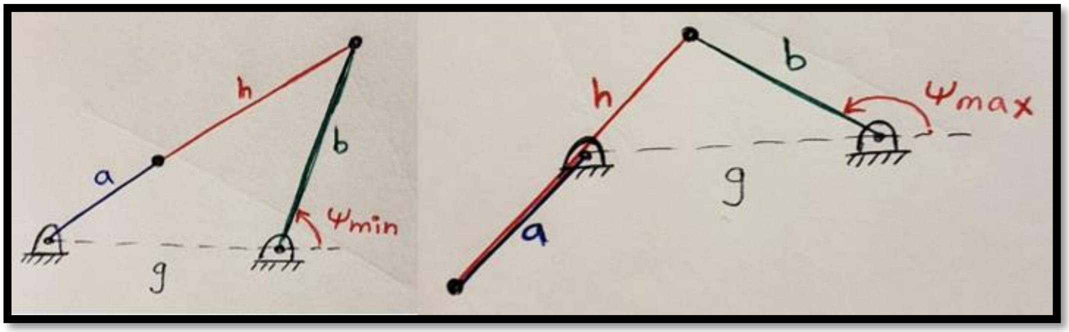
3) If you have both θ_{min} & θ_{max} then your input link is not passing zero or π! (You cannot go from positive to the negative range and this type has **two assembly modes**)



4) If you don't have θ_{min} & θ_{max} then your input link is a crank and it can fully rotate (crank).



We can do the same classification for Ψ (output angle)



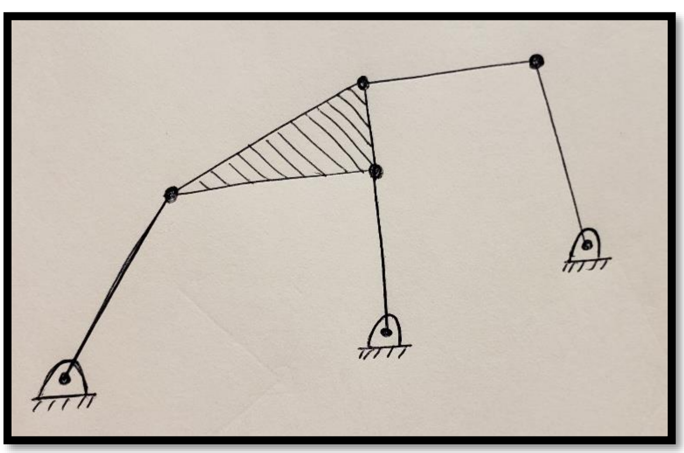
In this case, we can use the cos law same as before and just use $(\pi - \Psi)$ for the internal angle of triangles.
Exercise: Drive $cos(\Psi_{\min})$ & $cos(\Psi_{max})$

Full classification of a 4-bar linkage

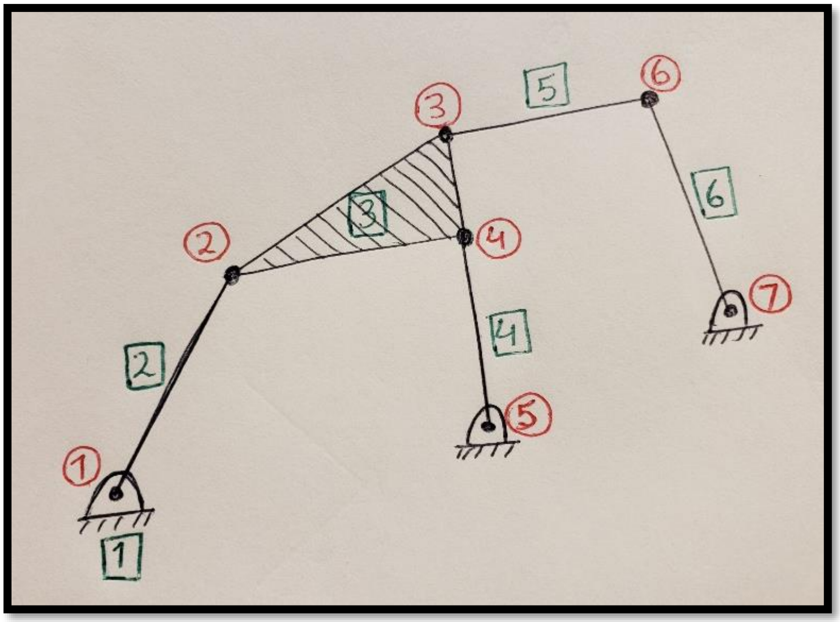
Grashof type	θ_{min}	θ_{max}	$\Psi_{\min}$	Ψ_{max}	Linkage	Description
(I) (Two assemblies)	∄	∄	∄	∄	Double Crank	Input & output fully rotate
	∄	∄	∃	∃	Crank-Rocker	Input fully rotate
	∃	∃	∄	∄	Rocker-Crank	Output fully rotate
	∃	∃	∃	∃	Double Rocker	None of the input or output fully rotate
(II) (Only one assembly)	∄	∃	∄	∃	Rocker (0)-Rocker (0)	Input & output Only pass (0)
	∄	∃	∃	∄	Rocker (0)-Rocker (π)	Input pass (0) Output pass (π)
	∃	∄	∄	∃	Rocker (π)-Rocker (0)	Input pass (π) Output pass (0)
	∃	∄	∃	∄	Rocker (π)-Rocker (π)	Input & output only pass (π)

- ∄: *Does not exist*
- There is a software named GIM for linkage analysis, you can download it for free and you use it to model the linkages that you design to see how they work.

Example: Position analysis for the following six-bar linkages



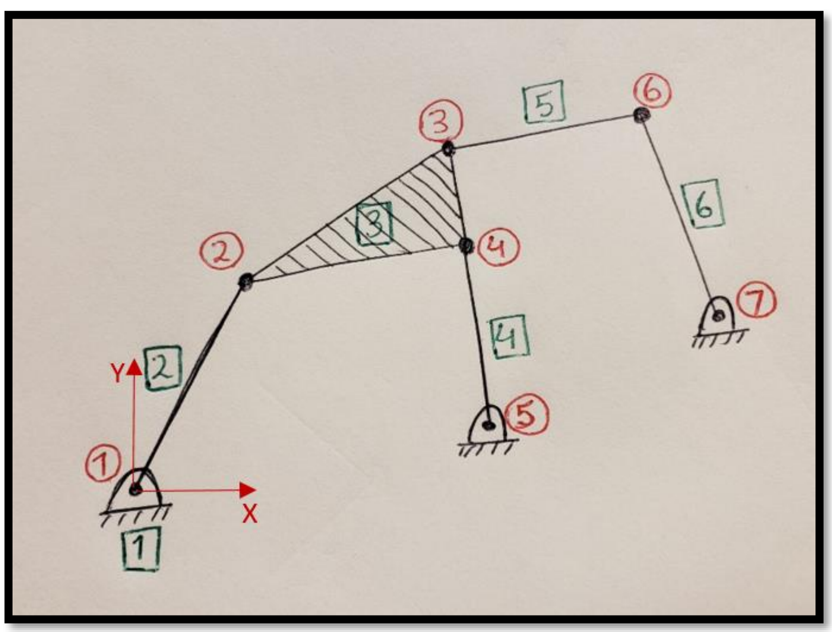
A) Mobility



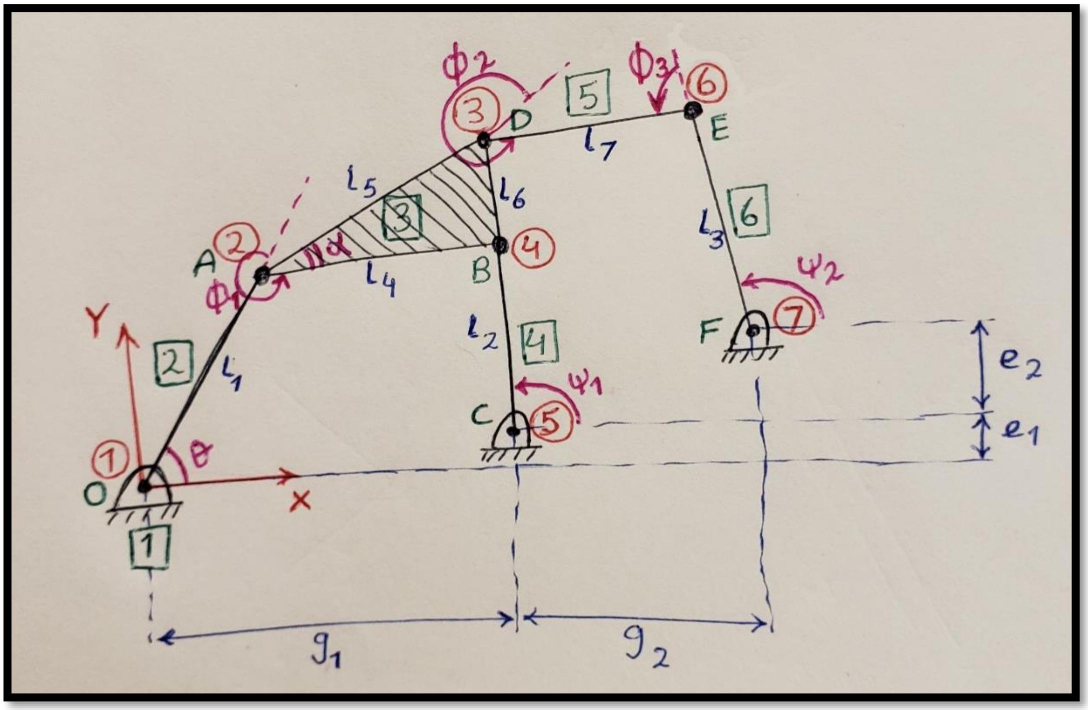
$$n = 6 \quad , \quad j = 7 \quad , \quad f_i = 1$$

$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (6-1) - 7 \times (3-1) = 15 - 14 = 1$$

B) Set our reference frame: Because we have 3 pivots on the ground, we won't be able to align them on a line, so we set up the reference frame as a general from one of the pivots.



C) Label the mechanism



D) Write the coordinate of points

$$\bar{A} = \begin{Bmatrix} l_1\, cos\theta \\ l_1\, sin\theta \end{Bmatrix}$$

$$\overline{D} = \overline{OA} + \overline{AD} \;\; or\; \overline{OC} + \overline{CF} + \overline{FE} + \overline{ED}$$

$$\overline{D} = \begin{Bmatrix} g_1 + g_2 + l_3\cos(\Psi_2) + l_7\cos(\Psi_2 + \emptyset_3) \\ e_1 + e_2 + l_3\sin(\Psi_2) + l_7\sin(\Psi_2 + \emptyset_3) \end{Bmatrix}$$

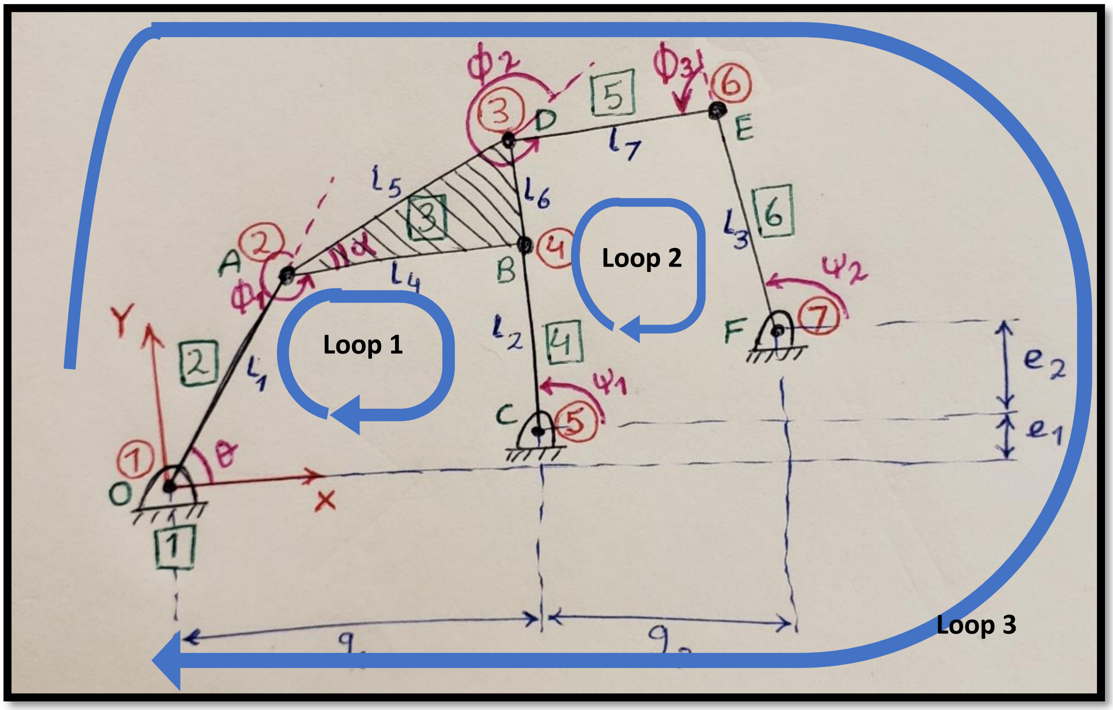
.
.
.

E) Find angular relation: because mobility is equal to one, so all angles are a function of θ.

$$\Psi_1(\theta),\;\; \Psi_2(\theta),\;\; \emptyset_1(\theta), ...$$

Use loop equations

1) We have to find identify the loops (independent)



2) We have 3 loops and only 2 independent loops. We will choose the loops that we have information about them. In this case, we don't know the angle between l_2 & l_6, so it would be better to use loop 1 and loop 3 because all angles are labeled for them.

Loop 1:

$$\overline{OA} + \overline{AB} = \overline{OC} + \overline{CB}$$

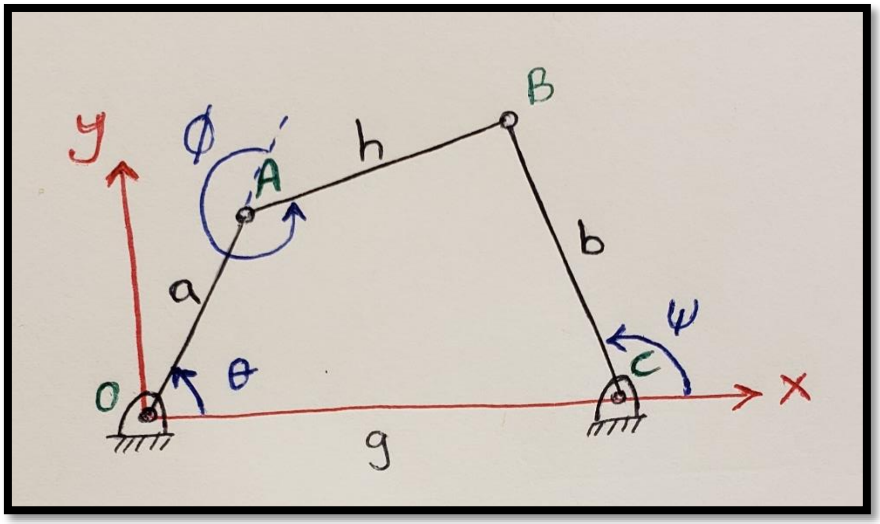
Loop 3:

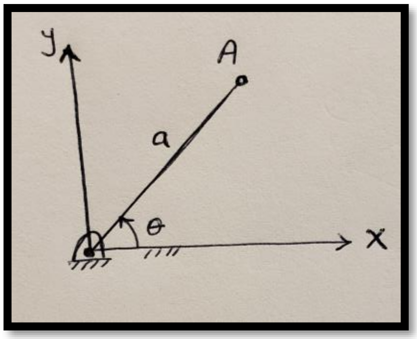
$$\overline{OA} + \overline{AD} + \overline{DE} = \overline{OC} + \overline{CF} + \overline{FE}$$

You will end up with 2 vector equations which are four equations in total and you have four unknowns, so you can find the angles.

Velocity Analysis

Velocity: Time derivative of position.





$\bar{A} = \begin{Bmatrix} a\ cos\theta \\ a\ sin\theta \end{Bmatrix}$

$\dot{\bar{A}} = \frac{d\bar{A}}{dt} = \frac{d\bar{A}}{d\theta} \times \frac{d\theta}{dt} = \begin{Bmatrix} -a\ sin\theta\ \dot{\theta} \\ a\ cos\theta\ \dot{\theta} \end{Bmatrix}$

Chain rule

$\frac{d\theta}{dt} = \dot{\theta} = Angular\ velocity\ of\ input\ link\ (given\ variable)$

$\bar{B} = \begin{Bmatrix} a\ cos\theta + h\ cos(\theta + \emptyset) \\ a\ sin\theta + h\ sin(\theta + \emptyset) \end{Bmatrix}$ or $\bar{B} = \begin{Bmatrix} g + b\ cos\Psi \\ b\ sin\Psi \end{Bmatrix}$

$\dot{\bar{B}} = \begin{Bmatrix} -a\ sin\theta\dot{\theta} - h\sin(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \\ a\ cos\theta\dot{\theta} + h\cos(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \end{Bmatrix}$ or $\dot{\bar{B}} = \begin{Bmatrix} -b\ sin\Psi\dot{\Psi} \\ b\ cos\Psi\dot{\Psi} \end{Bmatrix}$

$\bar{C} = \begin{Bmatrix} g \\ 0 \end{Bmatrix}$ $\dot{\bar{C}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

What is $\dot{\theta}$? The angular velocity of the **input link**.

Note: points don't have angular velocity!

What is $\dot{\emptyset}$? The angular velocity of **coupler link** with respect to **input link**.

What is $\dot{\theta} + \dot{\emptyset}$? The **absolute** angular velocity of the **coupler link** with respect to the **fixed frame**.

Note: In the planar case because all the axes of rotation are parallel (perpendicular to the plane) you can add the angular velocities.

The D.O.F. for 4-bar linkage is 1, in the above equations we just have $\dot{\theta}$. We don't know $\dot{\emptyset}$ and $\dot{\Psi}$ (given by mechanical constraints).

To compute $\dot{\emptyset}$, $\dot{\Psi}$ we will take derivatives in the loop equation.

We only have one loop in a 4-bar linkage: OA+AB=OC+CB

$$\begin{Bmatrix} a\ cos\theta + h\ cos(\theta + \emptyset) \\ a\ sin\theta + h\ sin(\theta + \emptyset) \end{Bmatrix} = \begin{Bmatrix} g + b\ cos\Psi \\ b\ sin\Psi \end{Bmatrix}$$

$$\begin{Bmatrix} -a\ sin\theta\dot{\theta} - h\sin(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \\ a\ cos\theta\dot{\theta} + h\cos(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \end{Bmatrix} = \begin{Bmatrix} -b\ sin\Psi\dot{\Psi} \\ b\ cos\Psi\dot{\Psi} \end{Bmatrix}$$

Two equations, two unknowns!

Unknowns: $\dot{\Psi}$, $\dot{\emptyset}$

Knowns: $a, b, h, \theta, \dot{\theta}, \emptyset, \Psi$

2 linear equations in 2 unknowns: Linear system

$$\begin{bmatrix} -h\sin(\theta + \emptyset) & b\ sin\Psi \\ h\cos(\theta + \emptyset) & -b\ cos\Psi \end{bmatrix} \begin{Bmatrix} \dot{\emptyset} \\ \dot{\Psi} \end{Bmatrix} = \begin{Bmatrix} (a\ sin\theta + h\sin(\theta + \emptyset))\ \dot{\theta} \\ (-a\ cos\theta - h\ cos(\theta + \emptyset))\ \dot{\theta} \end{Bmatrix}$$

From linear algebra, there are several methods that you can solve this linear system such as using *Cramer's rule.*

Note: Example for *Cramer's rule:* If we have a simple linear system:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

We can write it in matrix format:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Assume a1b2 − b1a2 nonzero. Then, with help of determinants, x and y can be found with Cramer's rule as:

$$\boxed{x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1 b_2 - b_1 c_2}{a_1 b_2 - b_1 a_2} \quad , \; y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1 c_2 - c_1 a_2}{a_1 b_2 - b_1 a_2}}$$

After solving these equations, the $\dot{\Psi}$and $\dot{\emptyset}$ will be found as follow:

$$\dot{\emptyset} = \left(\frac{-a\,\sin(\Psi-\theta)}{h\,\sin(\Psi-\theta-\emptyset)} - 1\right)\dot{\theta} \qquad \dot{\Psi} = \left(\frac{a\,\sin\emptyset}{b\,\sin(\theta+\emptyset-\Psi)}\right)\dot{\theta}$$

Acceleration Analysis

Point Acceleration: Take second derivatives on point positions.

$$\dot{\bar{A}} = \begin{Bmatrix} -a\,sin\theta\dot{\theta} \\ a\,cos\theta\dot{\theta} \end{Bmatrix} \Rightarrow \ddot{\bar{A}} = \begin{Bmatrix} -a\,cos\theta\dot{\theta}^2 - a\,sin\theta\,\ddot{\theta} \\ -a\,sin\theta\dot{\theta}^2 + a\,cos\theta\,\ddot{\theta} \end{Bmatrix}$$

$\ddot{\theta}$: Input angular acceleration(given)

$$\dot{\bar{B}} = \begin{Bmatrix} -b\,sin\Psi\dot{\Psi} \\ b\,cos\Psi\dot{\Psi} \end{Bmatrix} \Rightarrow \ddot{\bar{B}} = \begin{Bmatrix} -b\,cos\Psi\dot{\Psi}^2 - b\,sin\Psi\,\ddot{\Psi} \\ -b\,sin\Psi\dot{\Psi}^2 + b\,cos\Psi\,\ddot{\Psi} \end{Bmatrix}$$

$$\dot{\bar{C}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \ddot{\bar{C}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We need to calculate $\ddot{\Psi}$ and $\ddot{\emptyset}$ as a function of θ, $\dot{\theta}$, Ψ, $\dot{\Psi}$,$\emptyset$, $\dot{\emptyset}$. So, we have to take second derivatives in the loop equation and solve linearly for $\ddot{\Psi}$ and $\ddot{\emptyset}$.

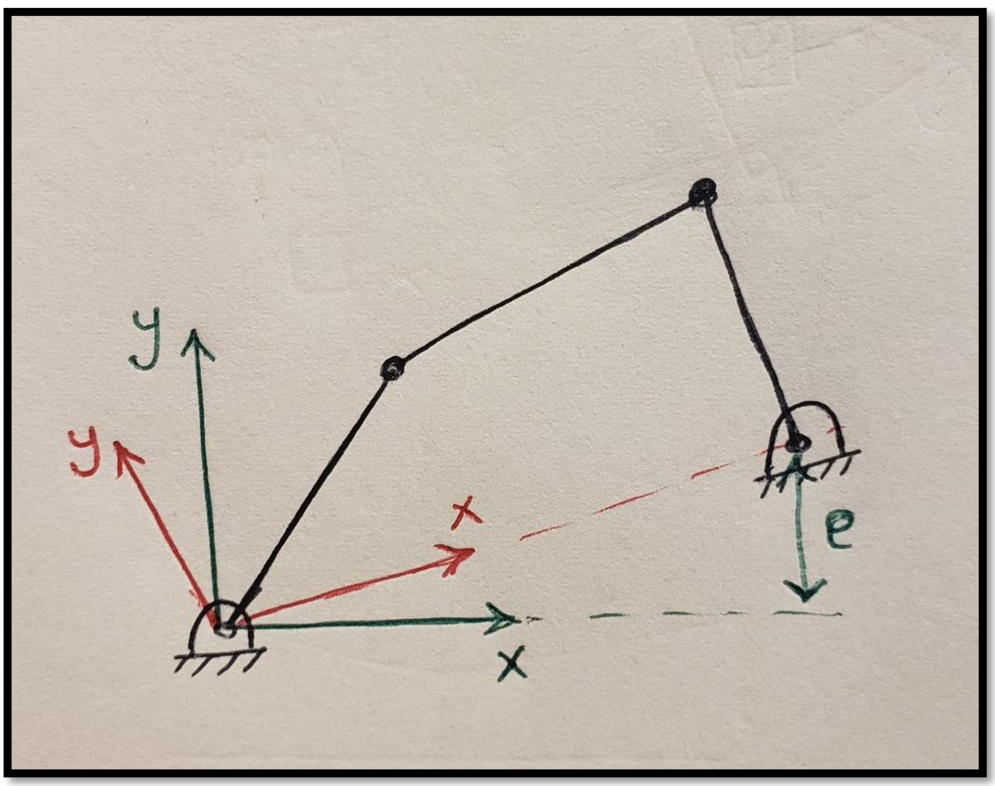
$$[big\ expression\ here]\begin{Bmatrix}\ddot{\emptyset}\\ \ddot{\Psi}\end{Bmatrix} = \{big\ expression\ here\}$$

-
-
-

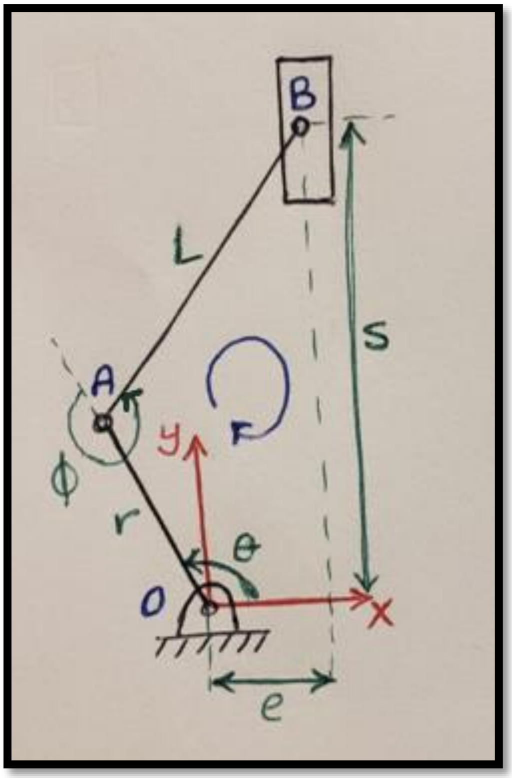
You can solve it in Maple/MATLAB to find $\ddot{\Psi}$ and $\ddot{\emptyset}$.

- Question: These equations for velocity and acceleration are consistent for all 4-bar linkages?

It depends on the reference frame that you choose. If your reference frame passes through input and output pivots, that would be the same for all 4-bar linkage. But if your reference frame for any reason does not pass through both of them, then $\emptyset$ and Ψ would be different (there is another parameter "**e**" in equations) and that will affect other dependent parameters ($\dot{\Psi}$, $\dot{\emptyset}$) as well.



Position Analysis Slider-Crank Mechanism



X-axis: perpendicular to sliding direction

θ : Input angle

$\emptyset$: Coupler angle

s: Slide

Variables

(We don't have output angle in this case)

Watch this video: https://www.youtube.com/watch?v=ZO8QEG4x0wY

Mobility:

$$n = 4 \quad , \quad j = 4 \quad , \quad f_i = 1 \text{ (All joints)}$$

$$M = 3 \times (4-1) - 4 \times (3-1) = 9 - 8 = 1$$

Mobility equals one, so we have one input variable (θ) and have to define two other variables (s and $\emptyset$) as a function of θ.
Find the position vectors for points:

$$\overline{\mathrm{A}} = \begin{Bmatrix} r\ cos\theta \\ r\ sin\theta \end{Bmatrix}$$

$$\overline{\mathrm{B}} = \begin{Bmatrix} r\ cos\theta + L\cos(\theta+\emptyset) \\ r\ sin\theta + L\sin(\theta+\emptyset) \end{Bmatrix} \quad \text{or} \quad \overline{\mathrm{B}} = \begin{Bmatrix} e \\ s \end{Bmatrix}$$

There is only one loop.

Loop equation: $\begin{Bmatrix} r\ cos\theta + L\cos(\theta+\emptyset) \\ r\ sin\theta + L\sin(\theta+\emptyset) \end{Bmatrix} = \begin{Bmatrix} e \\ s \end{Bmatrix}$

Solve for "s" and "$\emptyset$" using loop equation:

Step 1: Take $r\ cos\theta$ and $r\ sin\theta$ to the other side of the equation.

$$L\cos(\theta+\emptyset) = e - r\ cos\theta$$

$$L\sin(\theta+\emptyset) = s - r\ sin\theta$$

Step 2: Square both sides of the equations

$$(L\cos(\theta+\emptyset))^2 = (e - r\ cos\theta)^2$$

$$(L\sin(\theta+\emptyset))^2 = (s - r\ sin\theta)^2$$

Step3: Add the equations

$$(L\cos(\theta+\emptyset))^2 = (e - r\ cos\theta)^2$$

$$+ \quad (L\sin(\theta+\emptyset))^2 = (s - r\ sin\theta)^2$$

$$L^2\underbrace{\left(\cos^2(\theta+\emptyset) + \sin^2(\theta+\emptyset)\right)}_{1} = e^2 - 2er\ cos\theta + r^2cos^2\theta + s^2 - 2sr\ sin\theta + r^2sin^2\theta$$

$$r^2$$

$$L^2 = e^2 + r^2 - 2er\ cos\theta - 2sr\ sin\theta + s^2$$

$$s^2 - 2sr\ sin\theta + e^2 + r^2 - L^2 - 2er\ cos\theta = 0$$

This is a quadratic equation in the form of $as^2 + bs + c = 0$ with:

$a = 1$

$b = -2r\ sin\theta$

$c = e^2 + r^2 - L^2 - 2er\ cos\theta$

$$s = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{\cancel{2}r\ sin\theta\ \mp \sqrt{\cancel{4}r^2 sin^2\theta\ - \cancel{4}(e^2 + r^2 - L^2 - 2er\ cos\theta)}}{\cancel{2}}$$

$$s = r\ sin\theta\ \mp \sqrt{\underbrace{r^2 sin^2\theta}\ - e^2 \underbrace{- r^2} + L^2 + 2er\ cos\theta}$$

$$-r^2 cos^2\theta$$

$$s = r\ sin\theta\ \mp \sqrt{\underbrace{-(e^2 + r^2 cos^2\theta - 2er\ cos\theta) + L^2}_{(e - rcos\theta)^2}}$$

$$\boxed{s = \mathrm{r}\ \sin\theta \pm \sqrt{L^2 - (e - r\ cos\theta)^2}}$$

Once "s" is known, we can solve for $\emptyset$.

Divide two parts of the loop equation.

L is the length (always positive), so we can cancel it.

$$\cancel{L} \sin(\theta + \emptyset) = s - r\ sin\theta$$

$$\div$$

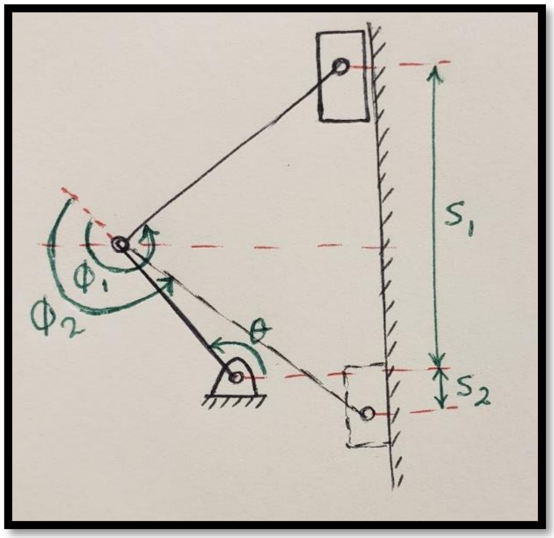
$$\cancel{L} \cos(\theta + \emptyset) = e - r\ cos\theta$$

$$\tan(\theta + \emptyset) = \frac{s - r\ sin\theta}{e - r\ cos\theta}$$

$$\boxed{\emptyset = arctan\left(\frac{s - r\ sin\theta}{e - r\ cos\theta}\right) - \theta}$$

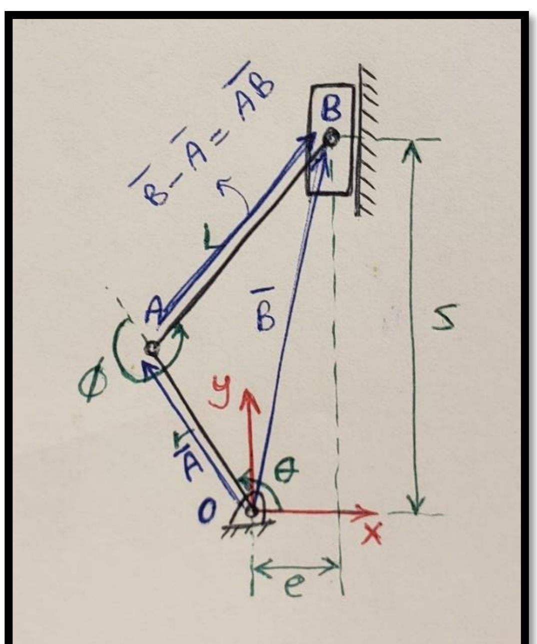
Note: use the same unit for $arctan\left(\frac{s-r\,sin\theta}{e-r\,cos\theta}\right)$ and θ (radian or degree)

Note: If you keep tracking sin and cos, then $arctan$ will have a single value (you don't need $\pm$)



Note: There are two values for ∅ (one for each "s" value)

Solve for "s" and "∅" using distance constraint



- Distance L is fixed

$$(\bar{B} - \bar{A}) \cdot (\bar{B} - \bar{A}) = L^2$$

$$\begin{Bmatrix} e - r\ cos\theta \\ s - r\ sin\theta \end{Bmatrix} \cdot \begin{Bmatrix} e - r\ cos\theta \\ s - r\ sin\theta \end{Bmatrix} = L^2$$

$$(e - r\ cos\theta)^2 + (s - r\ sin\theta)^2 = L^2$$

Rather than a second-degree equation, we find "s" from the following method:

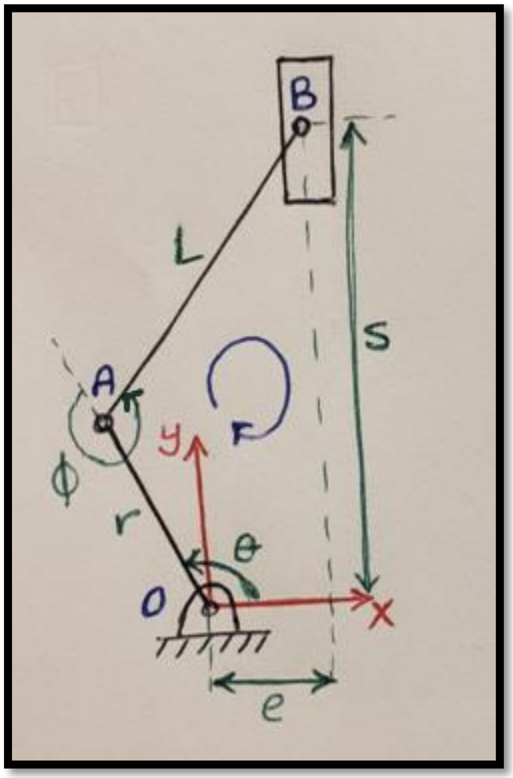
$$(s - r\ sin\theta)^2 = L^2 - (e - r\ cos\theta)^2$$

$$s - r\ sin\theta = \pm\sqrt{L^2 - (e - r\ cos\theta)^2}$$

$$\boxed{s = r\ sin\theta \pm \sqrt{L^2 - (e - r\ cos\theta)^2}}$$

- For the $\emptyset$, we can use the loop equation method.

Position Analysis Slider-Crank Mechanism



After finding the equation for "s" and "∅", now we have to find the limits of this mechanism. In this part, we have to find the sliding range "s" and check if we have a limit angle for θ or if it can able to fully rotate.

$$s = r\ sin\theta \pm \sqrt{L^2 - (e - r\ cos\theta)^2}$$

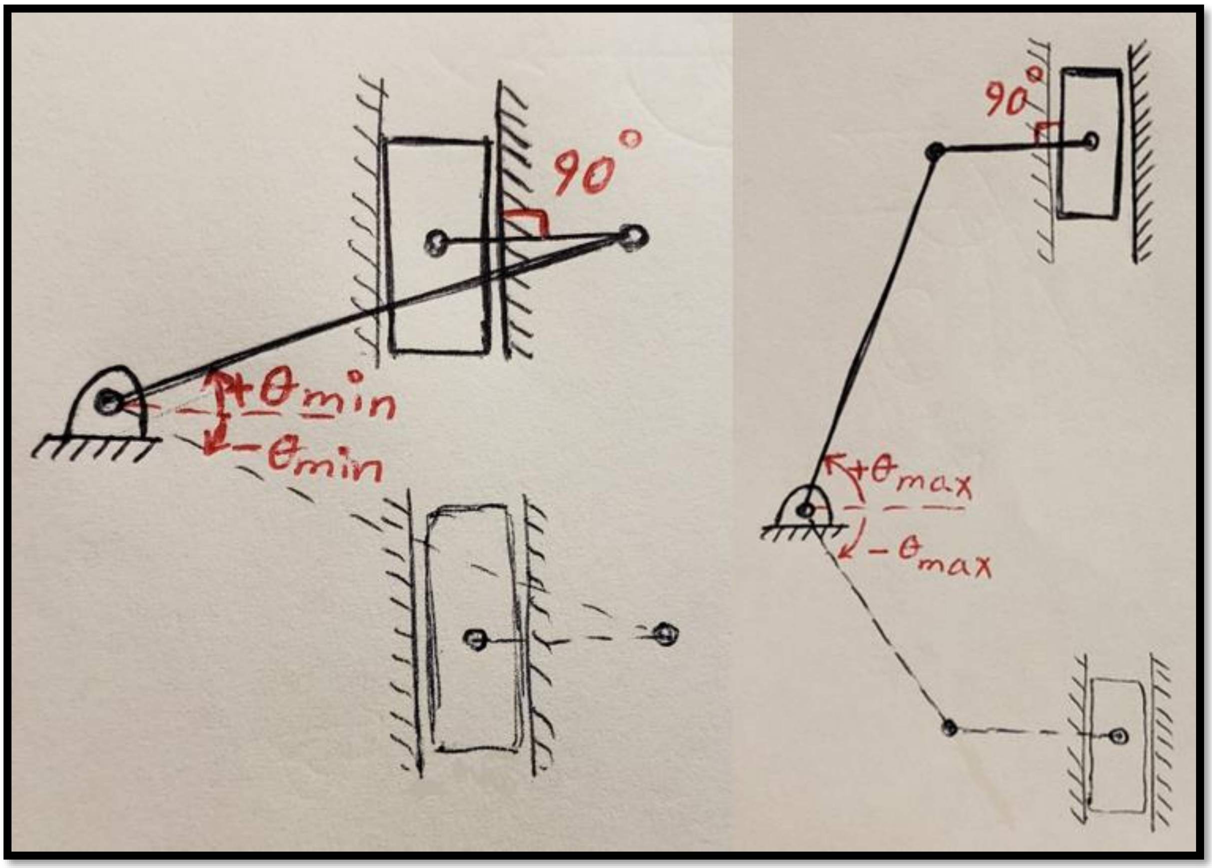
Limits on "θ"

- If from the above equation we find "s" is a complex number and that means it is beyond the maximum value (impossible).
- "s" would be real if $L^2 - (e - r\ cos\theta)^2 \geq 0$
- So, $L^2 - (e - r\ cos\theta)^2 = 0$ will give us the limit angle for θ.

$$\sqrt{(e - r\ cos\theta)^2} = \pm\sqrt{L^2}$$

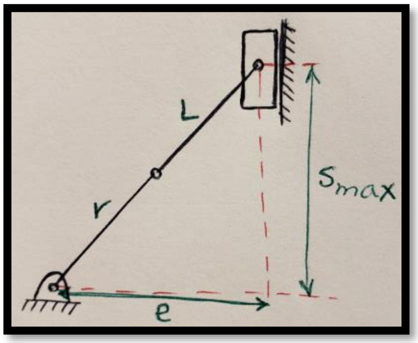
$$r\ cos\theta = e \pm L \Longrightarrow cos\theta = \frac{e \pm L}{r}$$

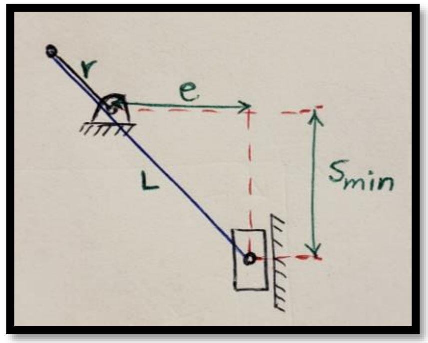
Limit Angles
$$\begin{cases} \theta_{min} = \pm arccos\left(\dfrac{e+L}{r}\right) \\ \theta_{max} = \pm arccos\left(\dfrac{e-L}{r}\right) \end{cases}$$



- Is it possible that $|e \pm L| > r$ (that causes $arccos\left(\frac{e+L}{r}\right) > 1$ or $arccos\left(\frac{e-L}{r}\right) < -1$)? If yes, what is the meaning of it?
- In the kinematic language, crank means fully rotate. In most cases, the slider-crank is made to fully rotate. So, in many cases θ_{min} and θ_{max} become complex and that means you don't have a limit angle and the input link can fully rotate (crank).

Limits on "s"



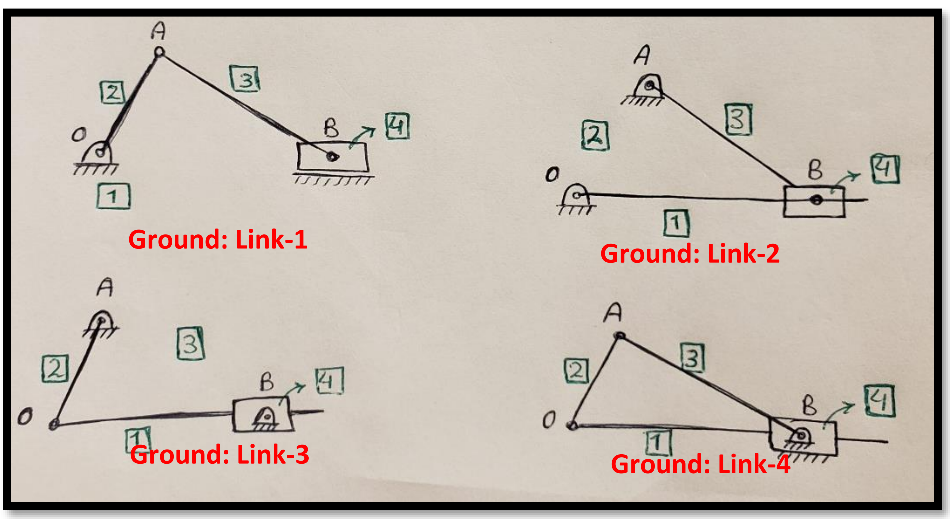


$$s_{max} = \pm\sqrt{(r+L)^2 - e^2}$$

$$s_{min} = \pm\sqrt{(L-r)^2 - e^2}$$

Slider-Crank Inversions

Inversions (in mechanisms): switch the ground link.

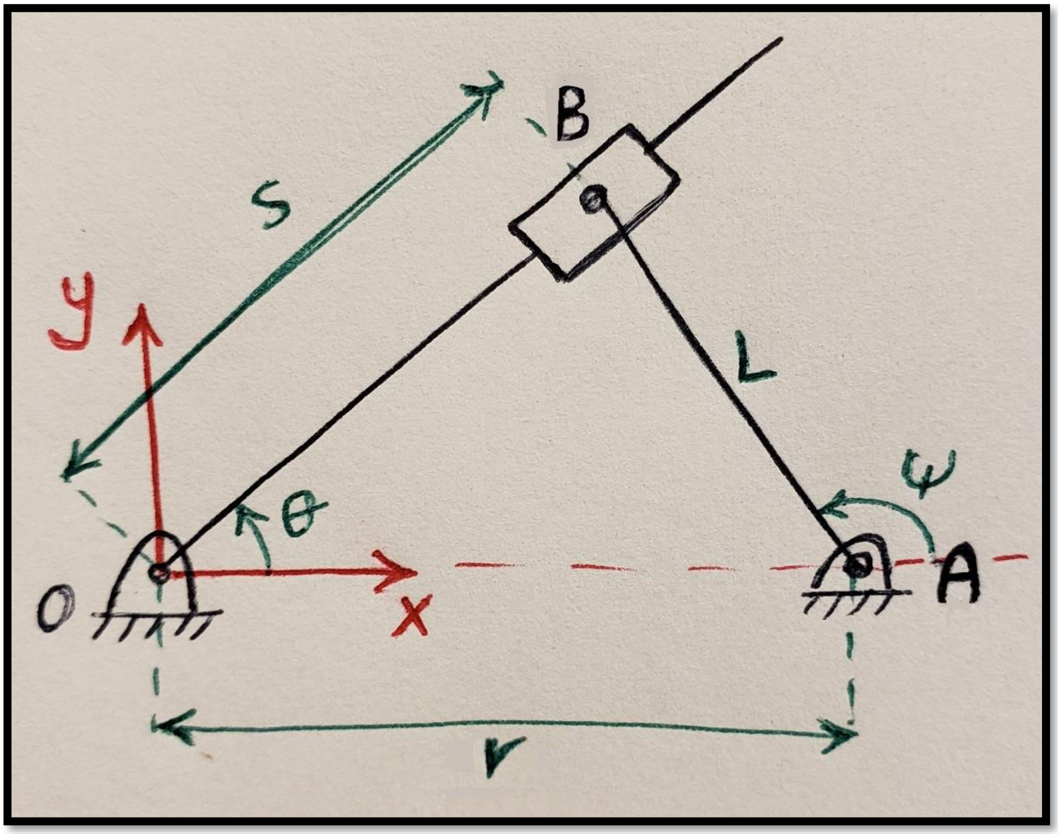


Some videos:

https://www.youtube.com/watch?v=7PYnsvVcCts

https://www.youtube.com/watch?v=B74fMbf5nek

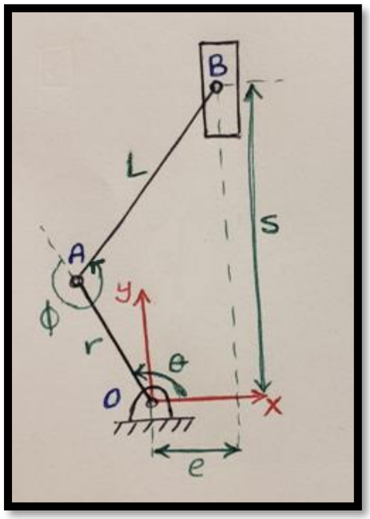
How to analyze other inversions?



$$\bar{B} = \begin{Bmatrix} s\ cos\theta \\ s\ sin\theta \end{Bmatrix} \qquad \text{or} \qquad \bar{B} = \begin{Bmatrix} r + L\ cos\Psi \\ L\ sin\Psi \end{Bmatrix}$$

- In this case: $e = 0$

Velocity Analysis



Take derivatives:

$$\bar{A} = \begin{Bmatrix} r\ cos\theta \\ r\ sin\theta \end{Bmatrix} \quad , \quad \dot{\bar{A}} = \begin{Bmatrix} -r\ sin\theta\ \dot{\theta} \\ r\ cos\theta\ \dot{\theta} \end{Bmatrix}$$

$$\bar{B} = \begin{Bmatrix} r\ cos\theta + L\ cos(\theta + \emptyset) \\ r\ sin\theta + L\ sin(\theta + \emptyset) \end{Bmatrix} \quad , \quad \dot{\bar{B}} = \begin{Bmatrix} -r\ sin\theta\ \dot{\theta} - L\sin(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \\ r\ cos\theta\ \dot{\theta} + L\cos(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \end{Bmatrix}$$

$$\bar{B} = \begin{Bmatrix} e \\ s \end{Bmatrix} \quad , \quad \dot{\bar{B}} = \begin{Bmatrix} 0 \\ \dot{s} \end{Bmatrix}$$

We have to find $\dot{s}(\theta, \dot{\theta})$ and $\dot{\emptyset}(\theta, \dot{\theta})$. For that, we can take the derivate of the loop equation ($\dot{\bar{B}}$) and solve for them.

Linear equations using ($\dot{\bar{B}}$):

$$\begin{Bmatrix} -r\ sin\theta\ \dot{\theta} - L\sin(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \\ r\ cos\theta\ \dot{\theta} + L\cos(\theta + \emptyset)(\dot{\theta} + \dot{\emptyset}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{s} \end{Bmatrix}$$

Simplify $\dot{s}$ to not be a function of $\emptyset$

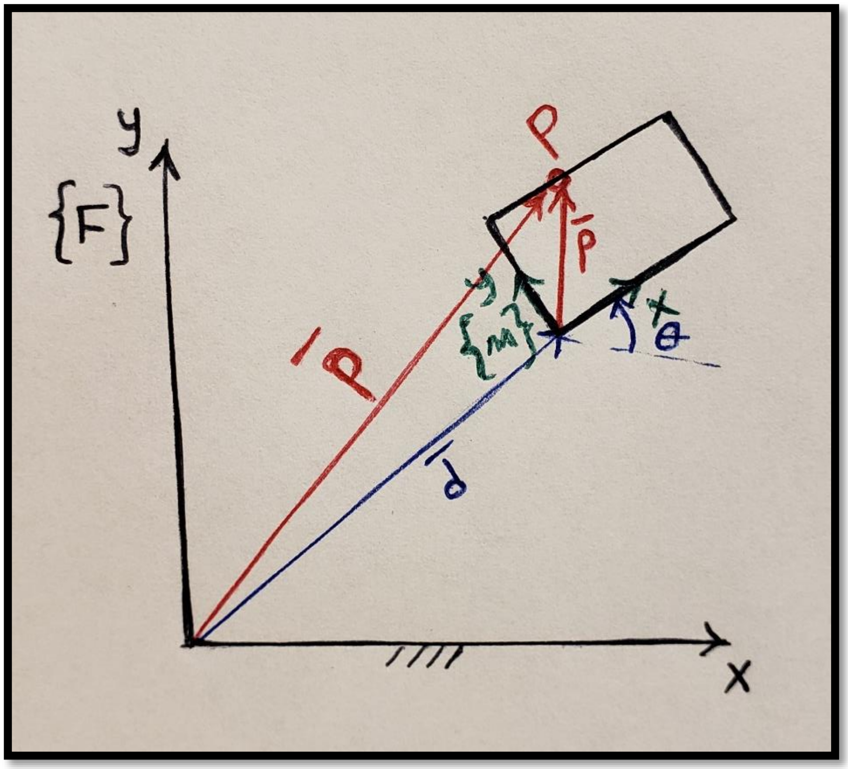
$$\dot{s} = \frac{r\ sin\emptyset}{\sin(\theta+\emptyset)}\ \dot{\theta} \quad \text{or} \quad \dot{s} = \frac{r\ (s\ cos\theta - e\ sin\theta)}{s - r\sin\theta}\ \dot{\theta}$$

$$\dot{\emptyset} = \frac{-r\ sin\theta - L\ \sin(\theta + \emptyset)}{L\ \sin(\theta + \emptyset)}$$

Acceleration Analysis: Take second derivatives and you can find $\ddot{S}$ and $\ddot{\emptyset}$.

Theoretical Kinematics

Position



$\{M\}$: Moving Frame

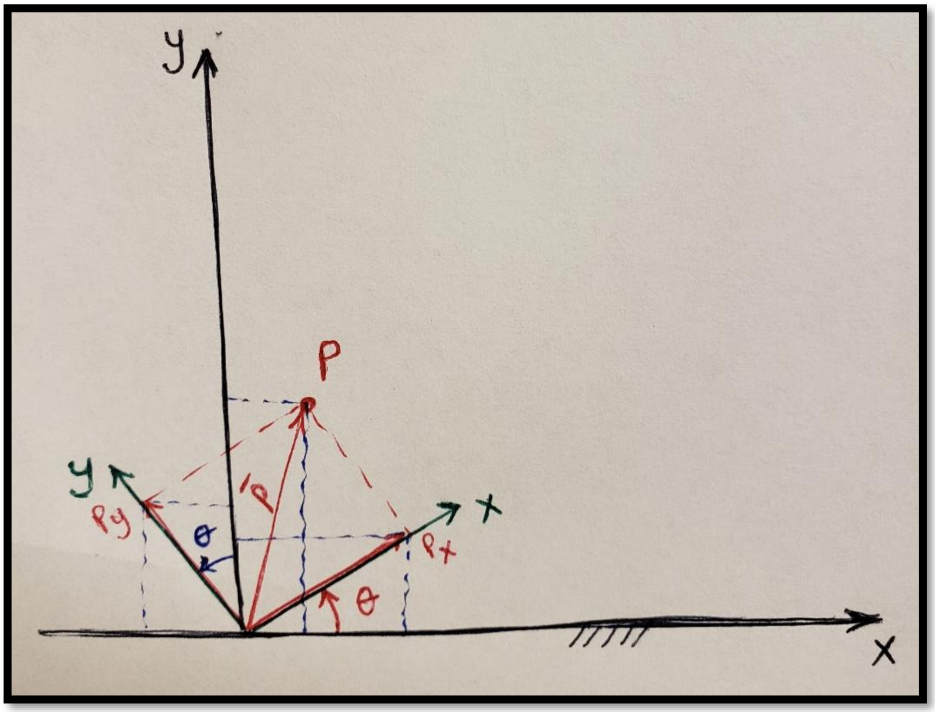
$\{F\}$: Fixed Frame

$\bar{p}$: Coordinate point P in the Moving frame

$\bar{P}$: Coordinate point P in the Fixed frame (it is the function of time and if the body is moving this coordinate would be changed)

To compute $\bar{P}$: Keep track of using $\{M\}$and position vector $\bar{d}$ and angle θ

$\bar{P} = \bar{d} + \bar{p}$ (But first, we need to express $\bar{p}$ in fixed frame)



$$\begin{Bmatrix} p_x \cos\theta - p_y\, sin\theta \\ p_x\, sin\theta + p_y\, cos\theta \end{Bmatrix}$$

$$\bar{P} = \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} + \begin{Bmatrix} p_x \cos\theta - p_y\, sin\theta \\ p_x\, sin\theta + p_y\, cos\theta \end{Bmatrix}$$

$$\bar{P} = \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} + \underbrace{\begin{bmatrix} \cos\theta & -\,sin\theta \\ sin\theta & cos\theta \end{bmatrix}}_{\text{Rotation Matrix } R(\theta)} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

Position vector in the fixed frame

Position vector in the moving frame

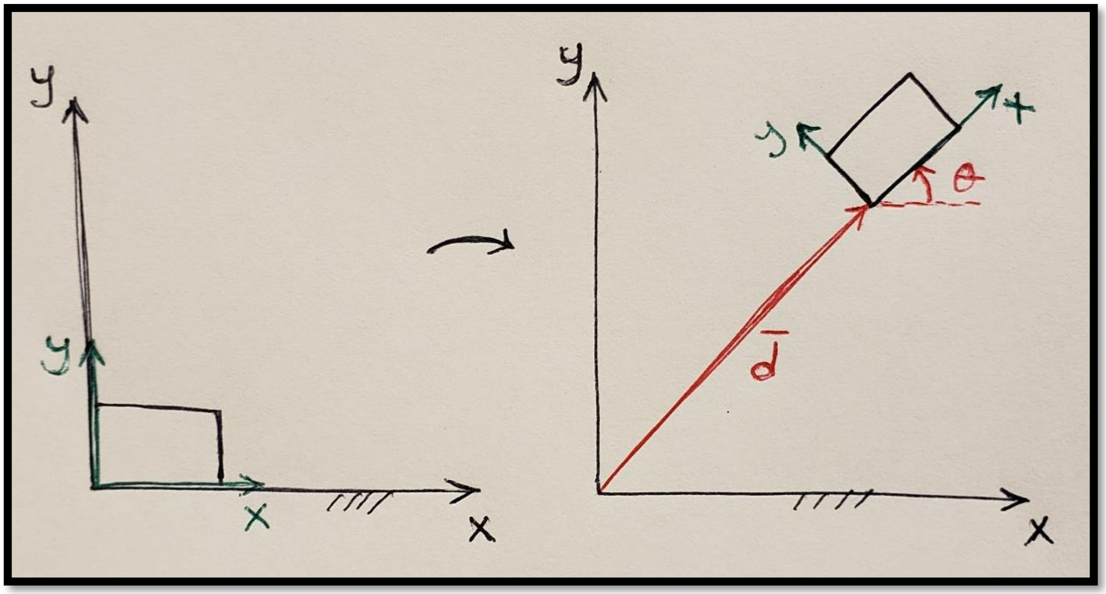
- Rotation Matrices are orthogonal, the determinant of them equals +1, it is invertible, and the inverse is transposed ($[R]^{-1} = [R]^T$), etc.
- $R(\theta)$= $\begin{bmatrix} \cos\theta & -\,sin\theta \\ sin\theta & cos\theta \end{bmatrix}$
- This rotation matrix can be used to solve any problem. Just we have to make sure to measure the angle (θ) CCW.

So, right now we can compute the $\overline{P}$ in the Fixed frame:

$$\boxed{\bar{P} = \bar{d} + R(\theta)\,\bar{p}}$$

This expression can have two meanings:

1. The changing of coordinate from moving frame to fixed frame.
2. Position of point "P" after displacing the body by $\bar{d}$ and θ.

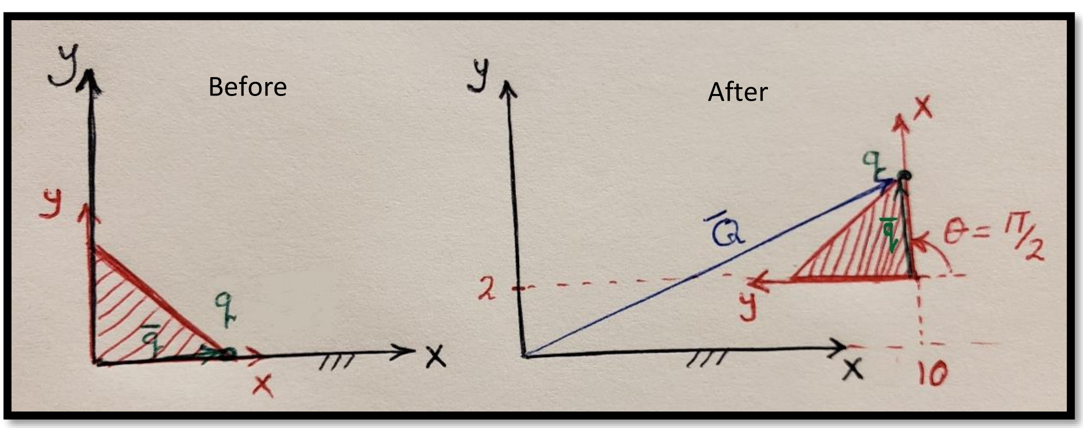


- The displacement will be given by the position vector of origin ($\bar{d}$) and rotation angle of the moving frame with respect to the fixed frame (θ)

$\bar{p}$: Coordinate point p before the motion

$\bar{P}$: Coordinate point p after motion given by $\bar{d}$ and θ.

Example: The following triangle after a rigid motion is reached a new position. Point "q" in this triangle has the vector of $\bar{q} = \begin{Bmatrix} 5 \\ 0 \end{Bmatrix}$ which is measured in the moving frame attached to the triangle. Find the $\bar{Q}$ (vector of point "q" with respect to the fixed frame) after this motion.



From the graph, $\bar{d} = \begin{Bmatrix}10\\2\end{Bmatrix}$ and the rotation angle is $\theta = \frac{\pi}{2}$. So, the rotation matrix would be:

$$R(\theta) = \begin{bmatrix}\cos\theta & -sin\theta\\ sin\theta & cos\theta\end{bmatrix} = \begin{bmatrix}\cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2})\\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

From the following equation, we can find the $\bar{Q}$.

$$\bar{Q} = \bar{d} + R(\theta)\bar{q} = \begin{Bmatrix}10\\2\end{Bmatrix} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{Bmatrix}5\\0\end{Bmatrix} = \begin{Bmatrix}10\\2\end{Bmatrix} + \begin{Bmatrix}0\\5\end{Bmatrix} = \begin{Bmatrix}10\\7\end{Bmatrix}$$

Velocity

$$\bar{Q} = \bar{d} + R(\theta)\bar{q}$$

$$\dot{\bar{Q}} = \frac{d\bar{Q}}{dt} = \frac{d}{dt}\left(\bar{d} + [R]\bar{q}\right) = \dot{\bar{d}} + [\dot{R}]\bar{q}$$

$\bar{d}$: It is the origin of the moving frame with respect to the fixed frame. So, it is changing with time.
$\bar{q}$: It is attached to the moving frame. So, it's not changing with time.
$[R]$: It is the combination of $sin\,\theta$ and $cos\,\theta$, and θ is changing with time. So, $[R]$ will change with time.

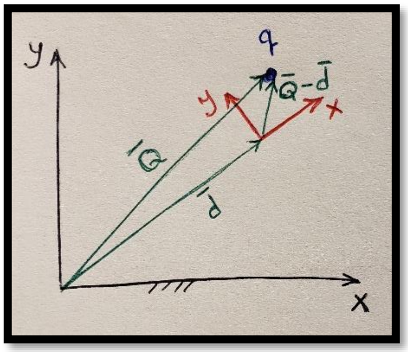
Everything in the above equation is expressed in the fixed frame except $\bar{q}$. So, let's express $\bar{q}$ in fixed frame:

$$\bar{Q} = \bar{d} + R(\theta)\bar{q}$$

$$\bar{Q} - \bar{d} = [R]\bar{q} \Longrightarrow [R]^{-1}(\bar{Q} - \bar{d}) = \bar{q} \Longrightarrow [R]^{T}(\bar{Q} - \bar{d}) = \bar{q}$$

So, we can write the velocity equation in this form:

$$\boxed{\dot{\bar{Q}} = \dot{\bar{d}} + [\dot{R}][R]^{T}(\bar{Q} - \bar{d})}$$



$$[\dot{R}][R]^T = \dot{\theta}\begin{bmatrix} -sin\theta & -\cos\theta \\ cos\theta & -sin\theta \end{bmatrix}\begin{bmatrix} \cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} =$$

$$\dot{\theta}\begin{bmatrix} -sin\theta\ \cos\theta\ +\ \cos\theta\ sin\theta & -sin^2\theta - \cos^2\theta \\ \cos^2\theta + sin^2\theta & \cos\theta\ \ sin\theta - sin\theta\cos\theta \end{bmatrix}$$

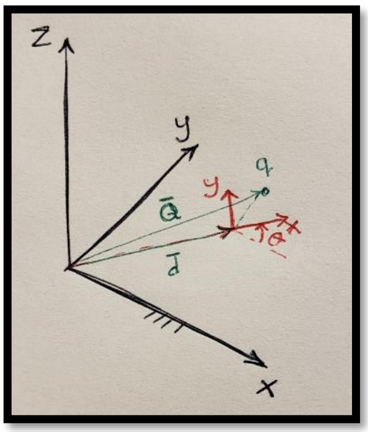
$$\boxed{[\dot{R}][R]^T = \dot{\theta}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$ Skew symmetric matrix

- We can find Skew symmetric matrix for higher dimensions (3, 4, etc.). If we are in **3-dimensions** this matrix is equivalent and acts as the cross product. (but remember the cross product is not defined in 2 dimensions, so we don't have a cross product in the planar case)

For our case (2-D), we can say we are in 3-D with Z=0

$$\dot{\bar{Q}} = \dot{\bar{d}} + \bar{\omega} \times (\bar{Q} - \bar{d}) = \begin{Bmatrix} \dot{d}_x \\ \dot{d}_y \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} \times \begin{Bmatrix} Q_x - d_x \\ Q_y - d_y \\ 0 \end{Bmatrix}$$

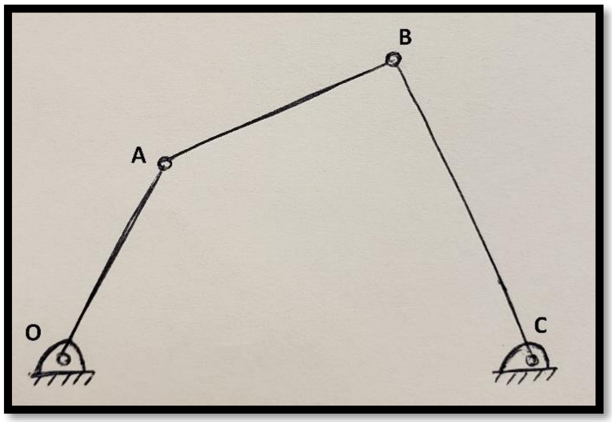
$$\bar{\omega} = \dot{\theta}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix}$$



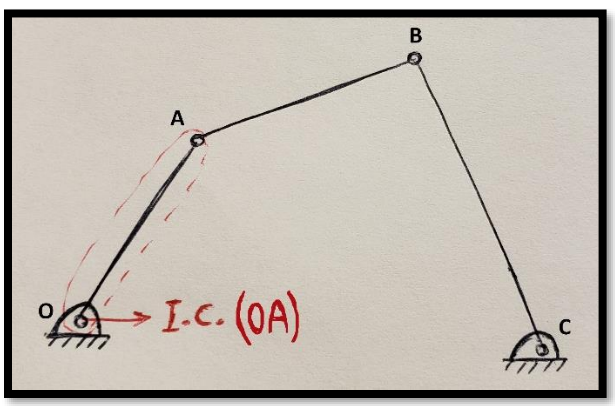
Instant center of velocity (instant center of rotation): There is always one point in a moving body that has zero velocity instantaneously (just for a second). The body instantaneously rotates about this point.

- We have an **Instant center of velocity** for the rigid bodies **Not** points!

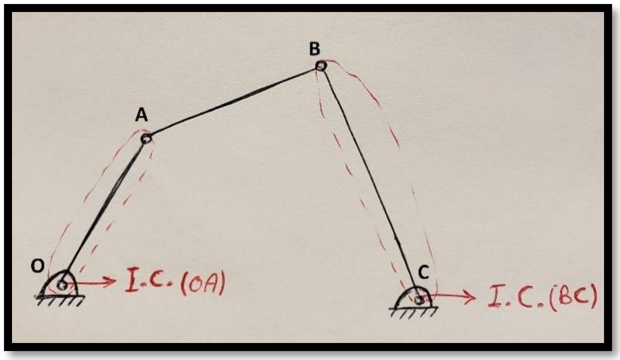
In the following 4-bar linkage:



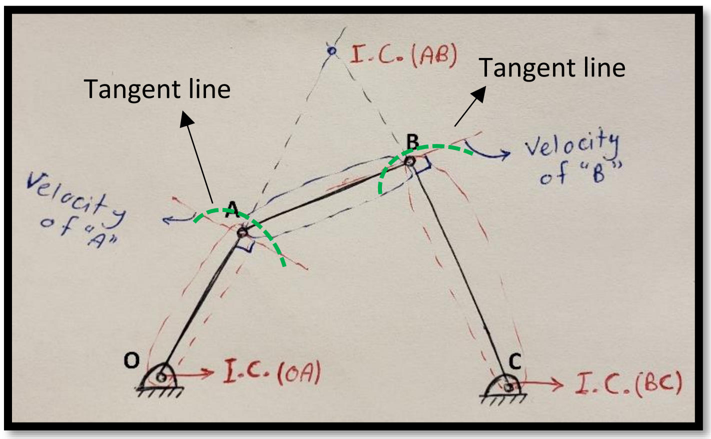
What is the Instant center of velocity for the input link? (What point has zero velocity?)



What is the Instant center of velocity for the output link?

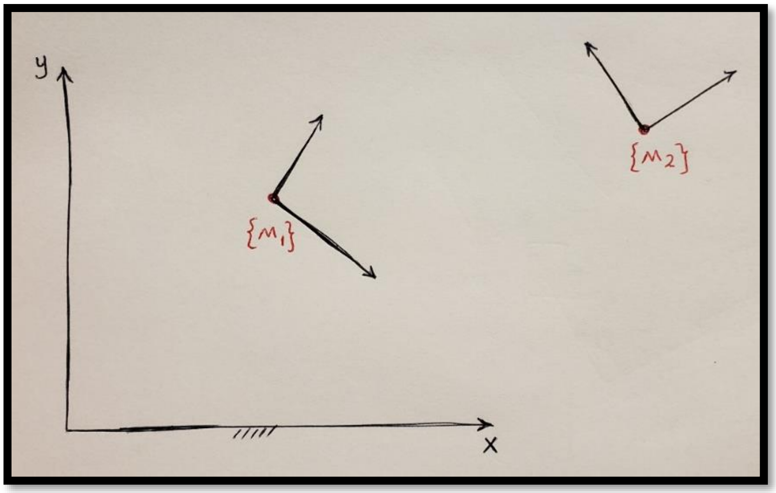


What is the Instant center of velocity for the coupler link?



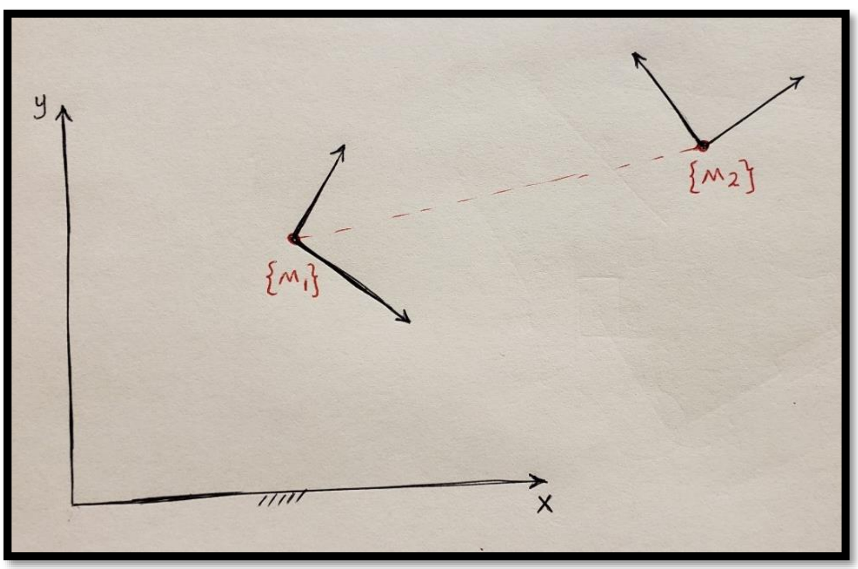
Pole: Point of rotation for a finite motion. For having a motion from frame $\{M_1\}$ to $\{M_2\}$, there is a point name "Pole" which the point can rotate about it with "minimal motion".

How to find the Pole?
We are trying to find the center of the circle that the frame is moving around it:
Step1: Select the same point on two positions of the moving frame $\{M_1\}$ & $\{M_2\}$



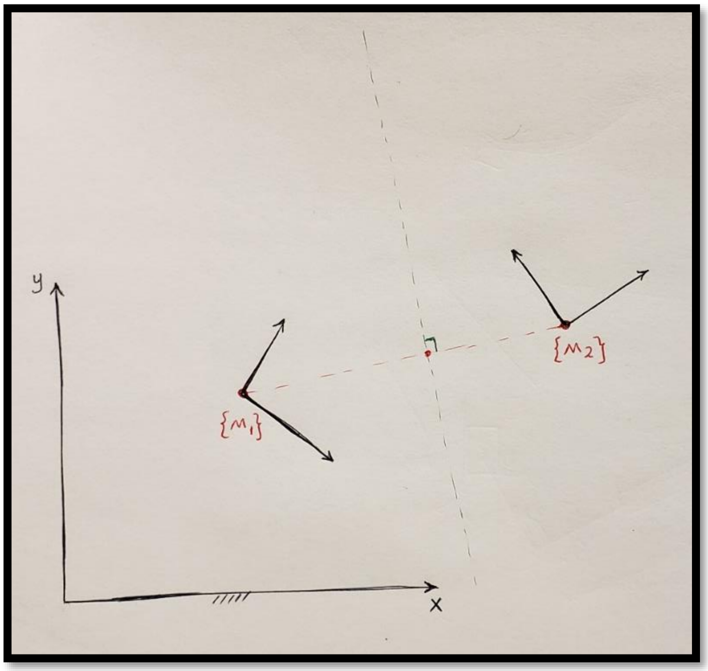
Q1: What is the minimum distance between two points? Straight line

Step2: Connect the points with a straight line.

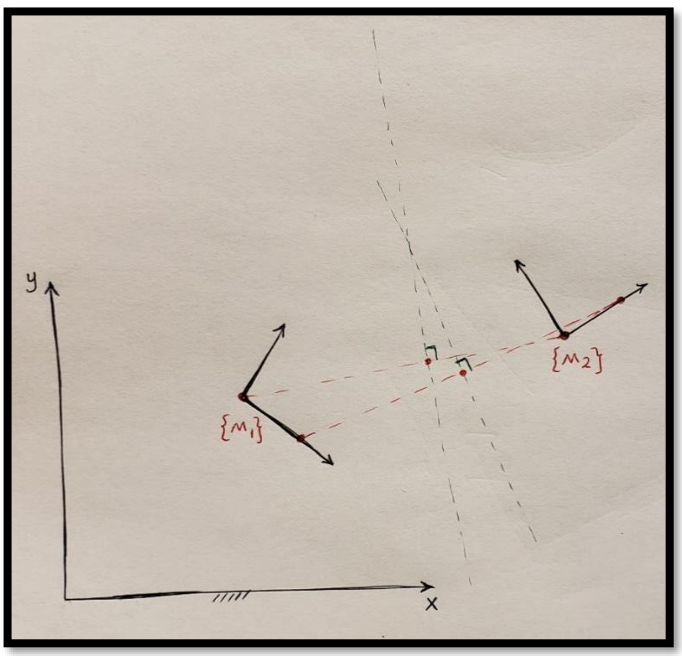


Q2: How many circles can pass through two points? Infinity

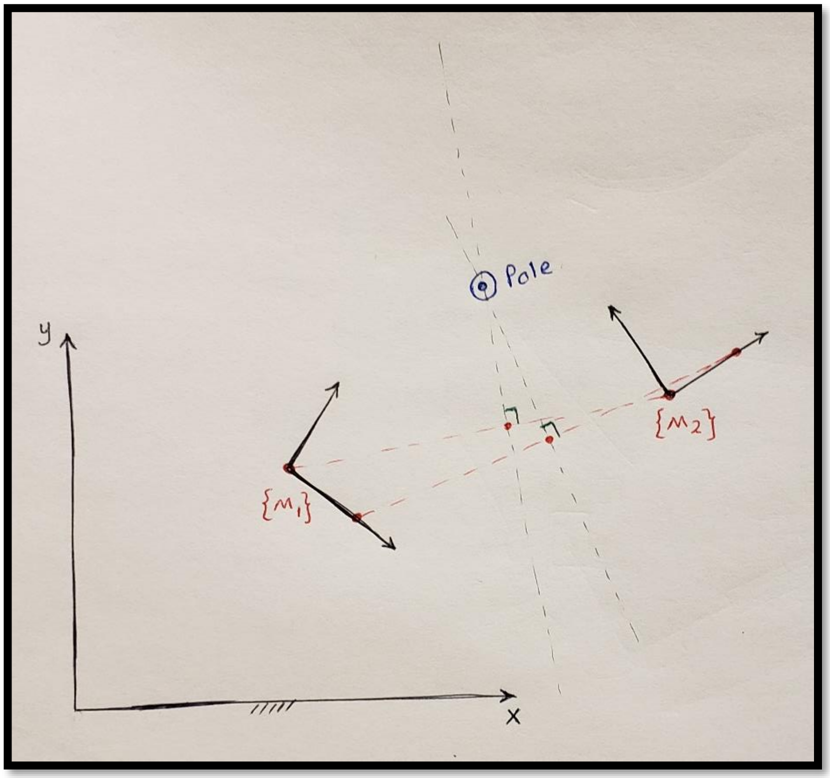
Step3: Find the mid-point on the line between two points and sketch a perpendicular line that passes this mid-point. The center of the circle definitely would be on this line.



Step4: For finding the exact location of the "Pole" we need to repeat this process with two other points on two positions of the moving frame $\{M_1\}$ & $\{M_2\}$



Step5: The location of the perpendicular lines reaching each other would be the "Pole"!

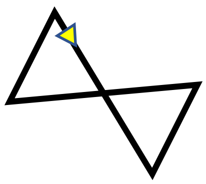

Pole
y
x
{M1}
{M2}

Synthesis of Linkages

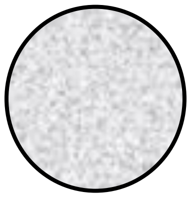
Design problem: The designer is given a task (A motion task) and is asked to design a mechanism to do the task.

1) **Task Definition:** Task can be defined in different forms

- A curve in the plane (for instance paint on a particular trajectory) (We will not cover in this class)



- An area of the plane (workspace) (We will not cover in this class)



- A set of finite positions (it will cover in this class)

- A set of input/output angle (For instance the motion of windshield wiper or design mechanism input fully rotate when the output racks between two particular positions) (it will cover in this class)

2) **Type Synthesis:** Select the most proper type of mechanism for doing the task.
   There is not any theory to answer this part yet. In this course, we just have to focus on the 4-bar linkages.
3) **Dimensional synthesis:** Calculate link lengths and position of fixed pivots for doing the task.
4) **Analyze the results:** Check limit angles, velocities, accelerations, size of mechanism, required force is not too high, etc. If the design is not good enough, repeat the steps and find a better design (solution).

Graphical Synthesis in the Plane for Rigid-Body Guidance

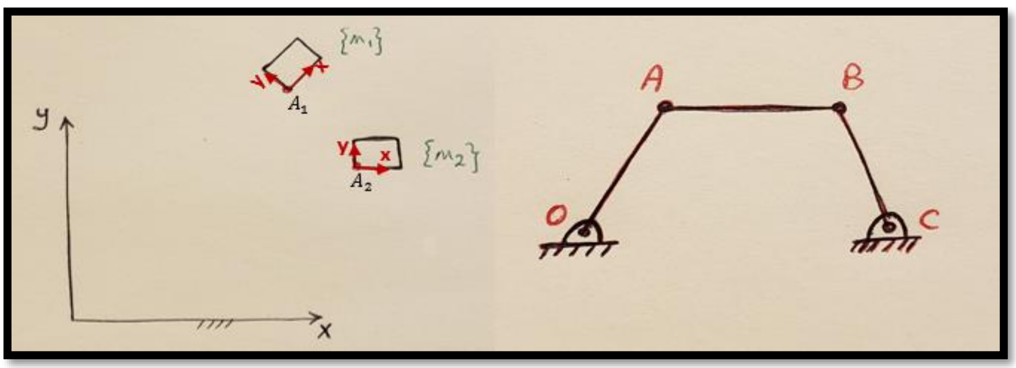
In this method, we always have a fixed frame and a rigid body that is moving to several different positions with respect to the fixed frame. We want to design the proper linkage that can able to do these tasks and guide the rigid body to these positions.

We are going to design a 4-bar linkage while the rigid body is attached to the coupler linkage. In other words, the coupler link can go through all task positions.

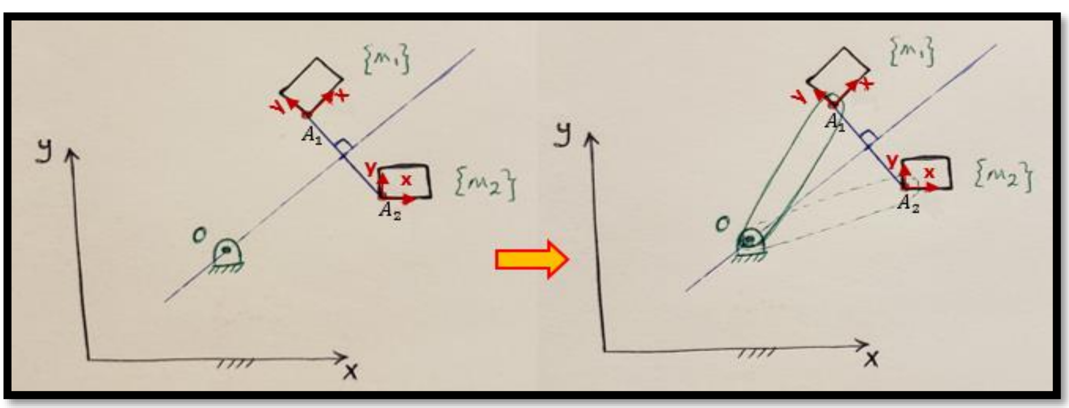
Why attached to the coupler linkage? Because we know the input and output linkage fully rotate so they cannot make a general motion and they have just rotation.

Two Position Graphical Synthesis:

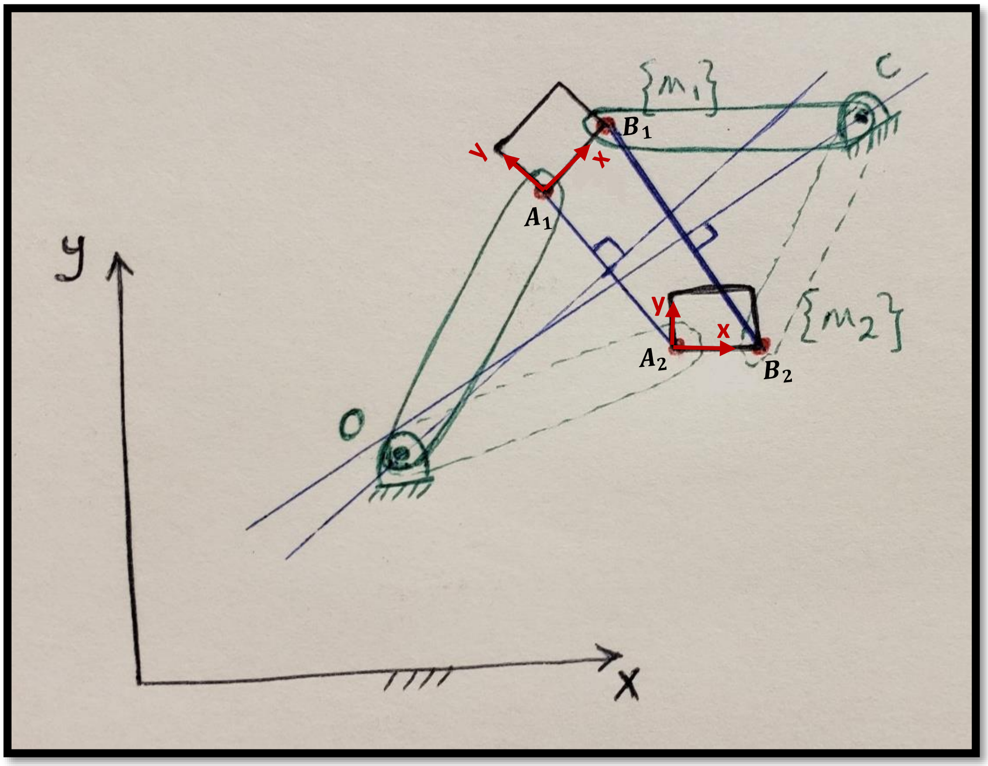
In this case, we have extra freedom and that means there are infinite possible solutions and many mechanisms that can be designed that are able to go through these two positions. So, in the first step, we have to select the moving pivot of the input link (A_i) (the joint that is not fixed).



We know pivot "A" rotates about the fixed pivot "O". As we discussed before, there are infinite circles that can pass through two points but the center of all of them would be on the line that is perpendicular to the line between these two points and passing through the midpoint. So, sketch the perpendicular bisector for points A_1 and A_2 and select **any** proper point on the line for locating pivot "O" based on other design preferences.

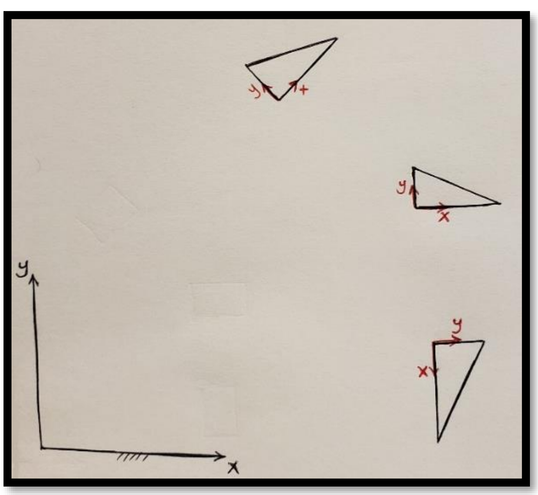


Repeat the process by selecting a proper point for the pivot "B_i". You can select any point inside or outside of the object. Just make sure the points for B_1 and B_2 having the same relation to frames $\{M_1\}$ and $\{M_2\}$ respectively. Then, with repeating what we did for finding the location of pivot "O" to find the location of pivot "C". This is the last step and with connecting pivots, the 4-bar linkage would be complete.

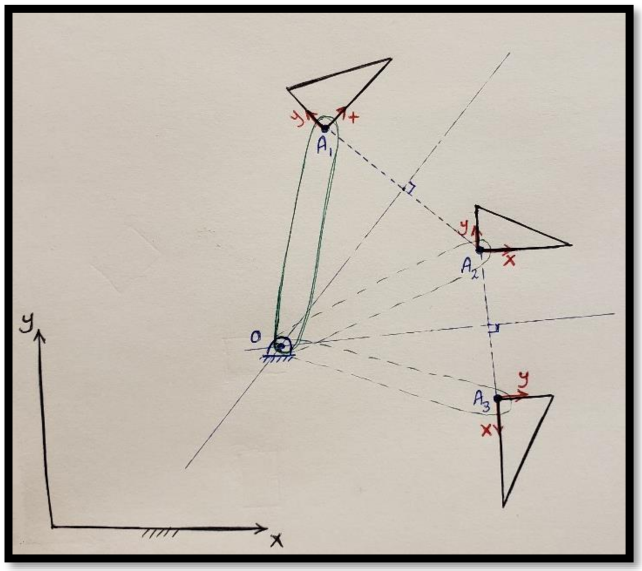


Three Position Graphical Synthesis:

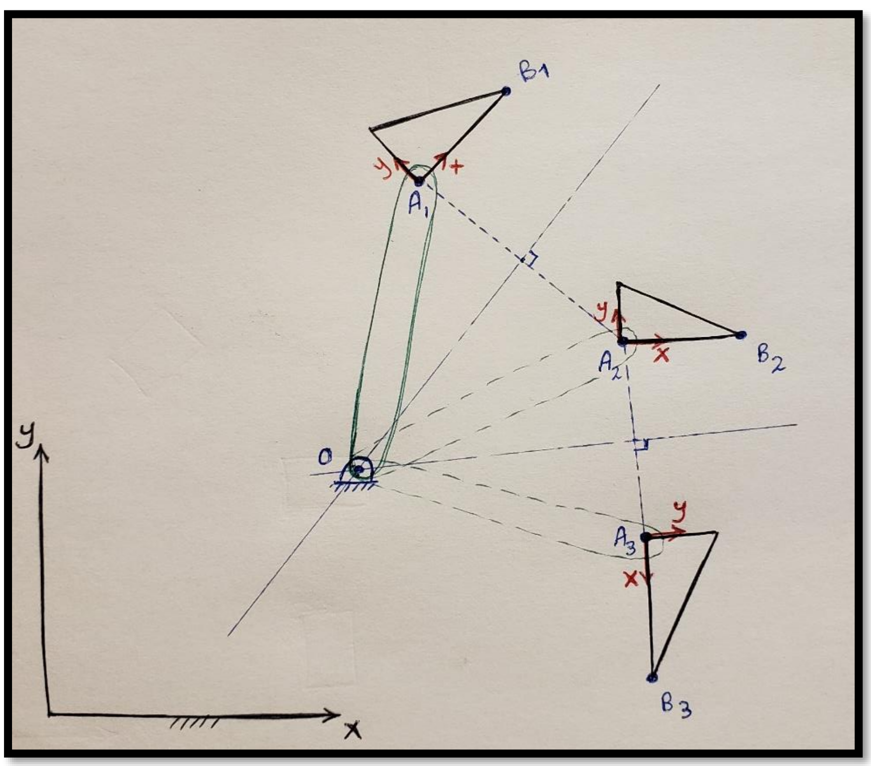
We still have extra freedom but less than two positions. Do the following steps:



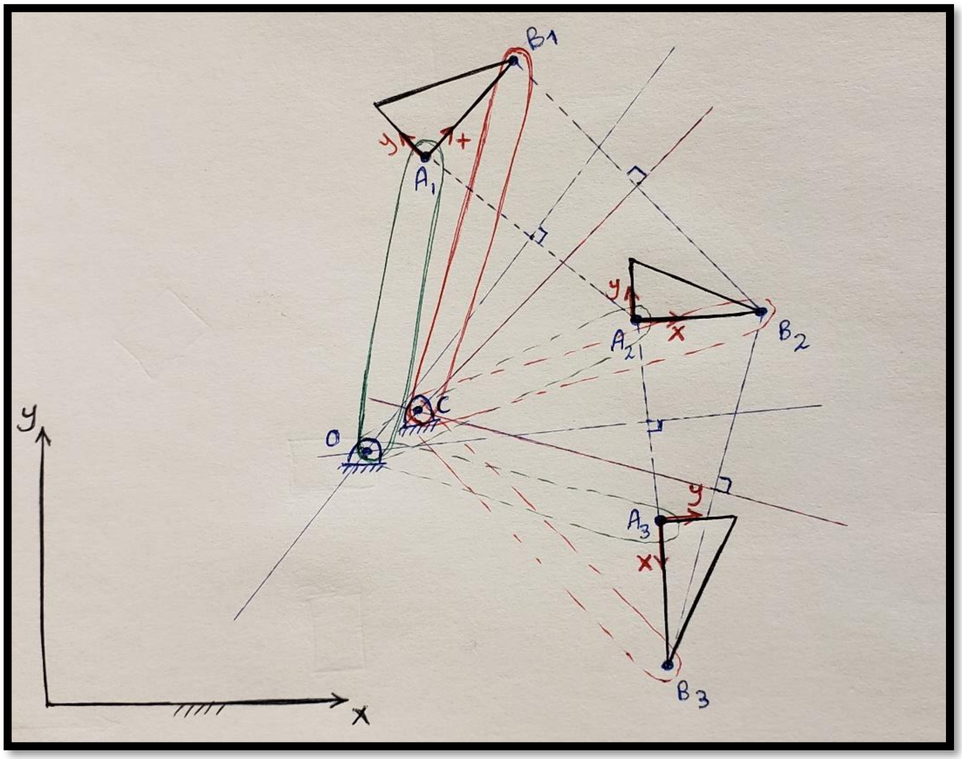
1) Select the moving pivot on the input link (A_i)
2) In this case, we have three points, so there is only one circle passing through these three points. The center of this circle (which is the position of pivot "O") would be the intersection of two perpendicular bisectors.



3) Select the moving pivot on the output link (B_i)



4) Repeat step 2 to find the location of pivot "C"
5) Connect all pivots and complete the 4-bar linkage!



Note: If pivots "O" and "C" coincide or are very close to each other or for any reason you dislike their location, you can change the position of moving pivots (A_i & B_i) and repeat the design steps and you will find completely different points for pivots "O" and "C"!

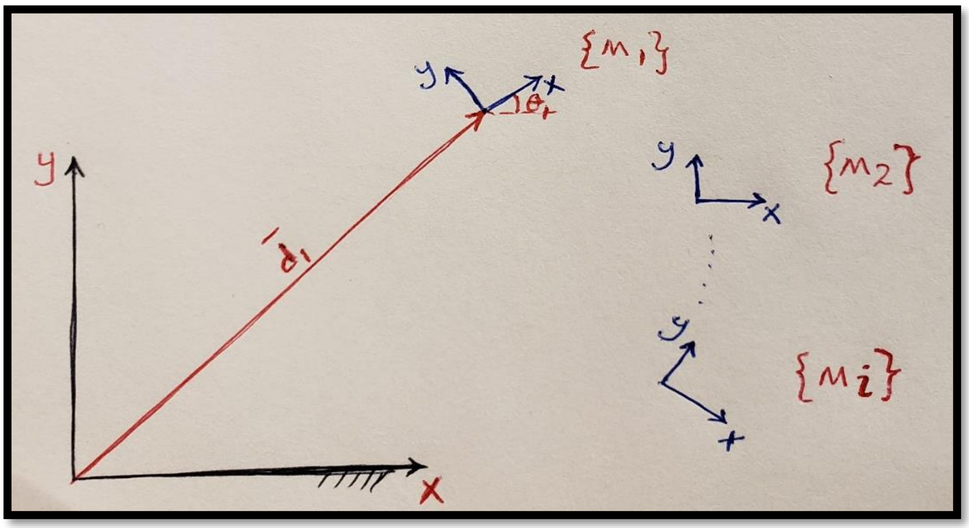
Note: We can define four positions but then we won't able to select the moving pivot and graphical calculation become more complex.

Note: For three positions, we can select the fixed pivot instead of the moving pivot. In that case, the process and step would be different and we need to find the pole, draw more lines and angles, etc. If you are interested, you can find the steps and explanation in either Norton or Waldron's books.

Algebraic Synthesis in the Plane for Rigid-Body Guidance (Exact Synthesis of 4-Bar Linkage)

- **Exact Synthesis:** We have to pass exactly through those positions.

In this method, the reference frame and a set of positions are given and we have to calculate and find the position of the joints (O, C, A_1, B_1). The shape of the links in the kinematic is not mattered.

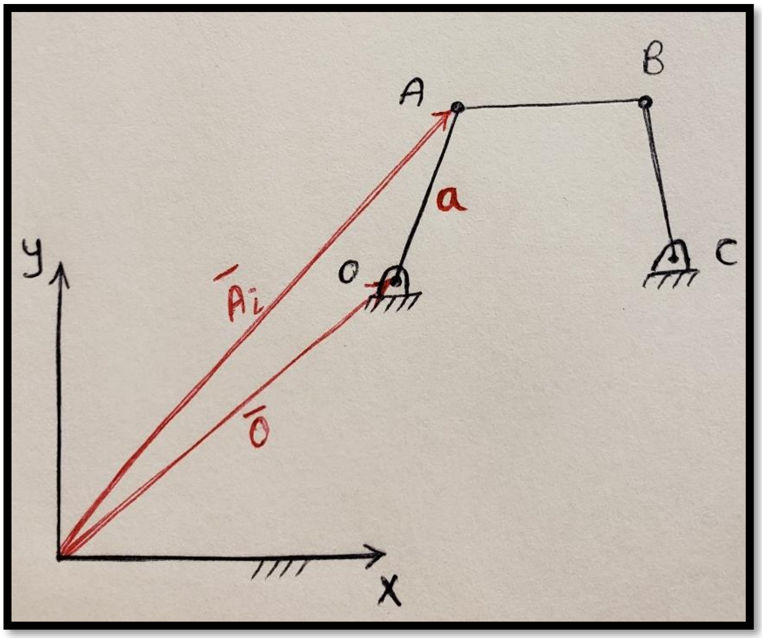


$$\bar{d}_1, \quad [R(\theta_1)]$$
$$\bar{d}_2, \quad [R(\theta_2)]$$
$$\vdots$$
$$\bar{d}_i, \quad [R(\theta_i)]$$

$$[R(\theta_i)] = \begin{bmatrix} \cos\theta_i & -sin\theta_i \\ sin\theta_i & cos\theta_i \end{bmatrix}$$

The moving pivots of the 4-bar linkage have a relation with the moving frame (we don't know what exactly this relation is). So, when the moving frame moves to a new position the moving pivots will move too. The moving frame move from position 1 to *i*.
Let's solve it for a random 4-bar linkage in the plane and find the location of pivots. In this case, the design parameters (unknowns) are O, C, A_i, B_i, a, b, h, etc. We just know the moving pivot rotates about the fixed pivot with the radius a!

For the input link, we can write the distance constraint equation for a general position of pivot A ($\bar{A}_i$).



$(\bar{A}_i - \bar{O}) \cdot (\bar{A}_i - \bar{O}) = a^2$

The dot product is distributive and commutative, so:

$\bar{A}_i.\bar{A}_i + \bar{O}.\bar{O} - 2\bar{A}_i.\bar{O} = a^2$

Now, for eliminating one of the unknown ("a") from the above equation, we write a similar equation for the first position of moving pivot A ($\bar{A}_1$) and subtract the above equation from it.

$$\begin{array}{r} - \begin{array}{l} \bar{A}_i.\bar{A}_i + \cancel{\bar{O}.\bar{O}} - 2\bar{A}_i.\bar{O} = \cancel{a^2} \\ \bar{A}_1.\bar{A}_1 + \cancel{\bar{O}.\bar{O}} - 2\bar{A}_1.\bar{O} = \cancel{a^2} \end{array} \\ \hline \end{array}$$

You have to memorize this equation!

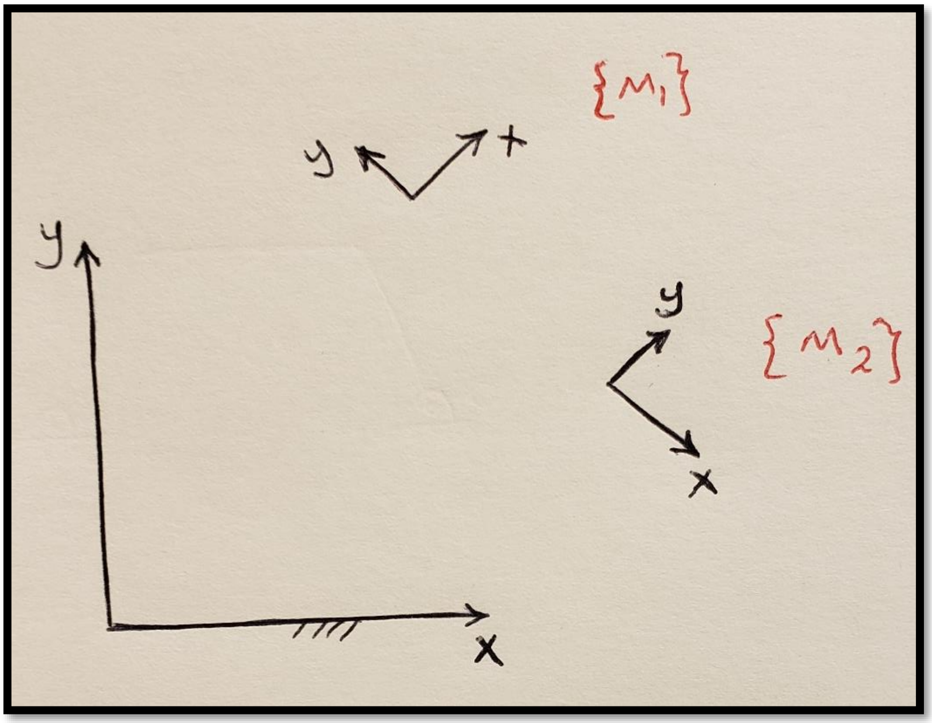
$$\boxed{\bar{A}_i.\bar{A}_i - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_i - \bar{A}_1).\bar{O} = 0}$$

The equation of the **perpendicular bisector** of $\bar{A}_1$ & $\bar{A}_i$

In the above equation:
$\bar{A}_i$: It is a general position and it can be $\bar{A}_2, \bar{A}_3, \dots, \bar{A}_n$

So, if we have n positions for the moving frame, we are able to create (n-1) perpendicular bisector lines.

Two Position Algebraic Synthesis:

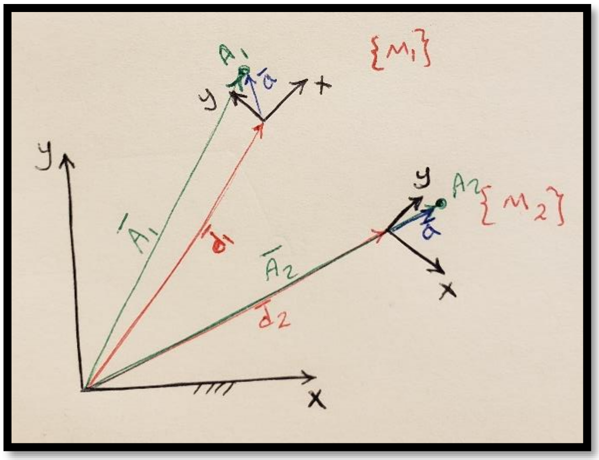


In this problem $\bar{d}_1, \theta_1, \bar{d}_2, \theta_2$ are given.

How many design equations do we have for this case? We have two positions and only one design equation.

$$\bar{A}_2.\bar{A}_2 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_2 - \bar{A}_1).\bar{O} = 0$$

How many unknowns do we have in this design equation?
Three vectors $\bar{A}_1$, $\bar{A}_2$, $\bar{O}$ seems all are unknown but vectors $\bar{A}_1$ & $\bar{A}_2$ are dependent on each other. But vector $\bar{A}$ is fixed in the moving frame. So, we can express it in the moving frame.



$$\bar{A}_1 = \bar{d}_1 + [R(\theta_1)]\bar{a}$$
$$\bar{A}_2 = \bar{d}_2 + [R(\theta_2)]\bar{a}$$

In these two equations, we know $\bar{d}_1$, $\bar{d}_2$, θ_1, θ_2, so there is only one unknown $\bar{a}$

So, we have four unknowns (two unknowns per vector) and one equation.

$\bar{O} = \begin{Bmatrix} O_x \\ O_y \end{Bmatrix}$ (In fixed frame) , $\bar{a} = \begin{Bmatrix} a_x \\ a_y \end{Bmatrix}$ (In moving frame)

$\bar{A}_2.\bar{A}_2 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_2 - \bar{A}_1).\bar{O} = 0$

(There is a linear relation between $\bar{A}_1$, $\bar{A}_2$, and $\bar{a}$)

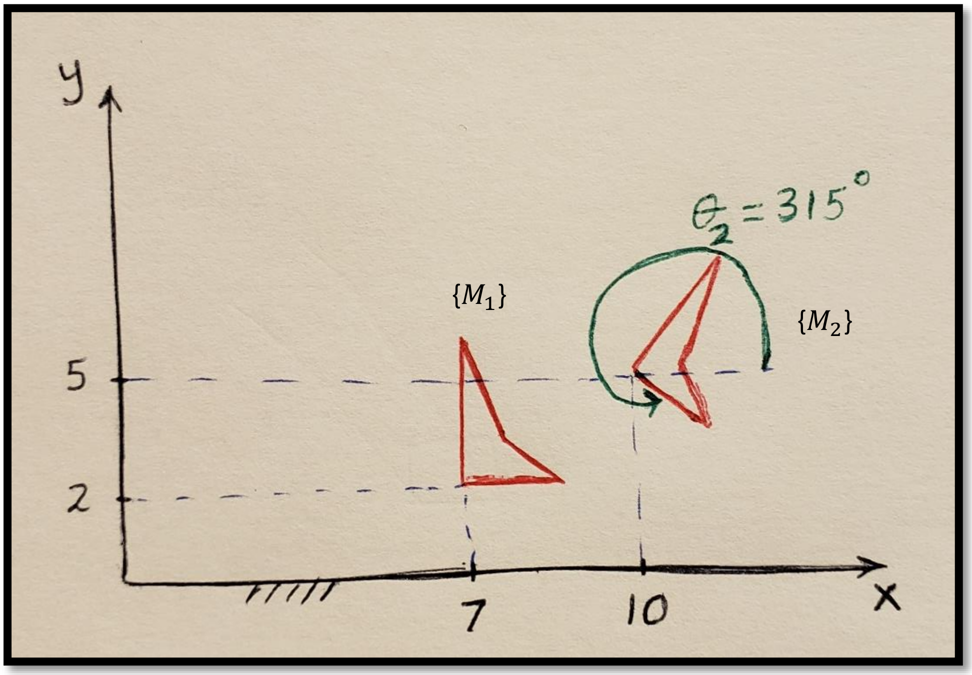
What we can do with four unknowns and only one equation? We can give value to 3 unknowns and solve for the remaining one.

Compare to graphical synthesis (which was easier to choose the moving pivot and find the fixed pivot), because we have an equation now, we have more flexibility to choose any parameters of the moving pivot or fixed pivot and find the remaining unknown (doesn't matter which value you choose).

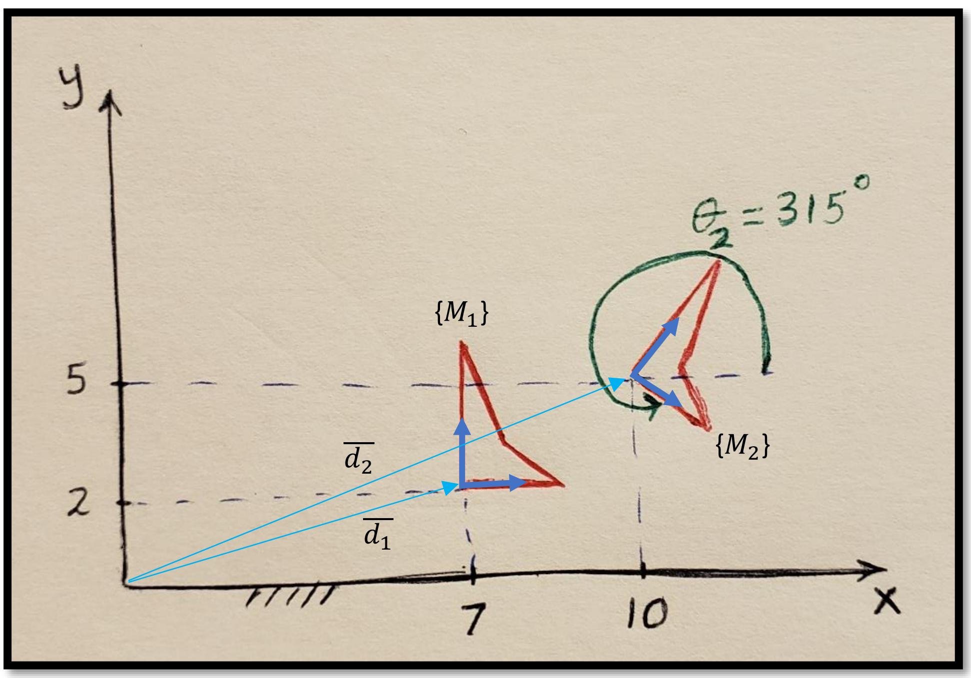
Then, you have to repeat the process for the output link to find the position of the pivots $\bar{B}$ and $\bar{C}$.

Synthesis of Linkages (Continue)

Example: Design a proper 4-bar linkage for the following two-position task.



Step 1: Select (define) the moving frame where ever you want and find the following information based on that:



Step 2: Write all position vectors ($\bar{d}_i$) and all rotation matrices ($[R(\theta_i)]$), for $i = 1,2,\ldots,n$ (n positions). In this case, we have two positions.

$$\bar{d}_1 = \begin{Bmatrix} 7 \\ 2 \end{Bmatrix}$$

$$[R(\theta_1)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\theta_1 = 0°)$$

$$\bar{d}_2 = \begin{Bmatrix} 10 \\ 5 \end{Bmatrix}$$

$$[R(\theta_2)] = \begin{bmatrix} \cos(315°) & -\sin(315°) \\ \sin(315°) & \cos(315°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (\theta_2 = 315°)$$

Step 3: Write design equations. In this case, we have only one equation.

$$\bar{A}_2.\bar{A}_2 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_2 - \bar{A}_1).\bar{O} = 0$$

Where

$$\bar{A}_1 = \bar{d}_1 + [R(\theta_1)]\bar{a} = \begin{Bmatrix} 7 \\ 2 \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{Bmatrix} 7 + a_x \\ 2 + a_y \end{Bmatrix}$$

$$\bar{A}_2 = \bar{d}_2 + [R(\theta_2)]\bar{a} = \begin{Bmatrix} 10 \\ 5 \end{Bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{Bmatrix} 10 + \frac{a_x}{\sqrt{2}} + \frac{a_y}{\sqrt{2}} \\ 5 - \frac{a_x}{\sqrt{2}} + \frac{a_y}{\sqrt{2}} \end{Bmatrix}$$

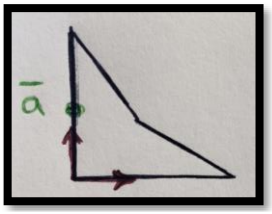
We have one equation and four unknowns (a_x, a_y, O_x, O_y). So, as a designer, we have to decide to give the value to three of these unknowns and solve the equation to find the last parameter. Based on the equation, that would be easier to give a number to the moving pivot and one of the parameters of the fixed pivot (The equation is linear in "O" and quadratic in "a").

Step 4: Select three parameters and solve for one.

For instance, let's select the value for a_x, a_y, and O_y and solve for O_x

$$\bar{O} = \begin{Bmatrix} O_x \\ O_y \end{Bmatrix} = \begin{Bmatrix} O_x \\ 5 \end{Bmatrix}$$

$$\bar{a} = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix} \Longrightarrow \bar{A}_1 = \begin{Bmatrix} 7 \\ 4 \end{Bmatrix} \quad , \quad \bar{A}_2 = \begin{Bmatrix} 11.4 \\ 6.4 \end{Bmatrix}$$



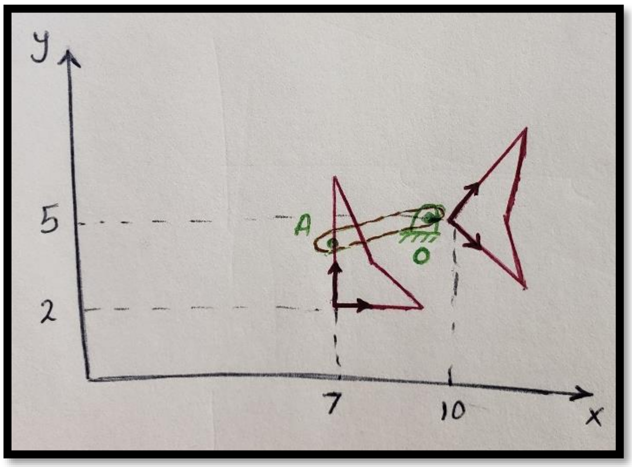
$\bar{A}_1.\bar{A}_1 = 65$, $\bar{A}_2.\bar{A}_2 = 171.4$

Design equation:

$$171.4 - 65 - 2 \times (11.4 - 7) \times O_x - 2 \times (6.4 - 4) \times 5 = 0$$

$$\boxed{O_x = 9.37}$$

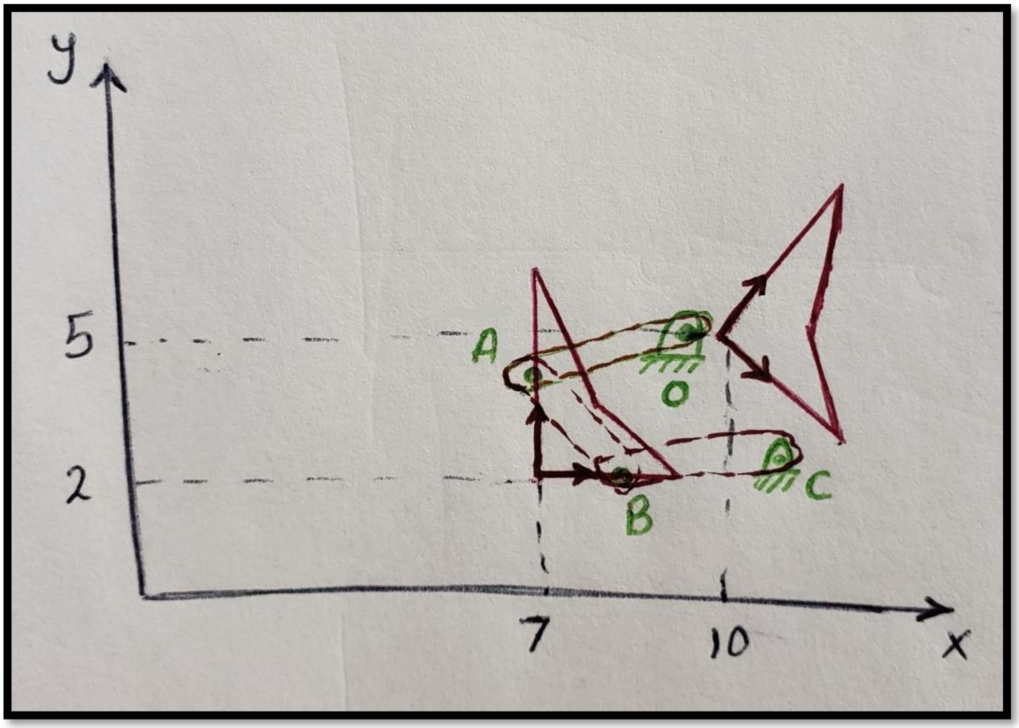
So, we have the position of two pivots:
A ($\bar{A}_1 = \begin{Bmatrix} 7 \\ 4 \end{Bmatrix}$) (attached to the moving object) and $O = \begin{Bmatrix} 9.37 \\ 5 \end{Bmatrix}$ (attached to the ground).



To complete the four-bar linkage, we need to repeat the process. We have to select a different moving pivot "$\bar{b}$" and select C_x or C_y (C is another fix pivot) and solve the design equation again.

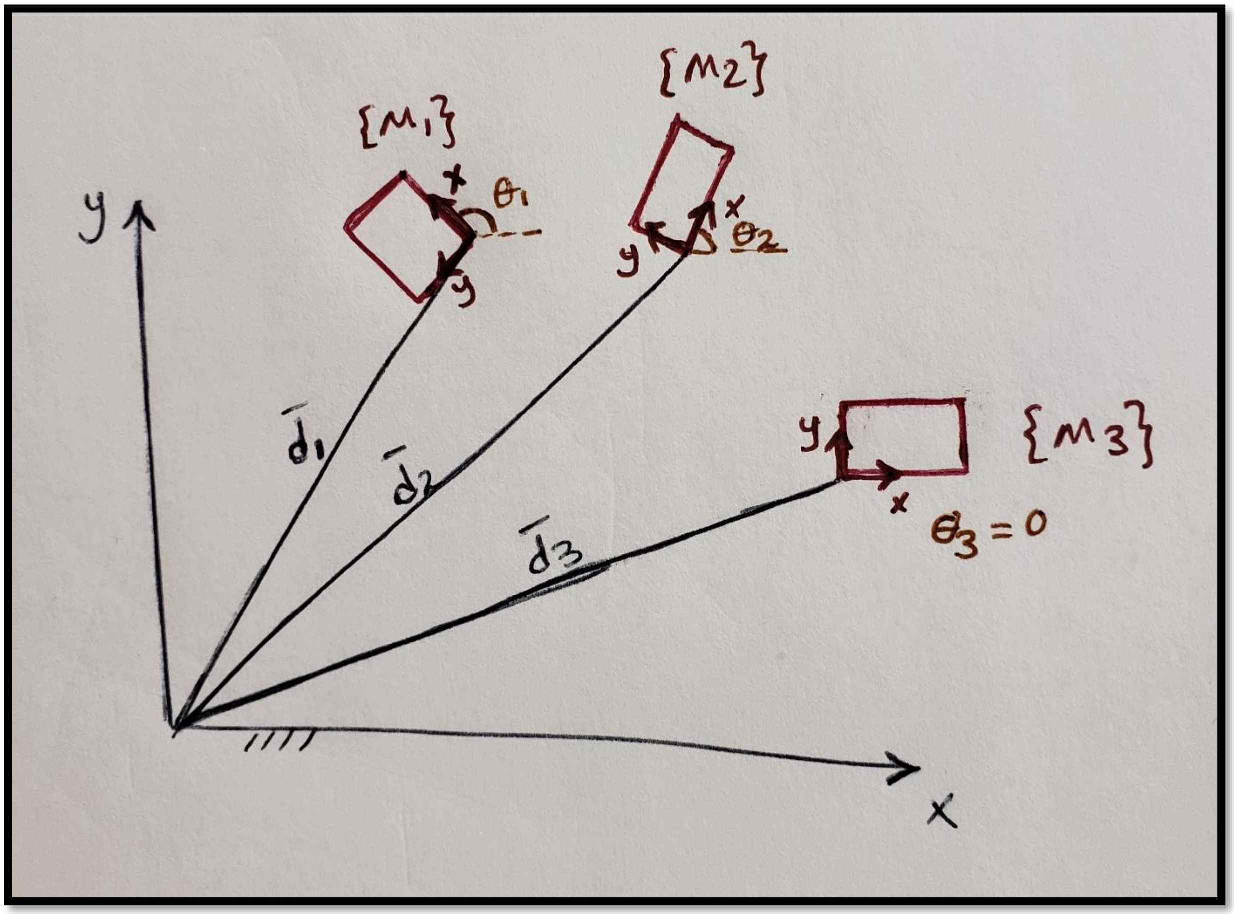
For instance:

$\bar{C} = \begin{Bmatrix} C_x \\ C_y \end{Bmatrix} = \begin{Bmatrix} 11 \\ C_y \end{Bmatrix}$ and $\bar{b} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$ and solve for C_y



Note: The benefit of using 4 bar linkage rather than a two links robot for making this motion: 1) the mechanism is stronger 2) you need only one actuator to move the object.

Algebraic 3-Position Synthesis



In this case, we have three positions and it means two design equations.

Eq1: $$\bar{A}_2.\bar{A}_2 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_2 - \bar{A}_1).\bar{O} = 0$$

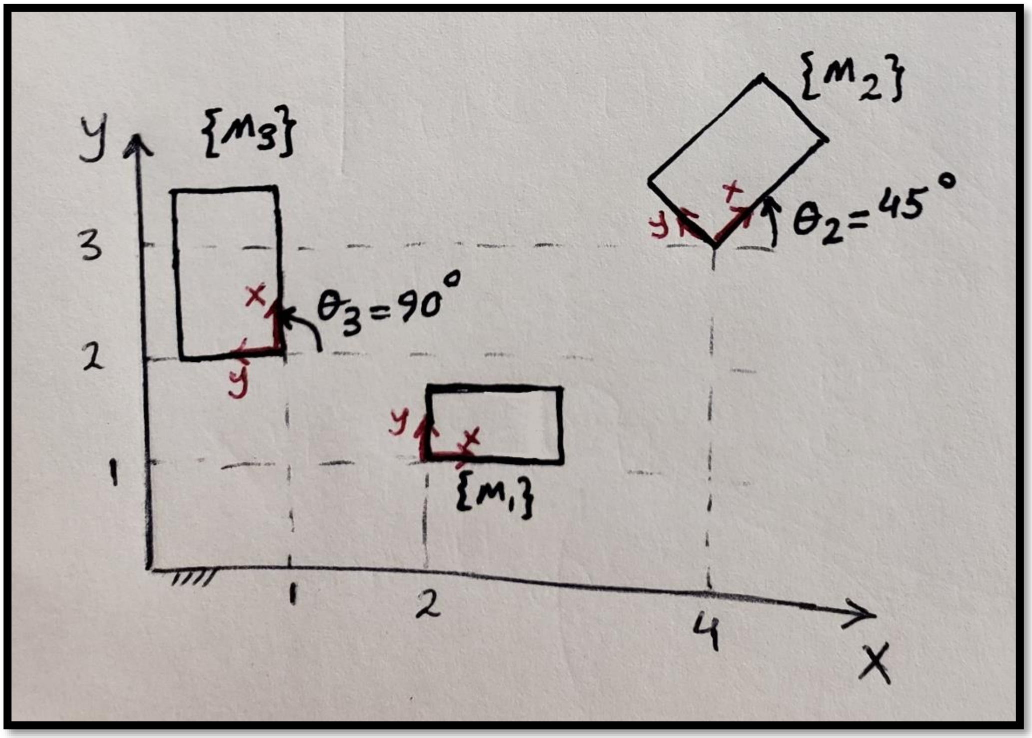
Eq2: $$\bar{A}_3.\bar{A}_3 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_3 - \bar{A}_1).\bar{O} = 0$$

Where $$\bar{A}_i = \bar{d}_i + [R(\theta_i)]\bar{a}$$

We have four unknowns (always four unknowns: two for moving pivots and two for the fixed pivots), and we can select two of these four parameters and solve for the other two.

It would be easier to select the number for parameters of moving pivot and solve for fix pivot.

Example:



$\bar{d}_1 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$, $[R(\theta_1)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(\theta_1 = 0°)$

$\bar{d}_2 = \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$, $[R(\theta_2)] = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $(\theta_2 = 45°)$

$\bar{d}_3 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, $[R(\theta_3)] = \begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $(\theta_2 = 90°)$

Do we know if the mechanism is going through these positions in the order it is presented (pos-1 to pos-2 to pos-3)? There is nothing in the equation that says that! Based on the equations we only know the 4-bar linkage will pass through all of these positions. So, after design if find the 4-bar linkage is not passing through the desired order we have to pick other values and solve equations again. (There are other methods but with using equations the only way is trial and error)

Step 1: Let's pick $\bar{a} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and solve for $\bar{O} = \begin{Bmatrix} O_x \\ O_y \end{Bmatrix}$

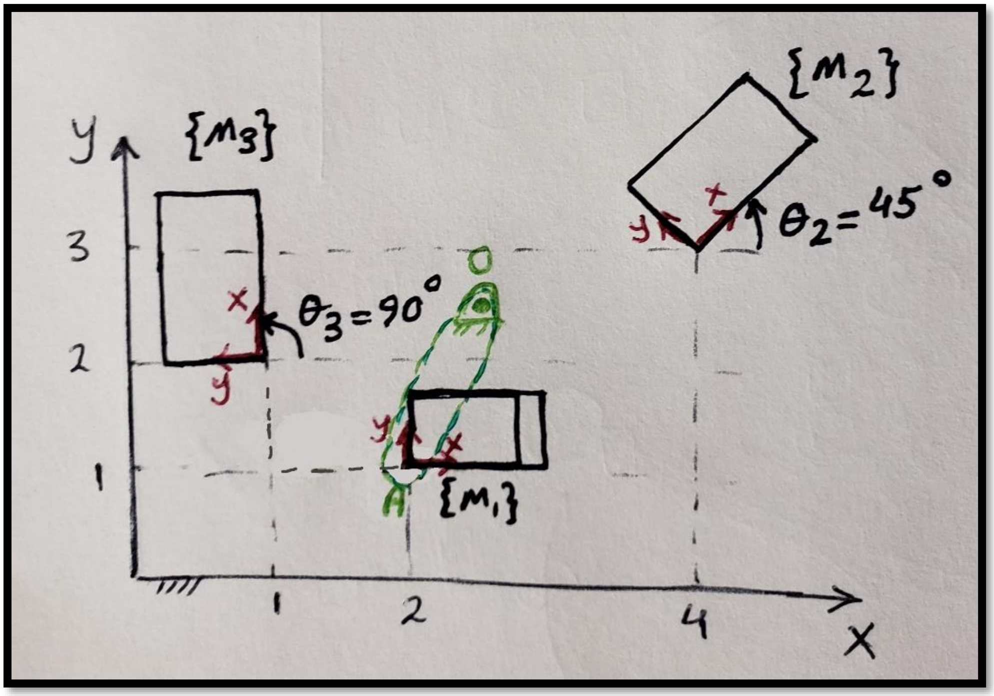
$\bar{A}_1 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$, $\bar{A}_2 = \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$, $\bar{A}_3 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

Write two design equations:

Eq1: $\bar{A}_2.\bar{A}_2 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_2 - \bar{A}_1).\bar{O} = 25 - 5 - 2\big(2\,O_x + 2\,O_y\big) = 0$

Eq2: $\bar{A}_3.\bar{A}_3 - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_3 - \bar{A}_1).\bar{O} = 5 - 5 - 2(-O_x + \,O_y) = 0$

$$\Rightarrow \begin{cases} O_x = 2.5 \\ O_y = 2.5 \end{cases}$$



Repeat the process for the output link: Let's pick $\bar{b} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ and solve for $\bar{C} = \begin{Bmatrix} C_x \\ C_y \end{Bmatrix}$

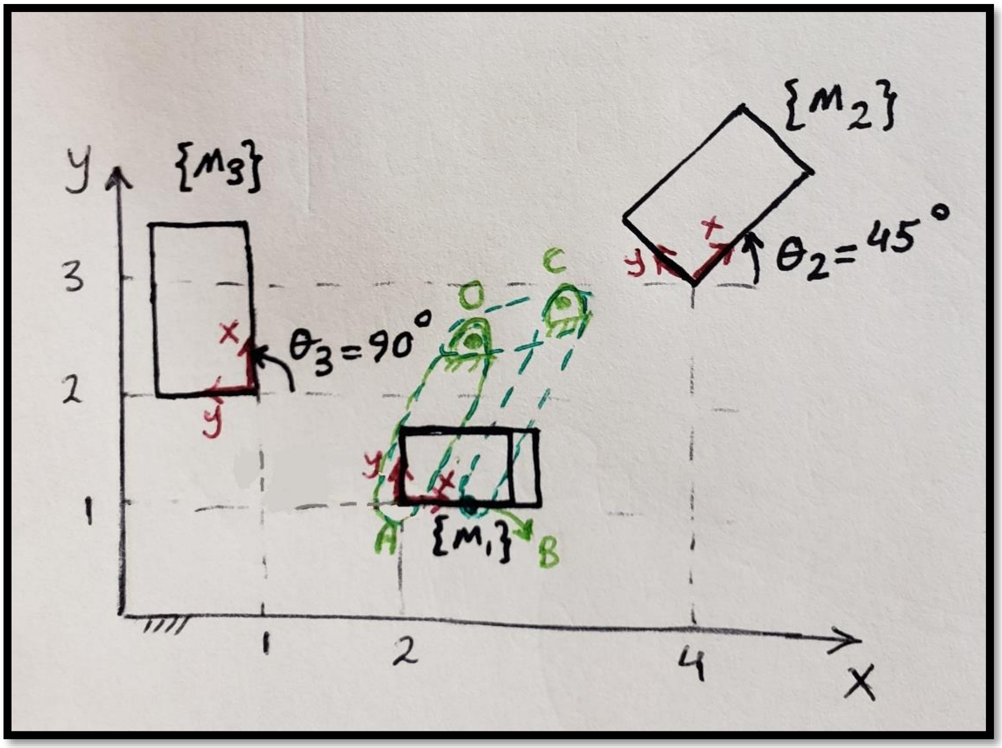
Just we will change the name of the $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ to $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ on the equations.

$\bar{B}_1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$, $\bar{B}_2 = \begin{Bmatrix} 4.7 \\ 3.7 \end{Bmatrix}$, $\bar{B}_3 = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

Eq1: $\bar{B}_2.\bar{B}_2 - \bar{B}_1.\bar{B}_1 - 2(\bar{B}_2 - \bar{B}_1).\bar{C} = 35.78 - 10 - 2\big(1.7\,C_x + 2.7\,C_y\big) = 0$

Eq2: $\bar{B}_3.\bar{B}_3 - \bar{B}_1.\bar{B}_1 - 2(\bar{B}_3 - \bar{B}_1).\bar{C} = 10 - 10 - 2(-2C_x + \,2C_y) = 0$

$$\Rightarrow \begin{cases} C_x = 2.93 \\ C_y = 2.93 \end{cases}$$



Algebraic 4-Position Synthesis

Define $(\bar{d}_i)$ & $([R(\theta_i)])$ for $i = 1,2,3,4$

That means we can define 3 design equations:

$$\bar{A}_i.\bar{A}_i - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_i - \bar{A}_1).\bar{O} = 0 \qquad i = 2,3,4$$

Select **One** parameter among $(a_x,\ a_y,\ O_x,\ O_y)$ and solve for the other three. Three quadratic equations and we need some software (MATLAB, Maple, etc.) to solve it. Also, you will have more than one solution (three quadratic equations, you will find **up to** $2 \times 2 \times 2 = 8$ solutions (it can be less)). Multiple solutions are good for us because we will have more designs to select.

Note: We have three unknown parameters of $\bar{O} = \begin{Bmatrix} O_x \\ O_y \end{Bmatrix}$ & $\bar{a} = \begin{Bmatrix} a_x \\ a_y \end{Bmatrix}$, so basically, we don't know the location of any of these points completely, and that is the reason we can't solve the problem for more than 3 positions graphically (bisector lines) and only we can solve it algebraically!

Algebraic 5-Position Synthesis

Define $(\bar{d}_i)$ & $([R(\theta_i)])$ for $i = 1,2,3,4,5$

That means we can define 4 design equations:

$$\bar{A}_i.\bar{A}_i - \bar{A}_1.\bar{A}_1 - 2(\bar{A}_i - \bar{A}_1).\bar{O} = 0 \qquad i = 2,3,4,5$$

We have 4 unknowns and that means, we can't select any of the parameters (a_x, a_y, O_x, O_y) and no free choice. Four quadratic equations and we need some software (MATLAB, Maple, etc.) to solve them. Also, you will have more than one solution (four quadratic equations, you will find **up to** $2 \times 2 \times 2 \times 2 = 16$ solutions (in reality it only has **up to 6** because of structure and it is not a completely general second-degree equation (the proof is beyond this course) and maybe some of these solutions are complex). In this case, you don't have that much flexibility, and based on the selected positions you have to pick from the solutions and if you want to change the design you have to modify the input positions (these are nonlinear equations, so a little change in input positions may a big change in the final design).

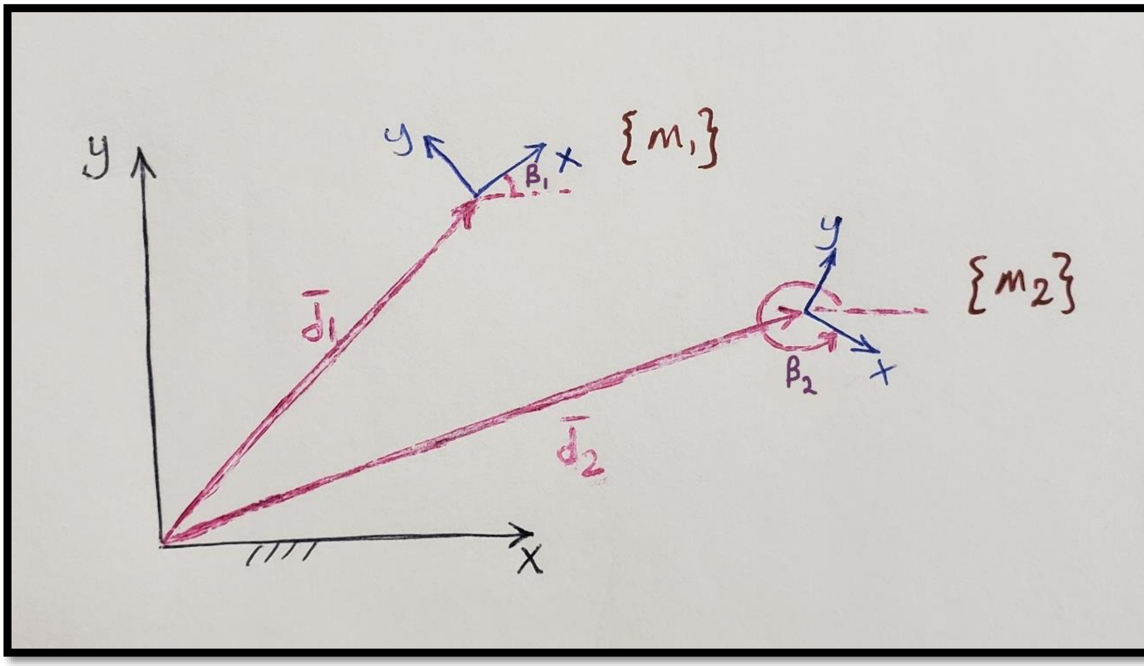
Note: **Five arbitrary positions** is the maximum limit number of positions for a 4-bar linkage!
Note: If you need to reach more positions, you can use 6 bar linkages, 8 bar linkages, etc.

Comments: After the design become complete maybe some of the requirements are not what you expect and you have to redesign the mechanism. Such as:

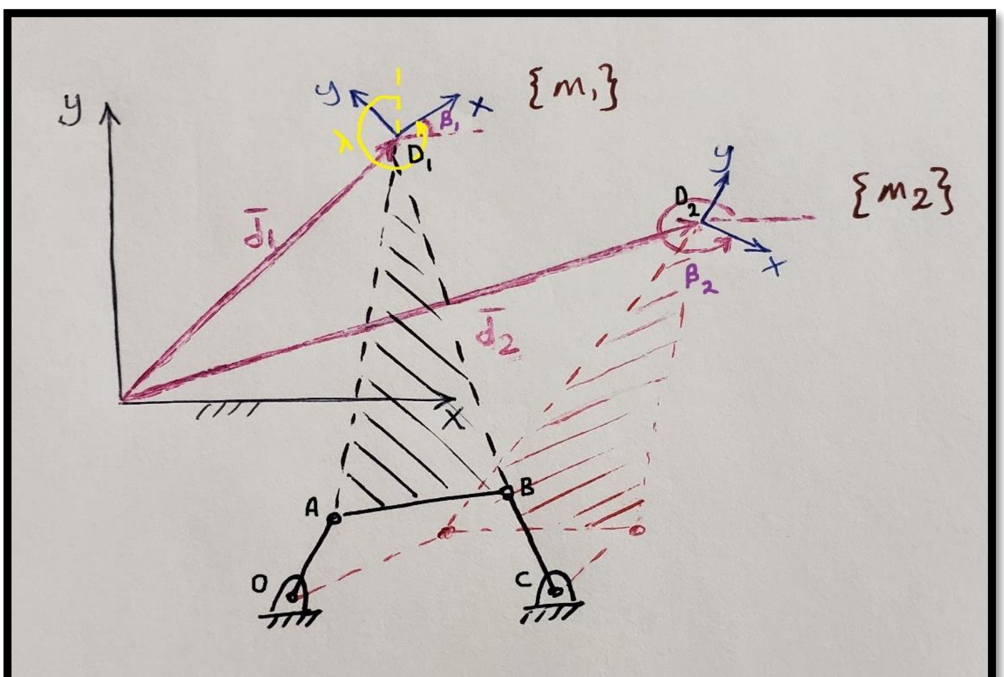
- Ordering of positions not specified.
- It may be you are passing singularity between two positions.
- It may be the mechanism can reach some of the positions in one assembly and some others in different assembly and you have to open and reassemble the mechanism (the position belong to different branches of the motion).
- Velocity and acceleration may not be the desired ones (you can add the velocity and acceleration equations to the design equations of the system).

Other Synthesis Method: Vector or Loop Equations

Just imagine that we have two positions as it is shown in the following figure.

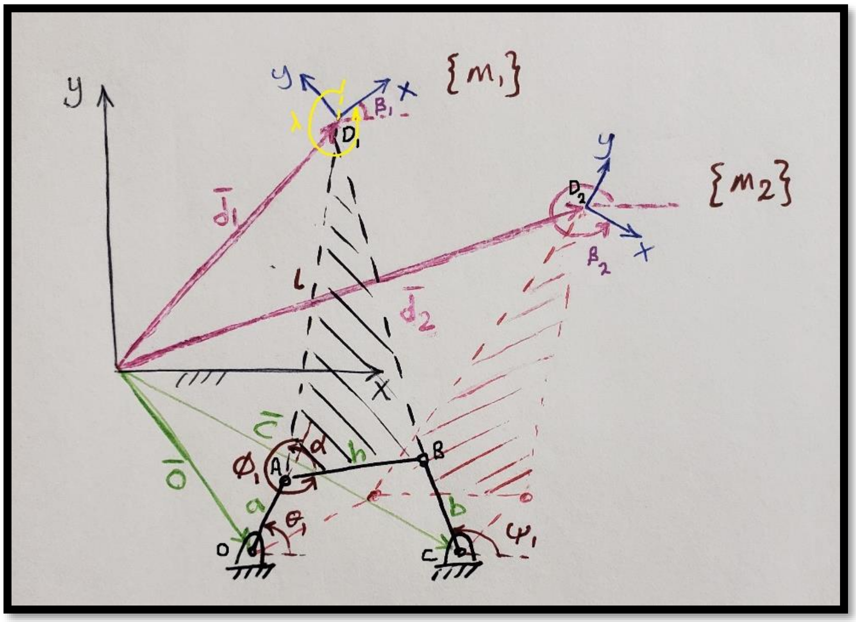


We don't know where the 4-bar linkage is, so we just sketch a random 4-bar linkage. However, we know the coupler link has to be connected to the moving frame.

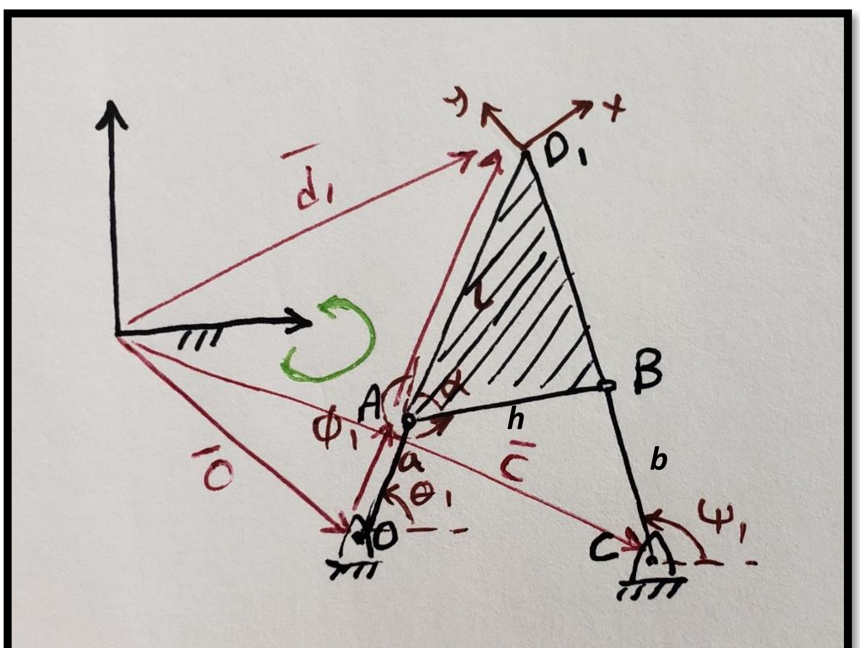


Input data: $\bar{d}_1, \beta_1, \bar{d}_2, \beta_2$

Unknown information about this 4-bar linkage: 1) We don't the location of pivot "O" 2) We don't know the location of pivot "C" 3) We don't know the link length "a" 4) We don't know the link length "h" 5) We don't know the link length "b" 6) We don't know the length of "l" 7) We don't know the angle of "α" 8) We don't know the angles "$\theta_1, \emptyset_1, \Psi_1$" for the first position 9) We don't know the angles "$\theta_2, \emptyset_2, \Psi_2$" for the second position 10) We don't know the angle "λ"



Position equation I:



Input: $\bar{d}_1, \beta_1$ (are given)

The rest of the parameters are unknown

Eq 1: Vector equation to point D_1

$$\begin{Bmatrix} O_x \\ O_y \end{Bmatrix} + \begin{Bmatrix} a\ cos\ \theta_1 \\ a\ sin\ \theta_1 \end{Bmatrix} + \begin{Bmatrix} l\ cos\ (\theta_1 + \emptyset_1 + \alpha) \\ l\ sin\ (\theta_1 + \emptyset_1 + \alpha) \end{Bmatrix} = \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$

Eq 2: This is a close mechanism so the standard loop equation for this mechanism has to be true.

$$\begin{Bmatrix} O_x \\ O_y \end{Bmatrix} + \begin{Bmatrix} a\ cos\ \theta_1 \\ a\ sin\ \theta_1 \end{Bmatrix} + \begin{Bmatrix} h\ cos\ (\theta_1 + \emptyset_1) \\ h\ sin\ (\theta_1 + \emptyset_1) \end{Bmatrix} = \begin{Bmatrix} C_x \\ C_y \end{Bmatrix} + \begin{Bmatrix} b\ cos\ (\Psi_1) \\ b\ sin\ (\Psi_1) \end{Bmatrix}$$

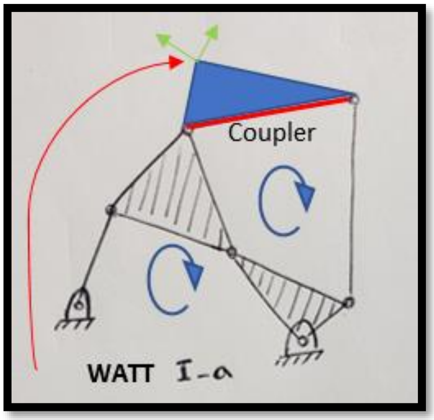
Eq 3: Angular equation

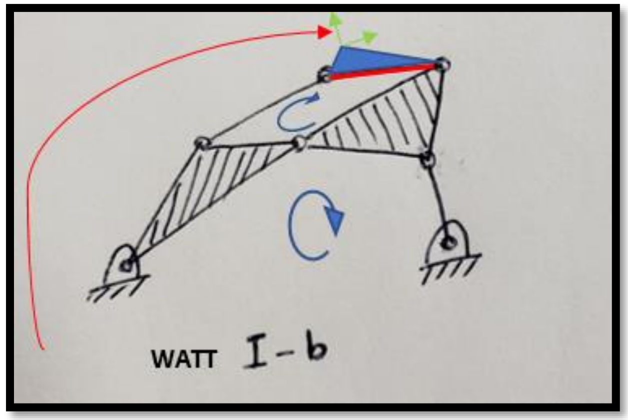
$$\theta_1 + \emptyset_1 + \alpha + \lambda = \beta_1$$

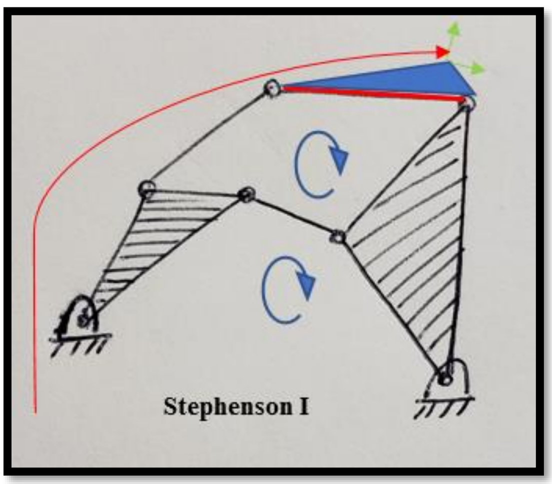
We will have 5 equations for each position. We can create 5 equations for the second position but the loop equation would be different because angles are different for a new position. For 4-bar linkage, we can write and solve these equations for a maximum of five positions (if you consider the orientation of the points but for points, without orientation you can reach more positions).

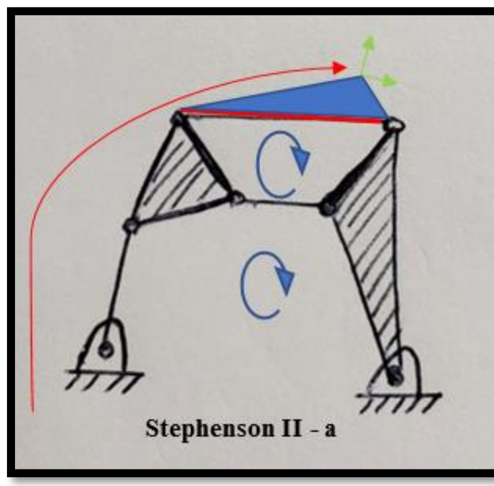
Why use this method? This method gives us a systematic way of creating design equations that we don’t have when we use simpler distance constraint equations. This is good for more complex mechanisms.

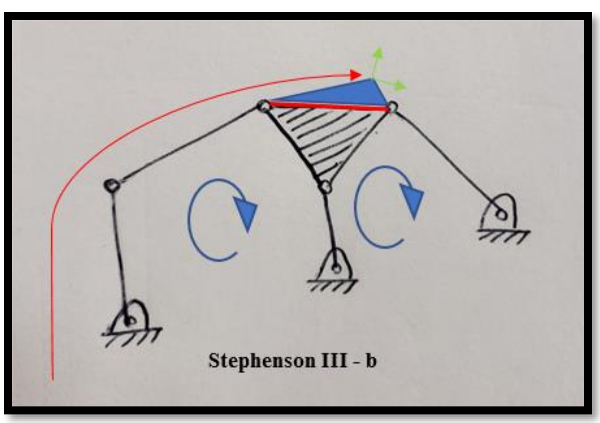
Example: 6 bar linkages (there are several designs with one D.O.F.). As you can see there are one vector equation and two loop equations for each position and it is a little complicated but more systematic way to create a design equation.









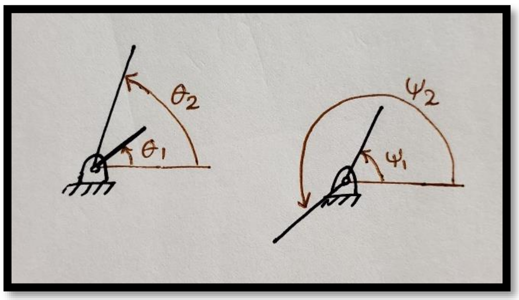


There are a few more 6-bar linkages.

Algebraic Synthesis for Input-Output Angles (Function Generation)

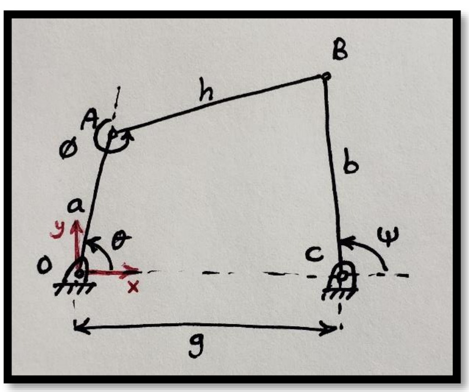
So far we worked on the rigid body guidance (guiding the rigid body through certain positions on the plain) based on the distance constraint design equation (we can use the loop design equations to solve it too). In this part, we are trying to define the output motion as the function of the input. In the other words, give a set of input/output angles.

$\boldsymbol{\theta}$	$\boldsymbol{\Psi}$
$\boldsymbol{\theta_1}$	$\boldsymbol{\Psi_1}$
$\boldsymbol{\theta_2}$	$\boldsymbol{\Psi_2}$
. . .	. . .
$\boldsymbol{\theta_n}$	$\boldsymbol{\Psi_n}$



We need to create some kind of design equation (unknown: all the dimensions of the 4bar linkage and the above table is our input data). There are several methods, but we will use Freudenstein's method.

Freudenstein's method: Loop equation



$$\begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} + \begin{Bmatrix} h\cos(\theta+\emptyset) \\ h\sin(\theta+\emptyset) \end{Bmatrix} = \begin{Bmatrix} g \\ 0 \end{Bmatrix} + \begin{Bmatrix} b\cos(\Psi) \\ b\sin(\Psi) \end{Bmatrix}$$

We want to solve for the dimension of the links: a, b, h, g

In this case, we can **scale** it, because the input-output angle relation will not change if the size of the links increases or decrease with the same ratio! So, we can start solving our problem with **one** of the links equal to one (for instance "$g = 1$") and after solving the problem, we can scale it back to any number we want. With this, we can eliminate one of the unknowns.

However, based on loop equations, we still have 4 unknowns ($a, b, h, \emptyset$). We can square the loop equations and add them up to eliminate $\emptyset$ and obtain a design equation.

$$[h \cos(\theta + \emptyset)]^2 = (1 + b \cos \Psi - a \cos \theta)^2$$
$$[h \sin(\theta + \emptyset)]^2 = (b \sin \Psi - a \sin \theta)^2$$

$$h^2 = 1 + b^2 \cos^2 \Psi + a^2 \cos^2 \theta - 2a \cos \theta + 2b \cos \Psi - 2ab \cos \Psi \cos \theta + b^2 \sin^2 \Psi + a^2 \sin^2 \theta - 2ab \sin \Psi \sin \theta$$

$$h^2 - 1 - b^2 - a^2 = -2a \cos \theta + 2b \cos \Psi - 2ab \cos \Psi \cos \theta - 2ab \sin \Psi \sin \theta$$

After simplifying:

$$\boxed{h^2 = 1 + b^2 + a^2 - 2a \cos \theta_i + 2b \cos \Psi_i - 2ab \cos(\Psi_i - \theta_i)}$$ **Design equation**

Three unknowns: a, b, h

We can solve for up to 3 sets of (θ_i, Ψ_i). If you define less, then you have to give value to one of the unknowns (links length $\boldsymbol{a}$ or $\boldsymbol{b}$ or $\boldsymbol{h}$)

Example: Let's imagine we have the relation between θ & Ψ with this equation: $\Psi = e^{\theta}$. (To find this function, we have to take many values in this function and then do approximate synthesis (pass approximately through these angles)).

We have to put three values for θ in this function and find values for Ψ.

$\boldsymbol{\theta}$	$\boldsymbol{\Psi}$
$\boldsymbol{\theta_1 = \frac{\pi}{4}}$	$\Psi_1 = 2.19$ rad
$\boldsymbol{\theta_2 = \frac{\pi}{3}}$	$\Psi_2 = 2.85$ rad
$\boldsymbol{\theta_3 = \frac{\pi}{2}}$	$\Psi_3 = 4.81$ rad

Now, just we need to put these values of θ & Ψ in the previous equation.

Eq-1: $$h^2 = 1 + b^2 + a^2 - 2a\cos\left(\frac{\pi}{4}\right) + 2b\cos(2.19) - 2ab\cos\left(2.19 - \frac{\pi}{4}\right)$$

Eq-2: $$h^2 = 1 + b^2 + a^2 - 2a\cos\left(\frac{\pi}{3}\right) + 2b\cos(2.85) - 2ab\cos\left(2.85 - \frac{\pi}{3}\right)$$

Eq-3: $$h^2 = 1 + b^2 + a^2 - 2a\cos\left(\frac{\pi}{2}\right) + 2b\cos(4.81) - 2ab\cos\left(4.81 - \frac{\pi}{2}\right)$$

Three equations and three unknowns (a, b, h)

Final answer (solved in Mathematica, MATLAB, Maple, etc.):

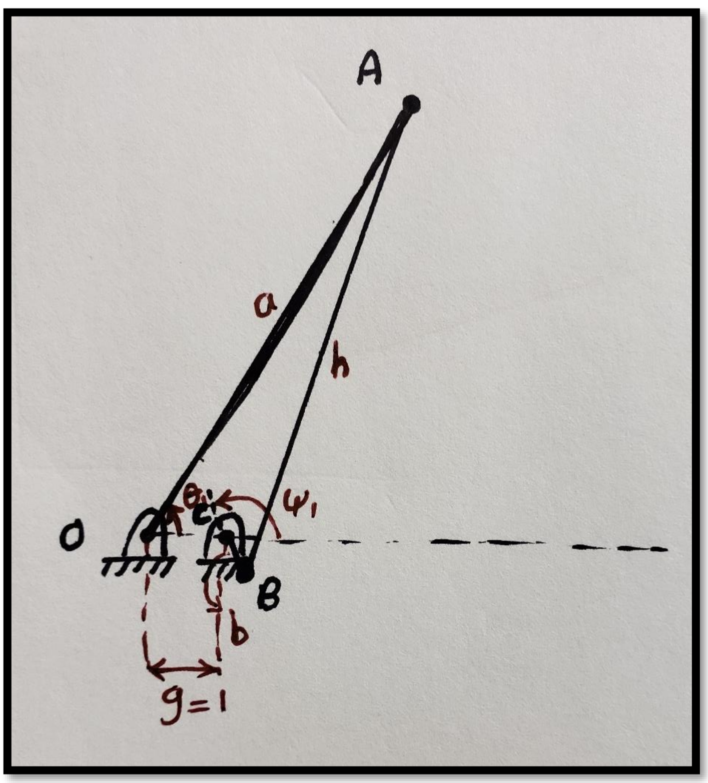
$a = 10.7$

$b = -0.6$

$h = 10.2$

What is the meaning of the negative link length (link b in this case)?

Let's take a look at the 4bar linkage for the first θ value.



The negative value of "b" means, the link $\overline{CB}$ is in the opposite direction of the angle of Ψ (with π difference).

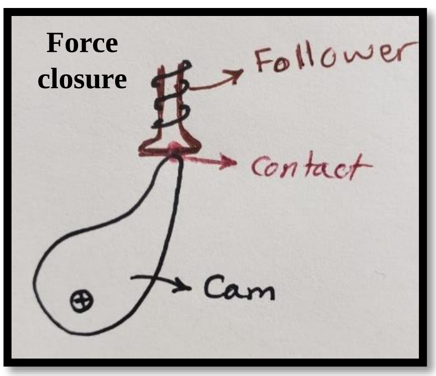
Cam-Follower Systems

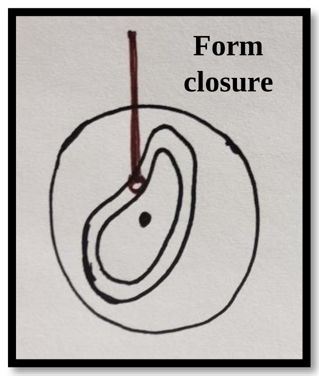
In linkages, if we want to make a more complex motion we need to add more links and that makes the mechanism more complicated (e.g. 4-bar linkage maximum can go through 5 positions). With a single cam, you can generate different motions. One typical motion that you can create by cam is “dual motion” which means you will move in part of your cycle and then stay with no motion for part of your cycle (that is very hard to create with linkages). In the cam, it is so easy and if you just keep the radius of the cam constant, that means the follower will not move.

https://www.youtube.com/watch?v=HsXWewecMLE

Cam-Follower systems: Encode motion on the cam profile (compact solution)

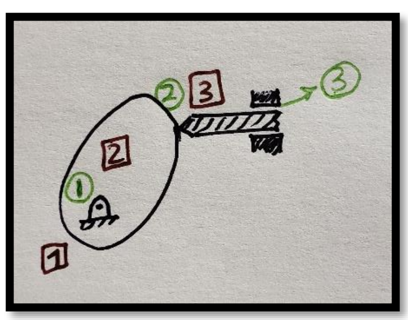
We need to keep the contact between cam and follower to have the motion based on the cam profile. For keeping the contact between cam and follower we need some force there. Two common solutions, in this case, are using a spring to apply force or making a slot on the cam to force the follower to go through that pattern.





The cam-follower systems have some problems compared to linkages, such as more friction and dynamic effects (for instance in the spring at high speed)

What is the mobility of Cam-Follower systems?



n = 3 , j = 3 , $f_i = 1$ (except joint #2)

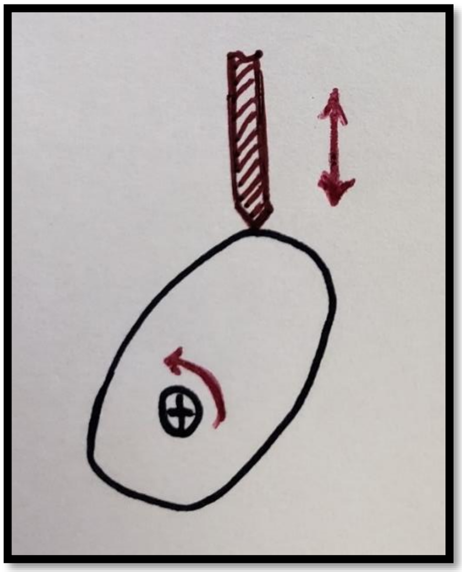
$$M = 3(n-1) - \sum_{i=1}^{j}(3 - f_i) = 3 \times (3-1) - 2 \times (3-1) - 1 \times (3-2) = 6 - 4 - 1 = 1$$

There are many different ways to classify cam-follower systems. These classifications will affect follower displacement motion.

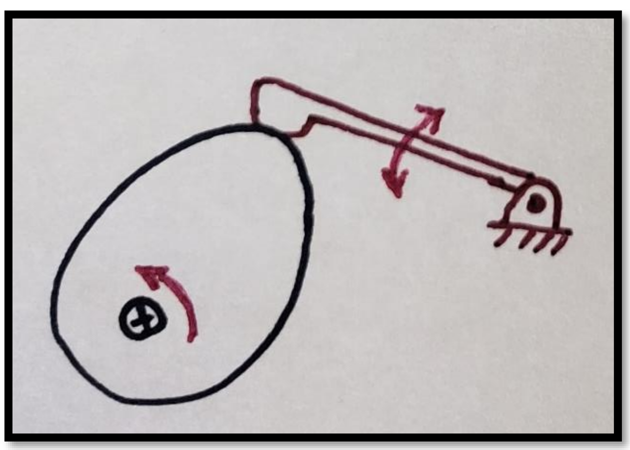
A) Classify based on the type of motion.

There are two types of motions:

1) Translating: cam rotate(input motion) but the follower translates on a straight line



2) Oscillating: cam rotate(input motion) and the follower will oscillate with some angles



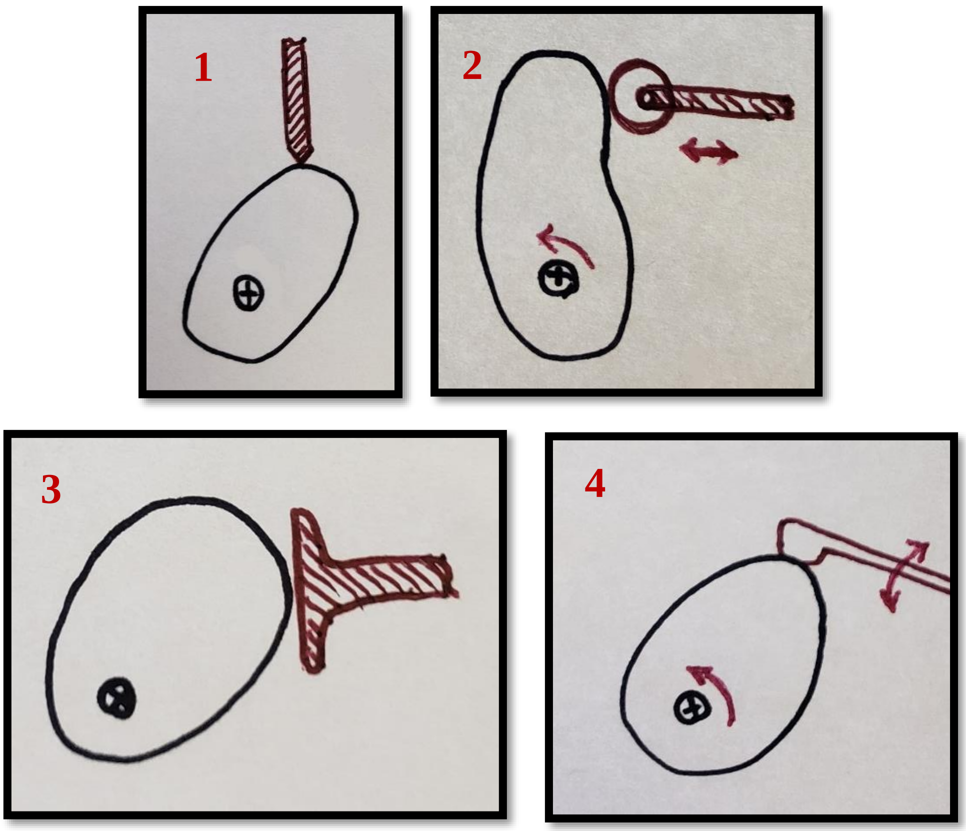
B) Classify based on types of followers.

1) Knife-Edged Follower: The follower will contact the cam only at one point.

2) Roller Follower: The roller is allowed to rotate (an idle degree of freedom)

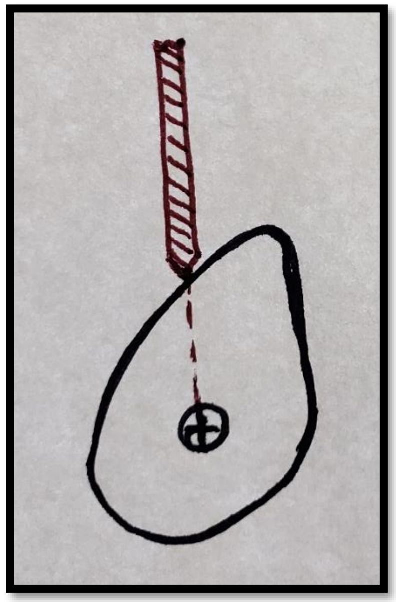
3) Flat-Faced Follower: Plane contacts cam

4) Cylindrical or Curved Follower: The follower has a particular shape for making a specific motion

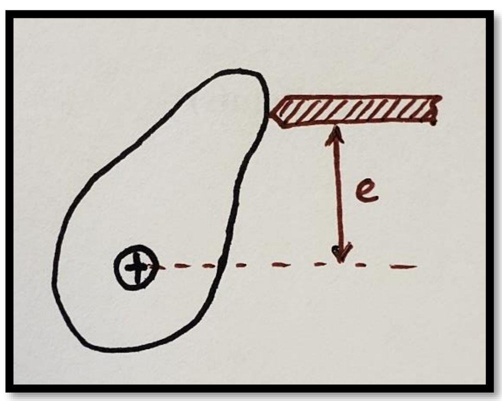


C) Classify based on the type of assembly: It is related to how the follower is placed with respect to the cam pin.

1) Radial: The follower is located aline with the pin joint of the cam



2) Offset: There is a distance between the follower and the pin of the cam

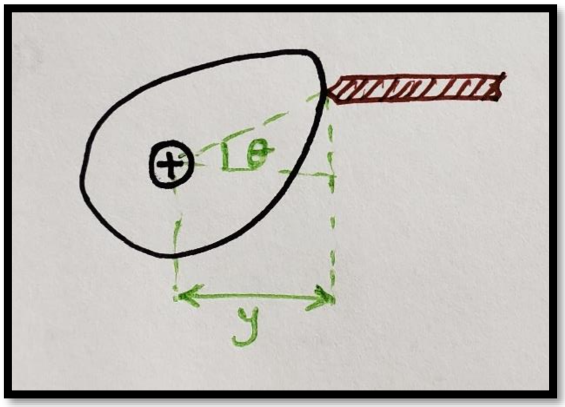


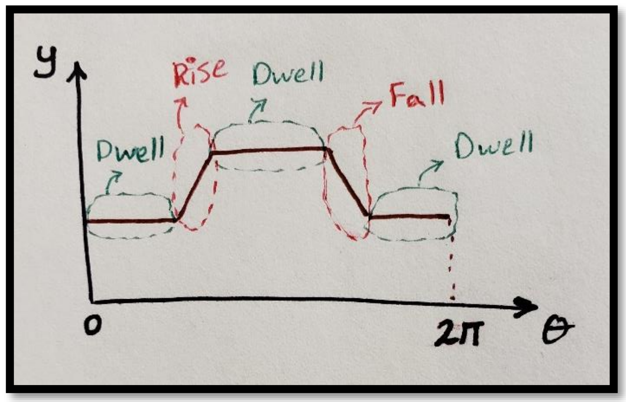
Follower Displacement Motion

The motion usually has:

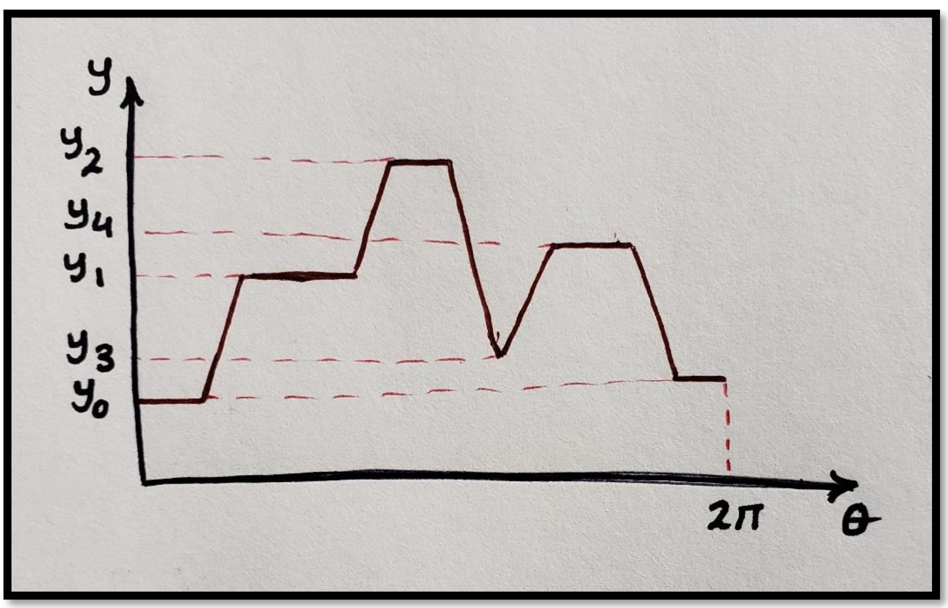
1) Rise (increase displacement)
2) Dwell (constant displacement)
3) Return/Fall (decrease displacement)

Cam in most of the applications fully rotates. So, one cycle would be a rotation of the cam. We are going to plot the motion of the follower (y) as a function of the angle of the cam (θ).

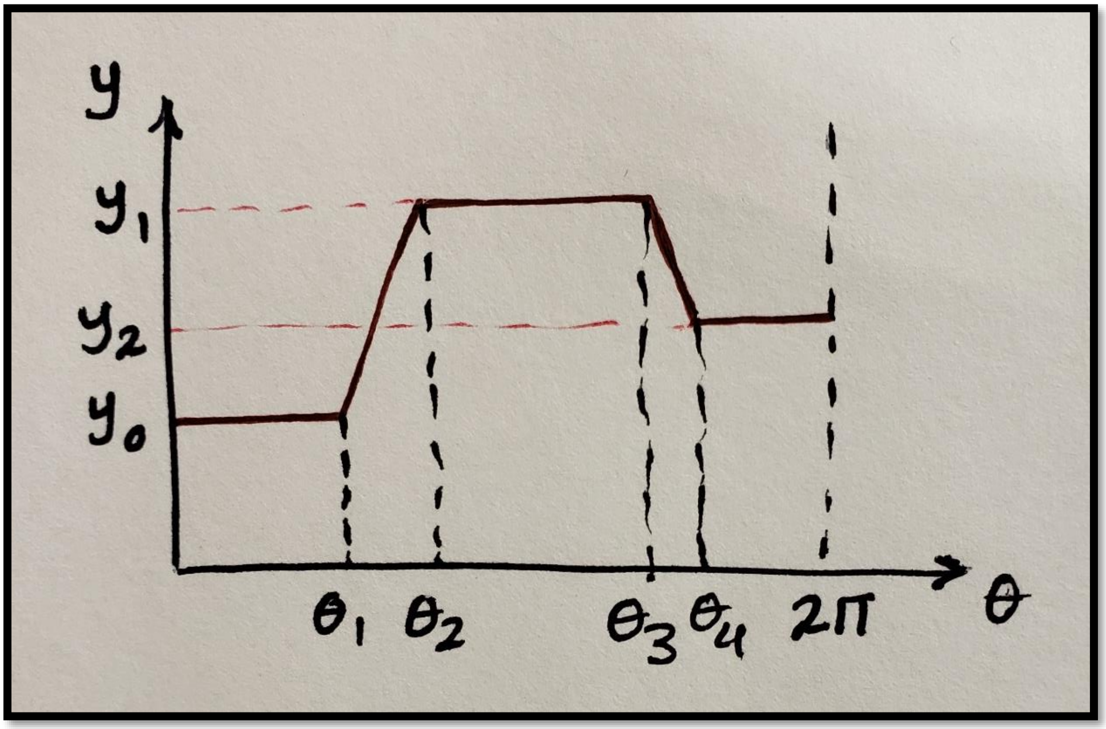




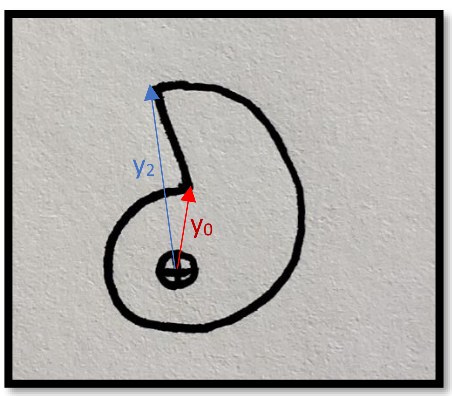
This motion based on the designed cam-follower can be completely different but in most cases at the end of the cycle, the follower has to back to the start point (y_0).



- What is the cam shape for the following plot?



Something like the following shape with a jump.



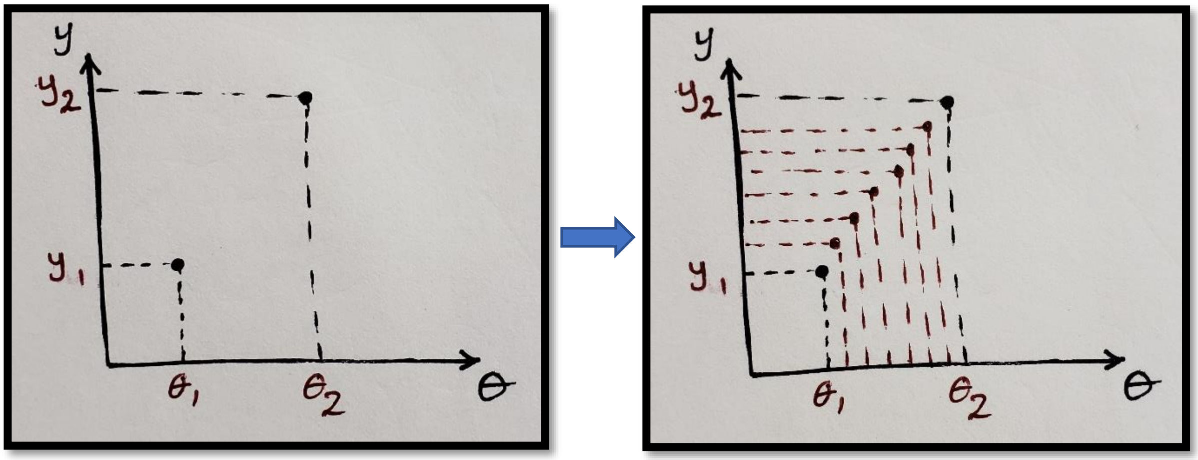
Piecewise function $y(\theta)$:

$$y(\theta) = \begin{cases} y_0 & 0 \le \theta < \theta_1 \\ Polynomial & \theta_1 \le \theta < \theta_2 \\ y_1 & \theta_2 \le \theta < \theta_3 \\ Polynomial & \theta_3 \le \theta < \theta_4 \\ y_2 & \theta_4 \le \theta < 2\pi \end{cases}$$

Note: The rise and return functions are very curtailed in high-speed applications and usually these are some of the industrial secrets of the companies (effect on the increased efficiency, etc.).

How do define the rise and the return functions? There are a couple of methods.

Just imagine we have the following plot and we want to reach from y_1 to y_2. One way of doing that is defining the set of points along the rise and defining the shape of the curve pointwise.



Based on these points we can define the constants of the polynomial.

$$y(\theta) = c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + \cdots$$

$$(c_0, c_1, c_2, c_3, c_4, \ldots): Coefficients\ shape\ the\ Polynomial$$

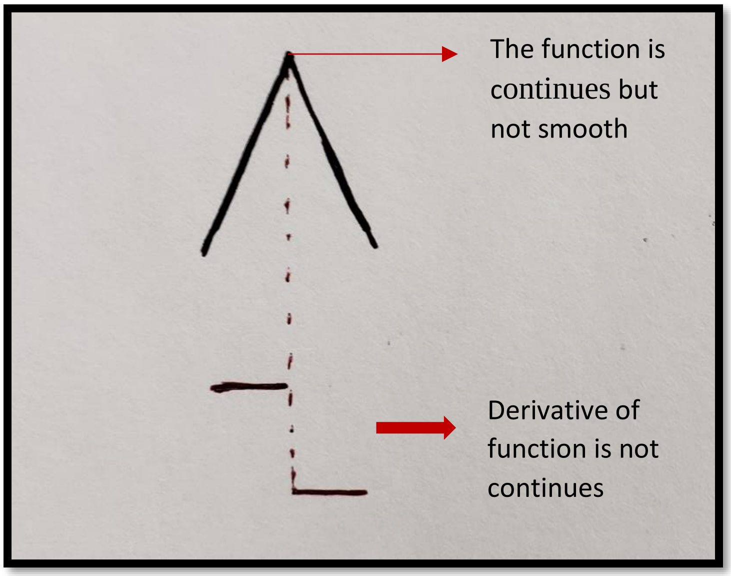
In the above example, we defined 8 points (6 points between the start and end points of the rise), so we can solve for 8 of these coefficients and have a 7-degree polynomial.

$$\left.\begin{aligned} y(\theta_1) &= c_0 + c_1\theta_1 + c_2{\theta_1}^2 + c_3{\theta_1}^3 + c_4{\theta_1}^4 + \cdots = y_1 \\ y(\theta_2) &= c_0 + c_1\theta_2 + c_2{\theta_2}^2 + c_3{\theta_2}^3 + c_4{\theta_2}^4 + \cdots = y_2 \\ &\vdots \end{aligned}\right\} \text{Set of linear equations in } C_i's$$

So if we have the plot and curve with this method we can find the mathematical function (polynomial) of that curve. As much as more points we have, we can find a better fit polynomial to the curve.

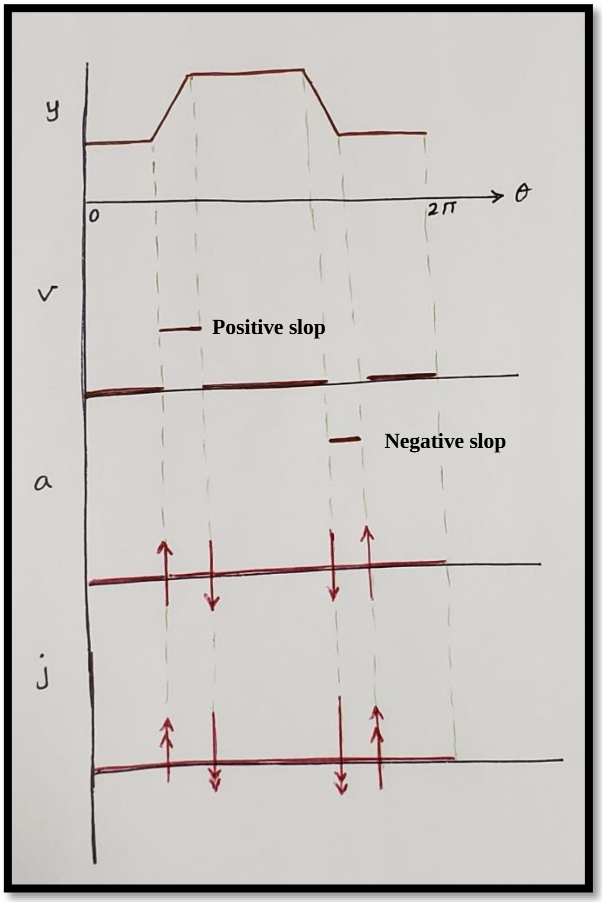
Defining boundary conditions for continuity and smoothness:

Another method for defining the rise/return curves is based on defining the boundary conditions. Continuity in the cam means there is not a big jump, in other words, there is not a big difference between "y" values of points before and after each θ value. Smoothness is different from continuity. For having a smooth motion, the derivative of the function has to be a continuous function! In general, we want to impose continuous displacement, velocity, acceleration, and third derivative (jerk) in cams. Jerk is related to impact. If you want to minimize the impact, you need to have continuous jerk.


The function is
continues but
not smooth
Derivative of
function is not
continues

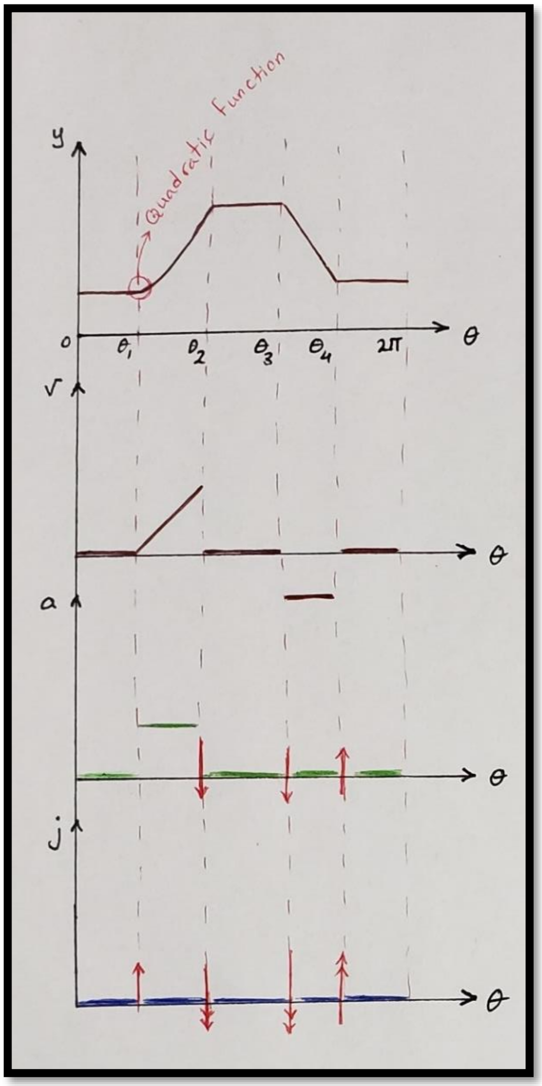
Cam-Follower Systems (Continue)

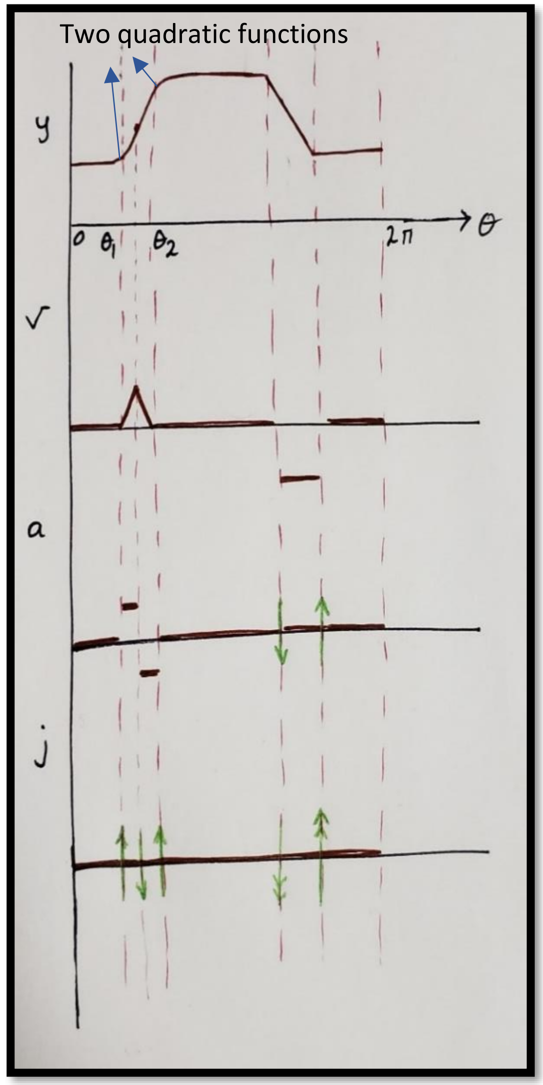
Example: In the following plot, we have a displacement function of a cam. Plot the velocity, acceleration, and jerk for this function. Which one is continuous and which one is discontinuous?



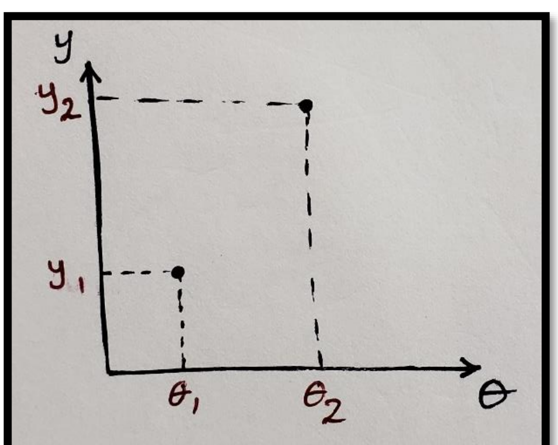
In this particular cam, we accomplish the continuity of the displacement but we don't have smoothness because the velocity curve is not continuous. In the acceleration plot, for zero velocity we will have zero acceleration. Also, for the constant velocity, we have zero acceleration but we have to have a jump in switching the velocity between zero and a constant value. This jump is shown with an arrow. In reality, it cannot happen because the physical system does not allow that, you will have a very deep change (peak) at that point. These jump points are the points where dynamic effects happen and they are important for us and have to be shown in plots. For the jerk plot, at the same points of the jump for acceleration, we will have very high jumps in jerk which are shown with double arrows.

Example: In the following plot, we have a displacement function of a cam. Plot the velocity, acceleration, and jerk for this function. Which one is continuous and which one is discontinuous?





Example: In the following plot we only know information about the initial and final point of a rise (we don't know anything about before and after these points).



The boundary conditions for continuous displacement

$$y(\theta_1) = y_1$$

$$y(\theta_2) = y_2$$

The boundary conditions for continuous velocity

$$v(\theta_1) = v_1$$

$$v(\theta_2) = v_2$$

Note: The v_1 & v_2 are the values of the velocity, so that $v(\theta)$ is continuous.

The boundary conditions for continuous acceleration

$$a(\theta_1) = a_1$$

$$a(\theta_2) = a_2$$

If "y" is the displacement function, velocity would be:

$$v = \dot{y} = \frac{dy}{d\theta}.\frac{d\theta}{dt} = y'.\dot{\theta}$$

$\dot{\theta}$: Angular velocity of the cam which is usually constant for cams because we can change the speed based on the cam profile.

For acceleration, we will have:

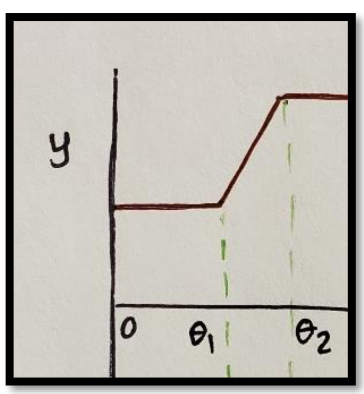
$$a = \ddot{y} = y''.\dot{\theta}^2 + \overset{0\ (\dot{\theta}\, =\, \text{constant})}{\cancel{y'.\ddot{\theta}}} = y''.\dot{\theta}^2$$

For jerk, we will have:

$$j = y'''.\dot{\theta}^3$$

Polynomial functions for the rise and return:

To make it easier to solve the rise (return) polynomial function, we will use **standard scaled and shifted polynomial functions**.

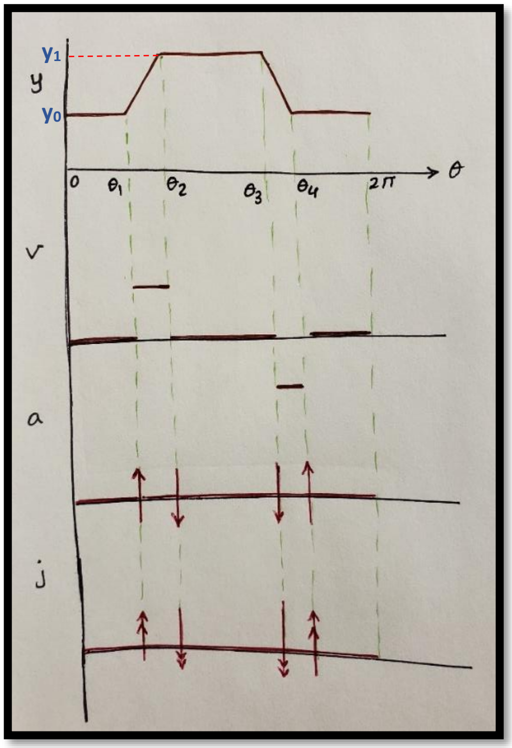


Rather than: $$y(\theta) = c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + c_4\theta^4 + \cdots$$

We will write it like this: $y\,(\theta) = c_0 + c_1\frac{\theta-\theta_1}{\theta_2-\theta_1} + c_2\frac{(\theta-\theta_1)^2}{(\theta_2-\theta_1)^2} + c_3\frac{(\theta-\theta_1)^3}{(\theta_2-\theta_1)^3} + c_4\frac{(\theta-\theta_1)^3}{(\theta_2-\theta_1)^3} + \cdots$

- $(\theta_2 - \theta_1)$: It is a constant value

A) Uniform Motion: it includes straight lines and is good for low-speed motion.



For uniform motion, the only condition is continuity of displacement.

For the rise, we have these two conditions:

$$y(\theta_1) = y_0$$
$$y(\theta_2) = y_1$$

What would be the degree of the polynomial? We have two conditions and we can only find two coefficients. You have to trim the polynomial according to the number of boundary conditions or points that you have.

$$y(\theta) = c_0 + c_1\frac{\theta - \theta_1}{\theta_2 - \theta_1}$$

- Also, if you look at the plot, for the rise, we have a line, and the above equation is a line equation!

c_0 & c_1: Unknown coefficients

θ_1 , θ_2 , y_0 , y_1: Known parameters

Impose the boundary conditions:

$$y(\theta_1) = c_0 + c_1 \cancelto{0}{\frac{\theta_1 - \theta_1}{\theta_2 - \theta_1}} = y_0 \Rightarrow \boxed{c_0 = y_0}$$

$$y(\theta_2) = c_0 + c_1 \cancelto{1}{\frac{\theta_2 - \theta_1}{\theta_2 - \theta_1}} = y_0 + c_1 = y_1 \Rightarrow \boxed{c_1 = y_1 - y_0}$$

For the return, we can repeat the same method. For the return, we have these two conditions:

$$y(\theta_3) = y_1$$

$$y(\theta_4) = y_0$$

The polynomial for the return would be:

$$y(\theta) = k_0 + k_1 \frac{\theta - \theta_3}{\theta_4 - \theta_3}$$

Impose the boundary conditions and solve for k_0 & k_1:

$$y(\theta_3) = k_0 + k_1 \cancelto{0}{\frac{\theta_3 - \theta_3}{\theta_4 - \theta_3}} = y_1 \Rightarrow \boxed{k_0 = y_1}$$

$$y(\theta_4) = k_0 + k_1 \cancelto{1}{\frac{\theta_4 - \theta_3}{\theta_4 - \theta_3}} = y_1 + k_1 = y_0 \Rightarrow \boxed{k_1 = y_0 - y_1}$$

Now, we have our polynomial function for the whole displacement!

$$y(\theta) = \begin{cases} y_0 & 0 \le \theta < \theta_1 \\ y_0 + (y_1 - y_0)\dfrac{\theta - \theta_1}{\theta_2 - \theta_1} & \theta_1 \le \theta < \theta_2 \\ y_1 & \theta_2 \le \theta < \theta_3 \\ y_1 - (y_1 - y_0)\dfrac{\theta - \theta_3}{\theta_4 - \theta_3} & \theta_3 \le \theta < \theta_4 \\ y_0 & \theta_4 \le \theta < 2\pi \end{cases}$$

This is the mathematical model of the cam profile.

Let's take the derivative and write the velocity function.

$$v(\theta) = \dot{y}(\theta) = \frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dy}{d\theta} \cdot \omega$$

$$v(\theta)=\begin{cases} 0 & 0\le\theta<\theta_1 \\ \dfrac{(y_1-y_0)}{\theta_2-\theta_1}.\dot{\theta} & \theta_1\le\theta<\theta_2 \\ 0 & \theta_2\le\theta<\theta_3 \\ -\dfrac{(y_1-y_0)}{\theta_4-\theta_3}.\dot{\theta} & \theta_3\le\theta<\theta_4 \\ 0 & \theta_4\le\theta<2\pi \end{cases}$$

If you check the plot of velocity for uniform motion, you will see the same results. Remember, in sections 2 & 4, the numerator of the velocity function is the same, based on the intervals $(\theta_2-\theta_1)$ & $(\theta_4-\theta_3)$, each one is bigger, and the velocity of that section would be smaller.

If we do the second derivative (acceleration),

$$a(\theta)=\begin{cases} 0 & 0\le\theta<\theta_1 \\ 0 & \theta_1\le\theta<\theta_2 \\ 0 & \theta_2\le\theta<\theta_3 \\ 0 & \theta_3\le\theta<\theta_4 \\ 0 & \theta_4\le\theta<2\pi \end{cases}$$

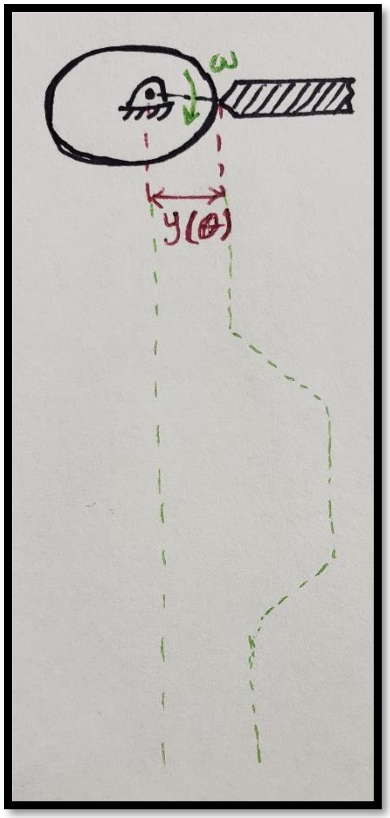
Would be the same for the third derivative (jerk):

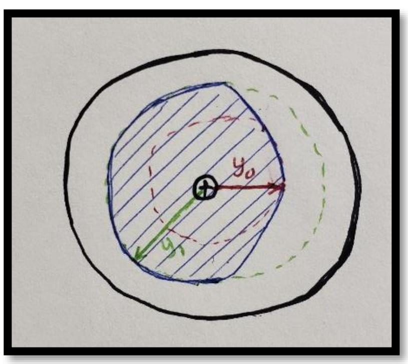
$$j(\theta)=\begin{cases} 0 & 0\le\theta<\theta_1 \\ 0 & \theta_1\le\theta<\theta_2 \\ 0 & \theta_2\le\theta<\theta_3 \\ 0 & \theta_3\le\theta<\theta_4 \\ 0 & \theta_4\le\theta<2\pi \end{cases}$$

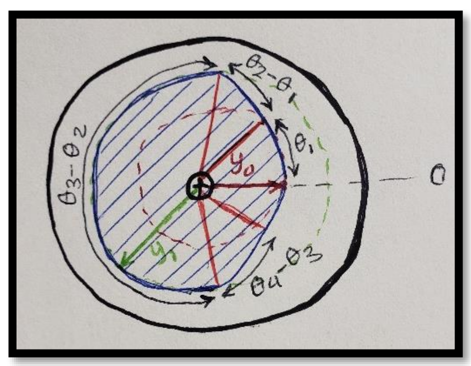
Remember to add the arrows (or double arrows) in your plots. Those are super important (they mean discontinuity). If you use MATLAB, Maple, etc. for plots, you have to add the arrows by hand.

- Do you think a cam with "Uniform Motion" would be a good design for a cam?
  Depends on the application. That may work for low-speed applications but for high-speed applications, it is not good because of the big impact effects.

- What would the cam profile look like for the previous example?


ω
y(θ)


y₀
y₁

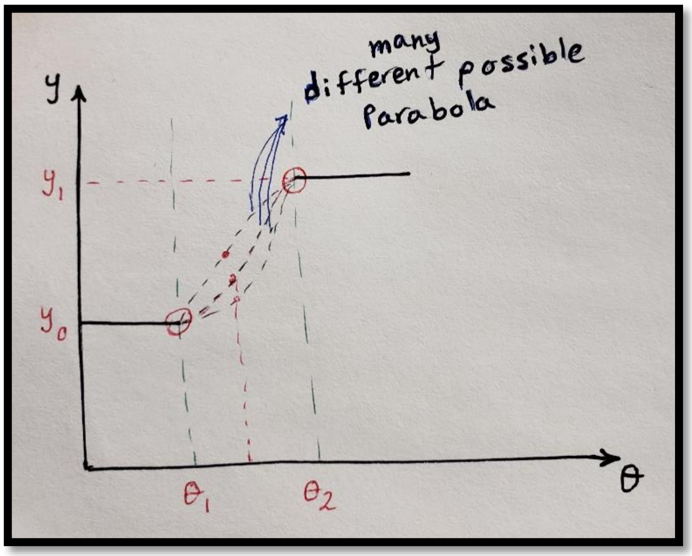

θ₂-θ₁
θ₁
y₀
0
θ₃-θ₂
y₁
θ₄-θ₃

Cam-Follower Systems (Continue)

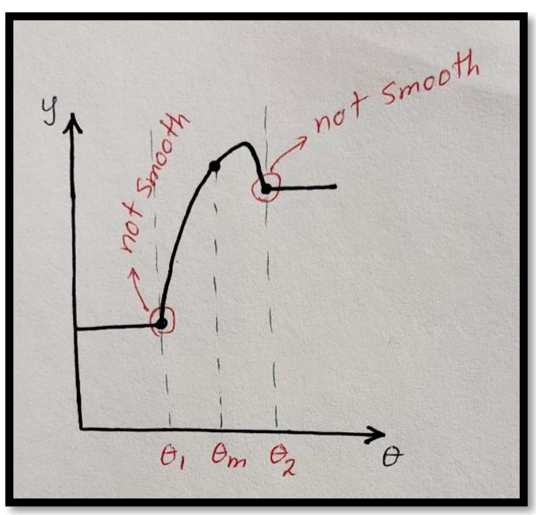
B) Parabolic Motion: It is a 2nd-degree polynomial

$$y(\theta) = c_0 + c_1 \frac{\theta - \theta_1}{\theta_2 - \theta_1} + c_2 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^2$$

We have different options to find the three coefficients for this parabola.



- Defining three points: $y(\theta_1)$, $y(\theta_2)$, and one point in the middle $y(\theta_m)$. But then we cannot able to check the smoothness of the boundary points. In other words, may we have 2nd order polynomial and continuous at both ends but it is not smooth!



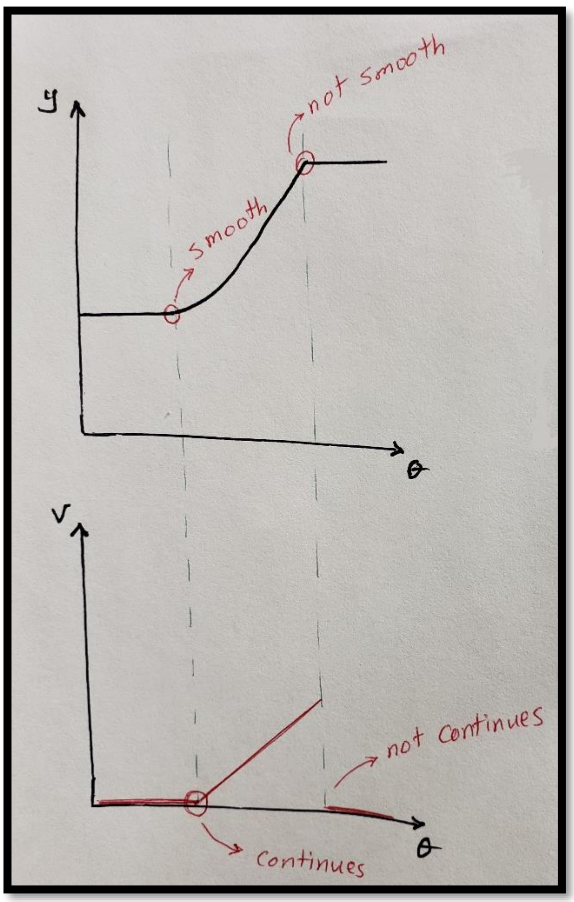
- Using boundary conditions: in this case, we need three boundary conditions

$$y(\theta_1) = y_0$$
$$y(\theta_2) = y_1$$

But we need one more boundary condition. It can be related to the velocity of the start or end point.

$$v(\theta_1) = 0$$

If we want to have a smooth motion, we need the velocity before and after that specific point (in this case θ_1) to be the same. In the following example, we have the velocity before and after θ_1 equal zero ($v_0 = 0$).



But for 2nd order polynomial, we can have control over one of the velocities (start or end point). Therefore, one of them become not smooth! If we want to have smoothness at both ends, we need to use a higher degree polynomial.

C) 3rd-degree polynomial

$$y(\theta) = c_0 + c_1 \frac{\theta - \theta_1}{\theta_2 - \theta_1} + c_2 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^2 + c_3 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^3$$

In this case, we will have 4 boundary conditions:

$$y(\theta_1) = y_0$$

$$y(\theta_2) = y_1$$

$$v(\theta_1) = 0$$

$$v(\theta_2) = 0$$

Solve for the coefficients:

$$y(\theta_1) = y_0 = c_0 + c_1 \overbrace{\frac{\theta_1 - \theta_1}{\theta_2 - \theta_1}}^{0} + c_2 \left(\overbrace{\frac{\theta_1 - \theta_1}{\theta_2 - \theta_1}}^{0}\right)^2 + c_3 \left(\overbrace{\frac{\theta_1 - \theta_1}{\theta_2 - \theta_1}}^{0}\right)^3$$

$$\boxed{c_0 = y_0} \qquad 1$$

$$y(\theta_2) = y_1 = y_0 + c_1 \overbrace{\frac{\theta_2 - \theta_1}{\theta_2 - \theta_1}}^{1} + c_2 \left(\overbrace{\frac{\theta_2 - \theta_1}{\theta_2 - \theta_1}}^{1}\right)^2 + c_3 \left(\overbrace{\frac{\theta_2 - \theta_1}{\theta_2 - \theta_1}}^{1}\right)^3$$

$$\boxed{y_1 = c_0 + c_1 + c_2 + c_3} \qquad 2$$

Take derivative to find the $v(\theta)$:

$$v(\theta) = \left(\frac{c_1}{\theta_2 - \theta_1} + \frac{2\, c_2 (\theta - \theta_1)}{(\theta_2 - \theta_1)^2} + \frac{3\, c_3 (\theta - \theta_1)^2}{(\theta_2 - \theta_1)^3}\right) \dot{\theta}$$

$$v(\theta_1) = 0 = \left(\frac{c_1}{\theta_2 - \theta_1} + \overbrace{\frac{2\, c_2 (\theta_1 - \theta_1)}{(\theta_2 - \theta_1)^2}}^{0} + \overbrace{\frac{3\, c_3 (\theta_1 - \theta_1)^2}{(\theta_2 - \theta_1)^3}}^{0}\right) \dot{\theta}$$

$$\frac{c_1}{\theta_2 - \theta_1} \dot{\theta} = 0 \Rightarrow \boxed{c_1 = 0} \qquad 3$$

$$v(\theta_2) = 0 = \left(\frac{c_1}{\theta_2 - \theta_1} + \frac{2\, c_2 \cancel{(\theta_2 - \theta_1)}}{(\theta_2 - \theta_1)^{\cancel{2}}} + \frac{3\, c_3 \cancel{(\theta_2 - \theta_1)^2}}{(\theta_2 - \theta_1)^{\cancel{3}}}\right) \dot{\theta}$$

$$(c_1 + 2\, c_2 + 3\, c_3) \frac{\dot{\theta}}{\theta_2 - \theta_1} = 0 \Rightarrow \boxed{c_1 + 2\, c_2 + 3\, c_3 = 0} \qquad 4$$

So,

$$c_0 = y_0$$

$$c_1 = 0$$

$$2\,c_2 + 3\,c_3 = 0 \Longrightarrow c_3 = -\tfrac{2}{3}c_2$$

$$c_2 + c_3 = y_1 - y_0 \Longrightarrow c_2 - \tfrac{2}{3}c_2 = y_1 - y_0 \Longrightarrow \tfrac{1}{3}c_2 = y_1 - y_0 \Longrightarrow \boxed{c_2 = 3(y_1 - y_0)}$$

$$c_3 = -\frac{2}{3} \times 3(y_1 - y_0) = \boxed{-2(y_1 - y_0)}$$

Therefore, for the rise we will have:

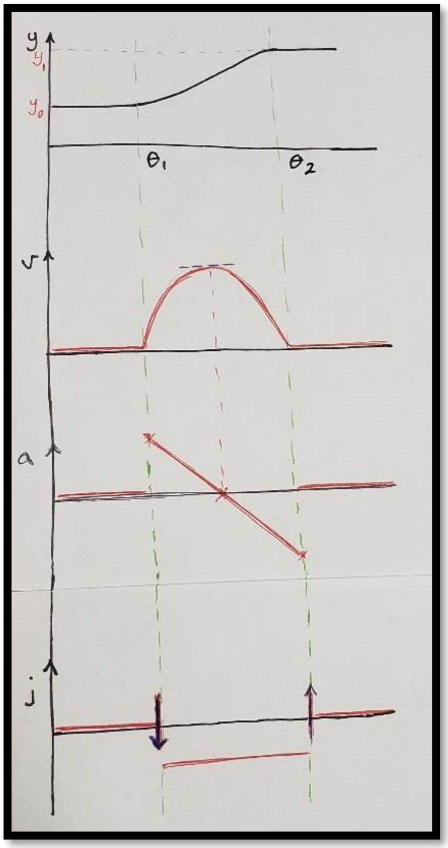
$$y(\theta) = y_0 + 3(y_1 - y_0)\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^2 - 2(y_1 - y_0)\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^3$$

$$v(\theta) = \left(6(y_1 - y_0)\frac{(\theta - \theta_1)}{(\theta_2 - \theta_1)^2} - 6(y_1 - y_0)\frac{(\theta - \theta_1)^2}{(\theta_2 - \theta_1)^3}\right)\dot{\theta}$$

- Both $v(\theta_1) = 0$ and $v(\theta_2) = 0$ as we want and the function is smooth at both ends.

$$a(\theta) = \underbrace{\left(\frac{6(y_1 - y_0)}{(\theta_2 - \theta_1)^2} - \frac{12(y_1 - y_0)}{(\theta_2 - \theta_1)^3}(\theta - \theta_1)\right)}_{\text{It is a line}}\dot{\theta}^2$$

Note: We assumed $\omega = constant$ so $\ddot{\theta} = 0$. Basically, in most of the cases, the cam is rotating with constant angular velocity but the follower based on the cam profile still can have linear acceleration!



D) 5th-degree polynomial: If we want to reach smoothness at displacement and velocity. In this case, we will have 6 boundary conditions.

Example: For the following boundary conditions: a) find the degree of the polynomial for the rise b) find the coefficient of the polynomial c) write the equation for the polynomial and d) plot displacement, velocity, acceleration, and jerk.

$$y(\theta_1) = y_0$$

$$y(\theta_2) = y_1$$

$$v(\theta_1) = 0$$

$$v(\theta_2) = 0$$

$$a(\theta_1) = 0$$

$$a(\theta_2) = 0$$

a) We have 6 boundary conditions, so we will have a 5^{th}-degree polynomial
b) If you solve for coefficients, you will find:

$$c_0 = y_0 \quad , \quad c_1 = 0 \quad , \quad c_2 = 0$$

$$c_3 = 10(y_1 - y_0)$$

$$c_4 = -15(y_1 - y_0)$$

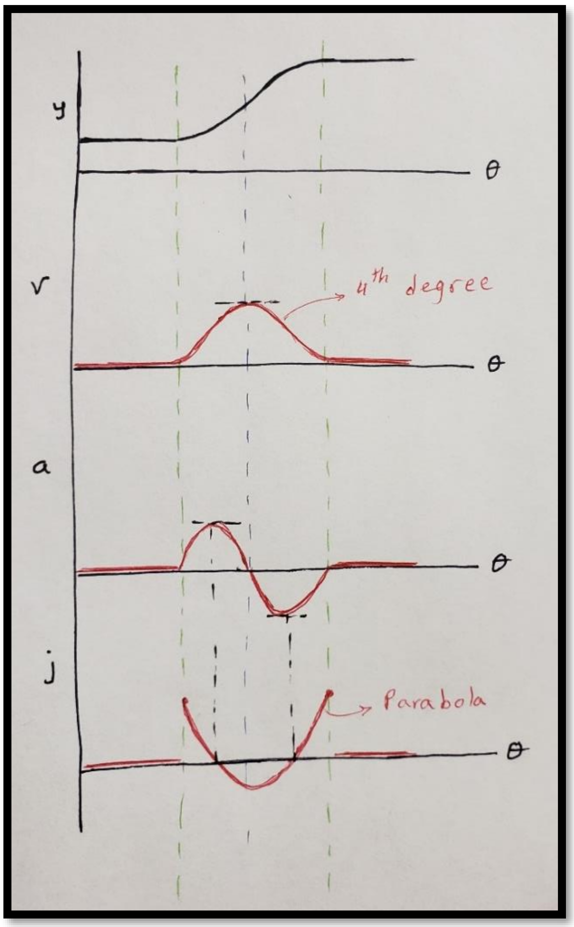
$$c_5 = 6(y_1 - y_0)$$

c)

$$y(\theta) = c_0 + c_1 \frac{\theta - \theta_1}{\theta_2 - \theta_1} + c_2 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^2 + c_3 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^3 + c_4 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^4 + c_5 \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^5$$

$$y(\theta) = y_0 + 10(y_1 - y_0) \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^3 - 15(y_1 - y_0) \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^4 + 6(y_1 - y_0) \left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^5$$

d) For plots we will have:



E) 7^{th}-degree polynomial: If we want to reach smoothness at displacement, velocity, and acceleration. In this case, we will have 8 boundary conditions. You can repeat the same process and find all coefficients.

Other possible requirements in the displacement function: Till now, we had a polynomial and check the information about displacement, velocity, acceleration, and jerk but we can look at the problem backward. For instance, find displacement for:

- Constant velocity rise/return
- Constant acceleration rise/return
- The constant velocity with smooth boundaries

  .
  .
  .

Example: Find the displacement for constant acceleration rise:

Just we know the acceleration is constant (not continuous). For instance, we can say $a = k_0$. Then velocity would be 1st-degree polynomial and displacement 2nd-degree polynomial.

$$a = k_0$$
$$v = k_0\theta + k_1$$
$$y = k_0\frac{\theta^2}{2} + k_1\theta + k_2$$

So, we can use a 2nd-degree polynomial! We can use the nice polynomial that we have from before for the 2nd-degree polynomial:

$$y(\theta) = c_0 + c_1\frac{\theta - \theta_1}{\theta_2 - \theta_1} + c_2\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^2$$

$$v(\theta) = \left(\frac{c_1}{(\theta_2 - \theta_1)} + \frac{2c_2}{(\theta_2 - \theta_1)^2}(\theta - \theta_1)\right)\dot{\theta}$$

$$a(\theta) = \frac{2c_2}{(\theta_2 - \theta_1)^2}\dot{\theta}^2$$

Impose the given conditions:

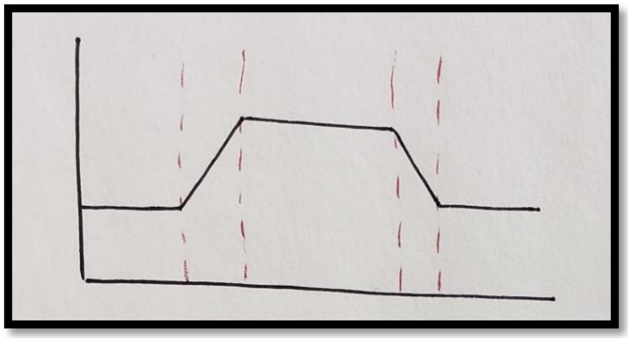
$a(\theta) = k_0 = \frac{2c_2}{(\theta_2-\theta_1)^2}\dot{\theta}^2 \quad \Longrightarrow \quad$ (c_2 can be found)

We still have to find two more coefficients (c_0 & c_1), but we don't have enough information. As a designer, we can select two other conditions to solve the problem and find values of c_0 & c_1. For instance, the velocity at one of the boundary points can be continuous or the velocity of the midpoint equal zero ($v\left(\frac{\theta_1+\theta_2}{2}\right) = 0$).

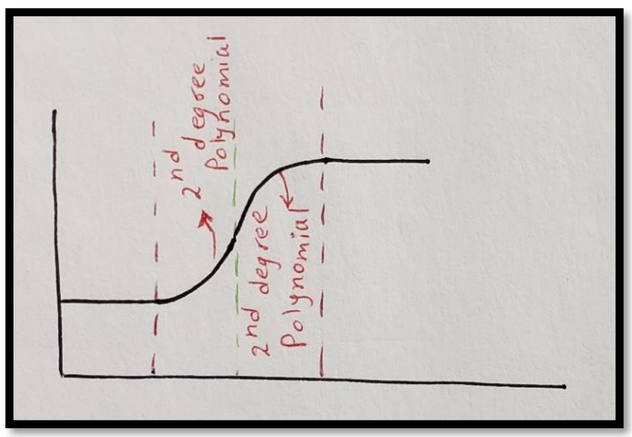
Cam-Follower Systems (Continue)

Cam Displacement Functions: (Waldron (3rd edition) Ch.10 & Norton Ch.8)

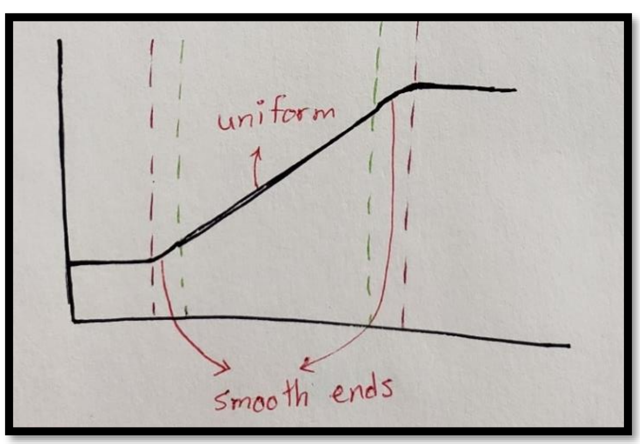
1) Uniform/Trapezoidal Motion (1st Degree polynomial)



2) Parabolic Motion (two 2nd degree polynomial)



3) Modified Trapezoid: Trapezoidal with smooth ends (combination of #1 & #2)

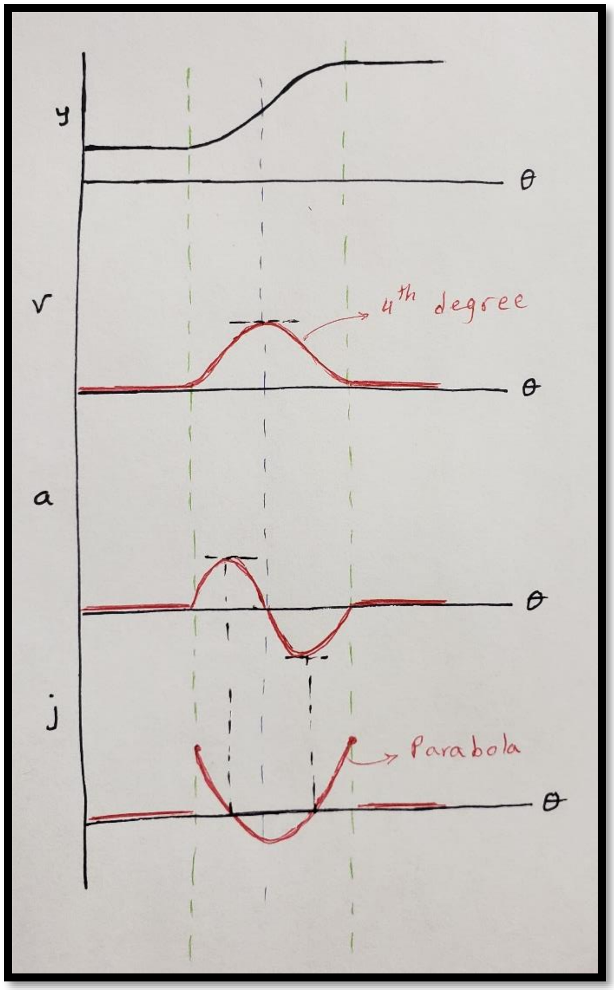


The fundamental law of cam design: "For cam designed for moderate to high speed, the displacement function must be continuous through the 1st and 2nd derivatives all along the motion." That means, for a high-speed cam, displacement, velocity, and acceleration have to be continuous. Therefore, none of the previous options are good for high-speed motion.

4) General Polynomial function

4a) 3-4-5 Polynomial (displacement, velocity, and acceleration are continuous)

$$y(\theta) = c_3\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^3 + c_4\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^4 + c_5\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^5$$

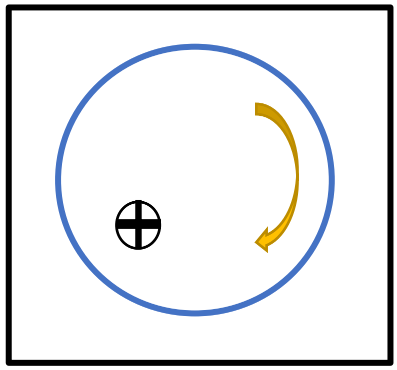


4b) 4-5-6-7 Polynomial (displacement, velocity, acceleration, and jerk are continuous)

$$y(\theta) = c_4\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^4 + c_5\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^5 c_6\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^6 + c_7\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right)^7$$

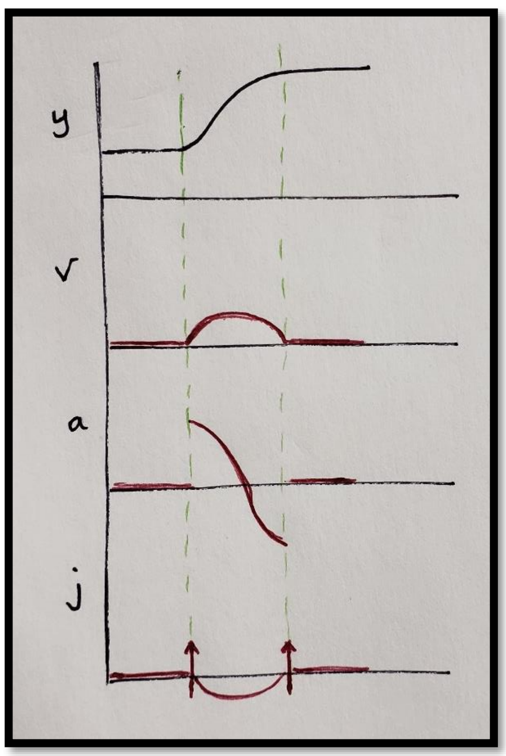
- These polynomial are for standard cam profile (from one dwell go to another dwell). However, if we are not working with a standard case, then the coefficients would be different.

5) Standard Harmonic Motion (SHM): Sinusoidal function (Sin & Cos)
   In this case, the center of rotation of the cam (point of the pin) has to be offset from the center of the circular wheel. The acceleration is not continuous, so it is not good for high speed.



   The general harmonic motion equation

$$\boxed{y(\theta) = c_0 + c_1 \cos(c_2\theta + c_3)}$$



6) Cycloidal Motion (based on sin & cos): Displacement, velocity, and acceleration are continuous

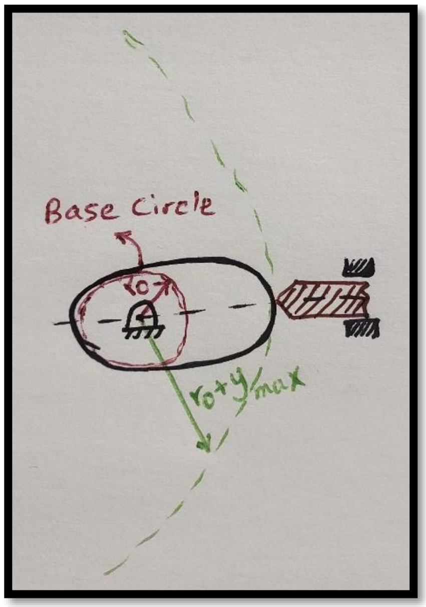
$$\boxed{y(\theta) = \cos\theta + c_1 \sin(c_2\theta + c_3)}$$

Determination of the Cam Profile

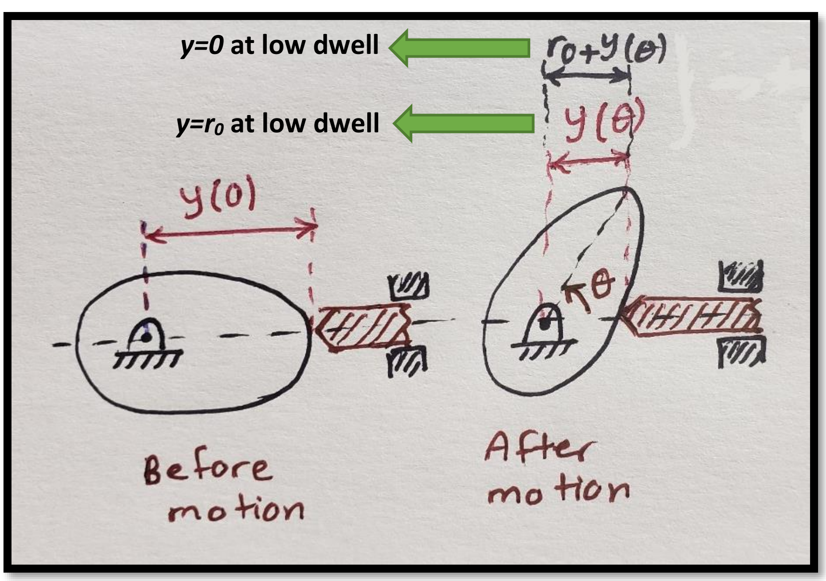
Our input for the design of the cam is $y(\theta)$. Depending on the type of follower and the location of the follower, the cam profile would be different.

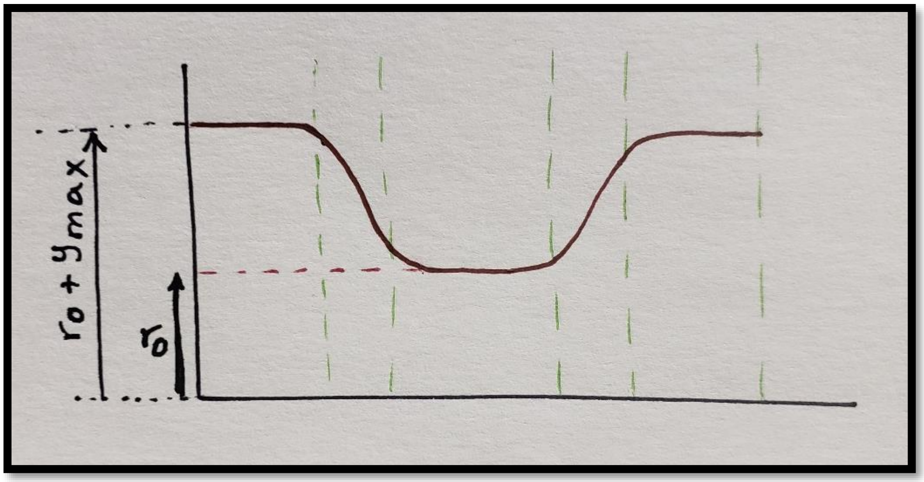
Types of followers:

1) Knife-Edged Follower-Radial (no offset)

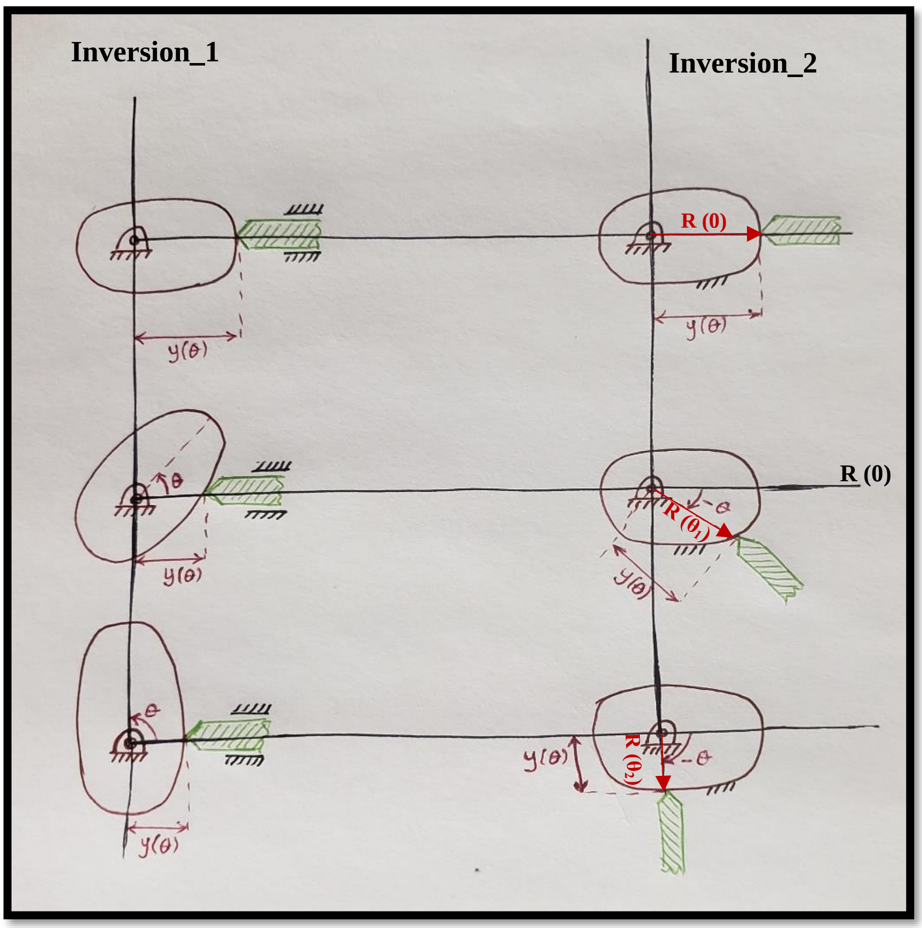


The base circle is really important in designing the cam because if your system allowed a certain offset, changing this base circle, would change the shape of your cam.

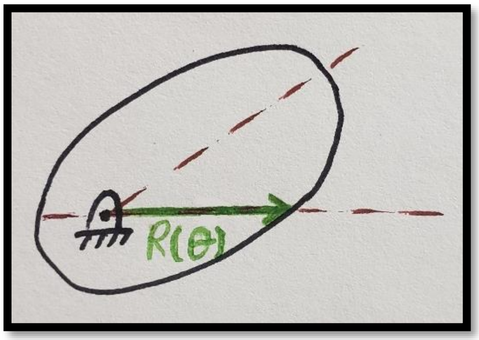




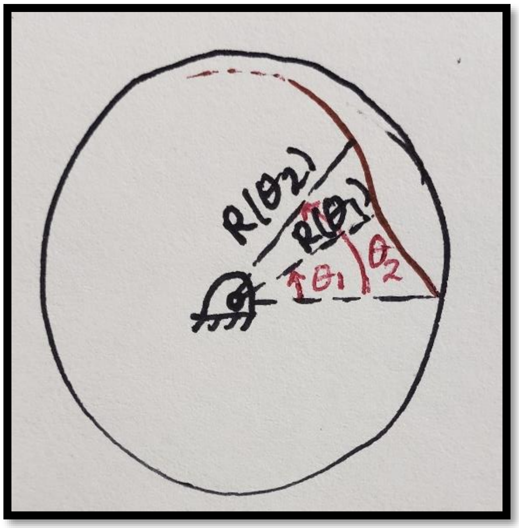
Invert the cam-follower system: The easier way to see how to carve the cam profile is using inversion (changing the ground link). So, we will fix the cam and move the follower around the cam.



Define cam profile as the radius vector $R(\theta)$



We will find the radius of the cam for different angles ($R(\theta_1)$ for θ_1, $R(\theta_2)$ for θ_2, etc.)



$$R(\theta) = \begin{Bmatrix} y(\theta)\, cos\,(-\theta) \\ y(\theta)\, sin\,(-\theta) \end{Bmatrix} = \begin{Bmatrix} y(\theta)\, cos\,(\theta) \\ -y(\theta)\, sin\,(\theta) \end{Bmatrix}$$ If cam rotates counterclockwise (CCW)

$$R(\theta) = \begin{Bmatrix} y(\theta)\, cos\,(\theta) \\ y(\theta)\, sin\,(\theta) \end{Bmatrix}$$ If cam rotates clockwise (CW)

$$R(\theta) = \begin{Bmatrix} \left(r_0 + y(\theta)\right) cos\,(\theta) \\ \left(r_0 + y(\theta)\right) sin\,(\theta) \end{Bmatrix}$$ If we use a base radius and cam rotate clockwise (CW)

Note: If the cam rotates CCW, in the inversion (fix the cam and moving follower) the follower will move CW and have a negative angle! If the cam rotates CW, the follower will move CCW and have a positive angle!

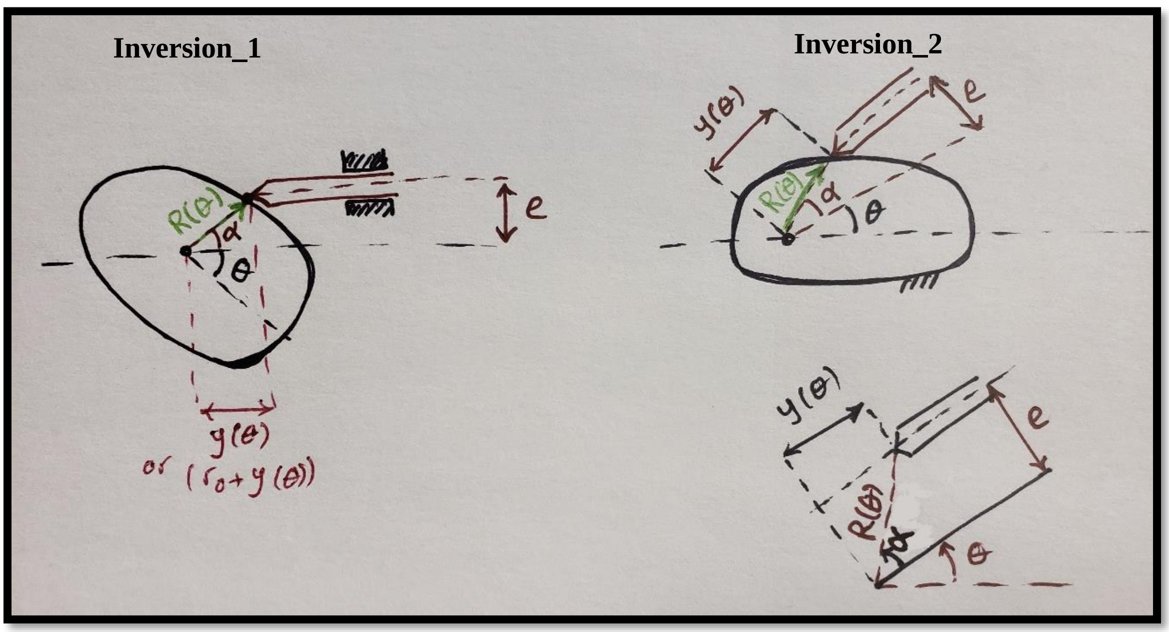
Note: Only for a radial knife-edged follower, the cam profile is just the polar coordinate plot of $y(\theta)$ or $\left(r_0 + y(\theta)\right)$. It is not true for other followers.

Another way to find the cam profile is by making a table for θ & $R(\theta)$:

θ	$R(\theta)$
1°	.
2°	.
.	.
.	.
.	.
360°	.

2) Knife-Edged Follower with Offset

For CW rotating cam: The follower has a constant offset "e". The radius is the touch point between cam and follower and shows with the angle of "$\theta + \alpha$"



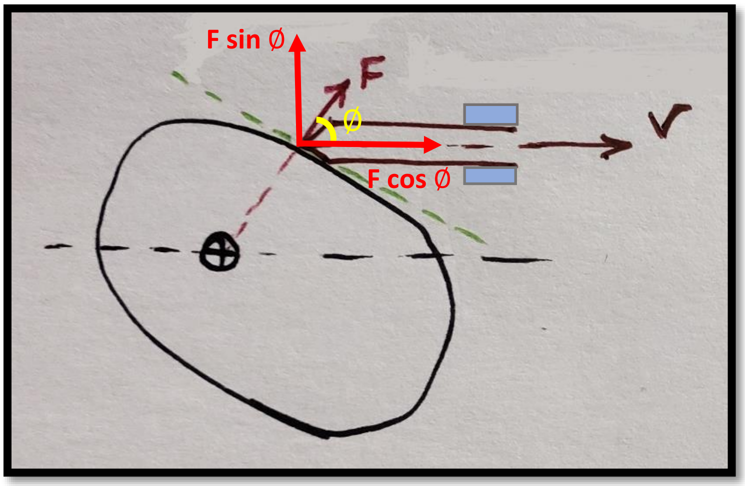
$$R(\theta) = \begin{Bmatrix} y(\theta)\cos(\theta) + e\cos\left(\theta + \frac{\pi}{2}\right) \\ y(\theta)\sin(\theta) + e\sin\left(\theta + \frac{\pi}{2}\right) \end{Bmatrix} = \begin{Bmatrix} y(\theta)\cos(\theta) - e\sin(\theta) \\ y(\theta)\sin(\theta) + e\cos(\theta) \end{Bmatrix} \quad \text{If cam rotates clockwise (CW)}$$

$$R(\theta) = \begin{Bmatrix} \left(r_0 + y(\theta)\right)\cos(\theta) - e\sin(\theta) \\ \left(r_0 + y(\theta)\right)\sin(\theta) + e\cos(\theta) \end{Bmatrix} \quad \text{If we use a base radius and cam rotate clockwise (CW)}$$

In the above equations by playing with the base radius or offset, we can modify our cam profile.

Cam-Follower Systems (Continue)

Transmission Pressure Angle: The pressure angle is the angle between the direction of application of the force and the direction of the velocity. So, it is related to the power ($Power = \bar{F} . \bar{V}$).



Ø: Transmission Pressure Angle

If the pressure angle is zero, you can transmit maximum power for the same force that you are inputting.

What happens if the transmission pressure angle is 90°? The system is not moving.

Design guideline: We want an approximate transmission pressure angle (Ø) between 0° to 30°. The transmission pressure angle is the function of θ and will change along the motion but we can modify it in the key parts of our cam profile where we want maximum power at the output. Modifying offset (e) will modify the transmission pressure angle (Ø).

Based on the type of follower, we will have different pressure angles.

$F\ cos\ \emptyset$: This component is in direction of V and creates the motion.
$F\ sin\ \emptyset$: This component is an undesirable reaction force.

To calculate the direction of the force, use $\bar{a}(\theta) = \frac{d^2\bar{R}(\theta)}{d\theta^2}$ ($\bar{R}(\theta)$ is the vector that defines your profile). Then, makes it a unit vector ($\bar{a}_{unit}(\theta) = \left(\frac{\bar{a}(\theta)}{\|\bar{a}(\theta)\|}\right)$) and dot product with the direction (unit vector) of velocity ($\bar{V}$). This will give you the cos (Ø).

Note: $\bar{F}.\bar{V} = \|\bar{F}\|\|\bar{V}\|\ cos\ \emptyset$

The pressure angle($\emptyset$) is a function of θ. In this plot, the pressure angle has to be positive ($\emptyset \geq 0$).

Note: The 1st derivative would be the tangent and 2nd derivative would be normal to the curve.

Analytical Determination of Cam Profile:

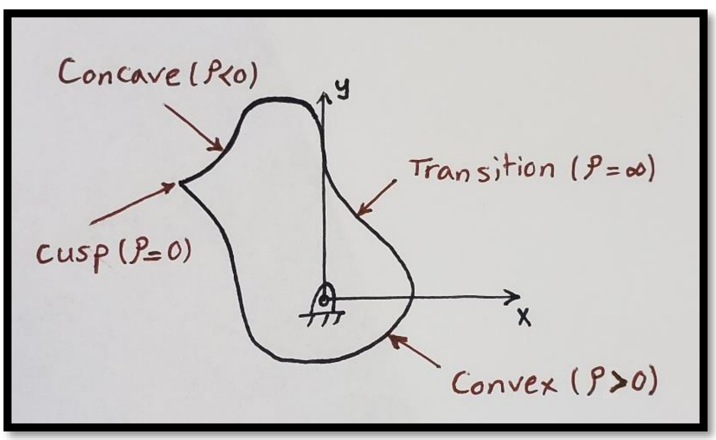
Cam Radius of Curvature (ρ): We will have four different values for this radius. (Curvature (k) is the inverse of the radius of curvature $k = \frac{1}{\rho}$)

Concave: $\rho < 0$

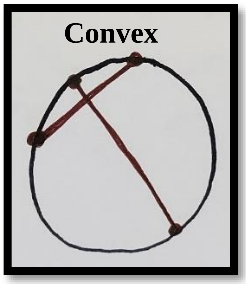
Cusp: $\rho = 0$

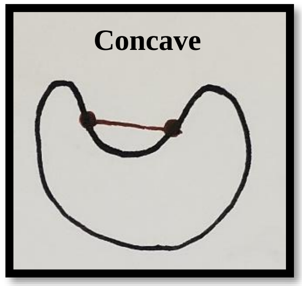
Convex: $\rho > 0$

Transition: $\rho = \infty$



Convex VS Concave: For Convex, the line between any two points will stay inside of the close curve but for concave the line between two points crosses outside of the close curve.

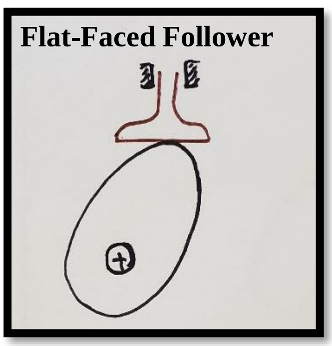


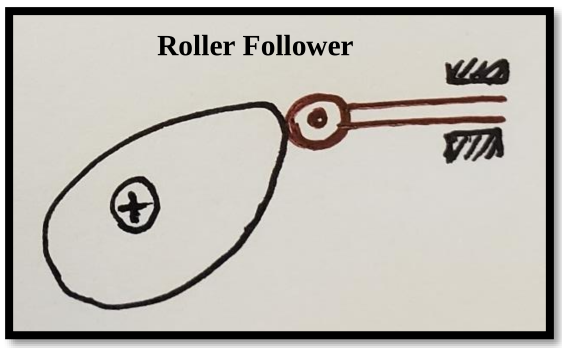


The cam radius of curvature is important for several reasons:

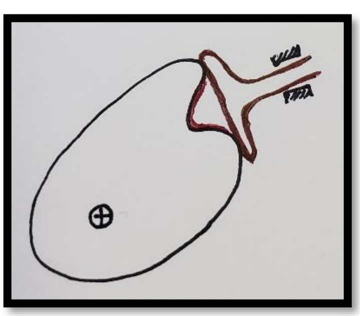
- If the cam is concave in a given area, the radius of curvature determines the minimum radius of the cutter that can be used to machine the cam and the minimum radius of the follower that can be used with the cam.
- The contact stresses between the cam and the follower are a function of the cam radius of curvature.
- Depending on your follower (Flat-Faced or Roller), you won't be able to follow the displacement in the concave areas.

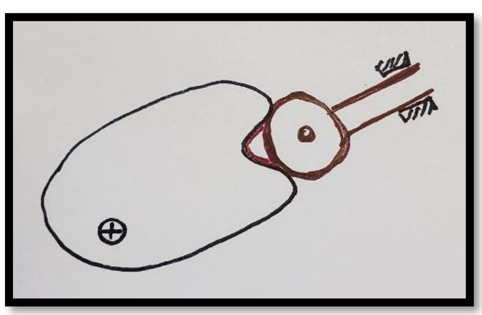
Flat-Faced followers and Roller followers are the two most common types of followers.





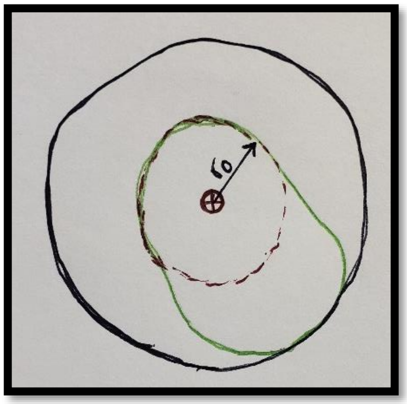
If we have a Flat-Faced follower, the follower on concave areas, will miss the return and rise part of the displacement function for the cam profile on that area. Same thing for roller follower with a bigger radius than the concave curvature.





- We don't want a cusp case ($\rho = 0$) in our cam.

Base radius (r_0): This is important for us to avoid problems of curvature in the cam profile. For instance, when we are creating the displacement function of our follower, we don't know if we have a cusp case ($\rho = 0$) or not, and this is not a type of discontinuity that we specified in our displacement function and it is related to selected base radius for the cam.



How to find the shape and radius of curvature? If we have a planar curve, for any three points (relatively close to each other) on the curve, we can find the center of curvature from the intersection of bisector normal lines.
The following formula is a general formula for finding the radius of any curvature (ρ).

Given: $\bar{R}(\theta) = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix}$

$\bar{v}\,(\theta) = \bar{R}'(\theta) = \frac{d\bar{R}(\theta)}{d\theta} = \begin{Bmatrix} v_x(\theta) \\ v_y(\theta) \end{Bmatrix}$

$\bar{a}\,(\theta) = \bar{R}''(\theta) = \frac{d^2\bar{R}(\theta)}{d\theta^2} = \begin{Bmatrix} a_x(\theta) \\ a_y(\theta) \end{Bmatrix}$

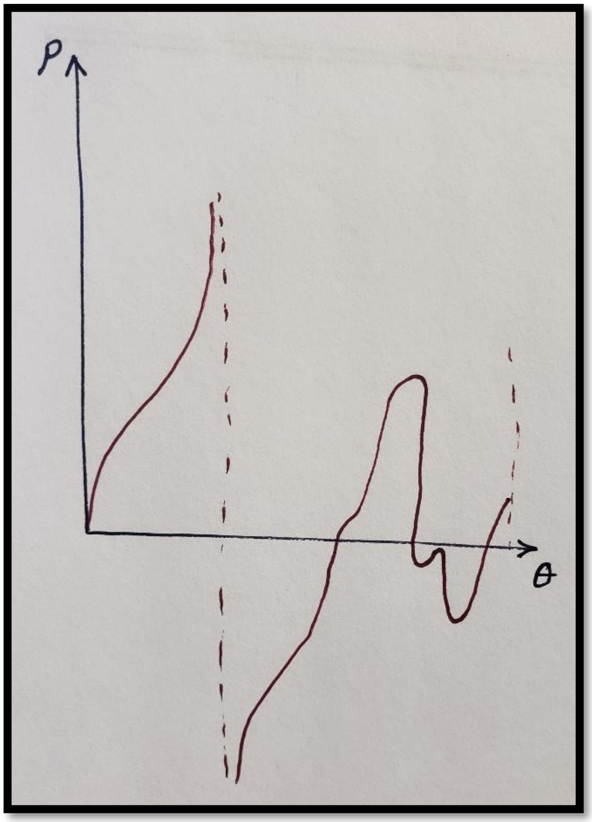
Note: $\bar{R}'(\theta)$ & $\bar{R}''(\theta)$ are 1st and 2nd derivatives of $\bar{R}(\theta)$ respect to θ (not time)

$$\boxed{\rho = \frac{(v_x{}^2 + v_y{}^2)^{3/2}}{v_x a_y - v_y a_x}}$$

$$\boxed{\rho = \frac{\left(\bar{v}\,(\theta).\bar{v}\,(\theta)\right)^{3/2}}{\bar{v}\,(\theta) \times \bar{a}\,(\theta)}}$$

In vector form (dot product: (.), cross product: (×))

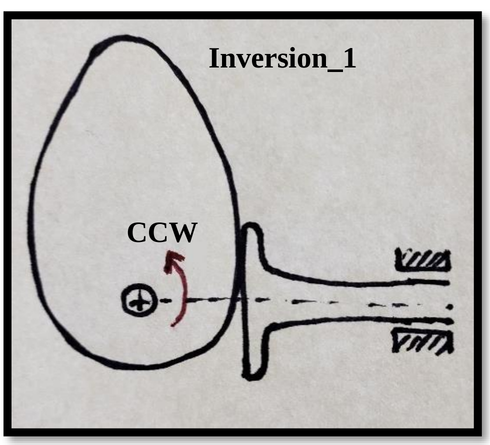
ρ is a function of θ. If we plot it based on θ values, you can see, 0, positive, negative, and infinity values for ρ.

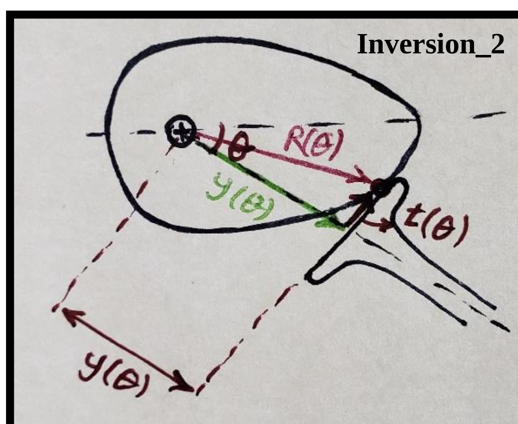


Design guideline: Always, we want $\rho > 0$. To avoid having a negative radius of curvature, we can select our base radius (r_0) 2-3 times of maximum rise (y_{max}). Still, you have to check but in most cases, you will have a positive value for the radius of curvature.

1) Flat-Faced Follower

The Flat-Faced Followers can be radial or not radial but the difference between them is very little.





$$\bar{R}(\theta) = \begin{Bmatrix} y(\theta)\ cos\ \theta + t(\theta)\ cos\ (\theta + \frac{\pi}{2}) \\ y(\theta)\ sin\ \theta + t(\theta)\ sin\ \left(\theta + \frac{\pi}{2}\right) \end{Bmatrix} = \begin{Bmatrix} y(\theta)\ cos\ \theta - t(\theta)\ sin\ \theta \\ y(\theta)\ sin\ \theta + t(\theta)\ cos\ \theta \end{Bmatrix}$$

If cam rotate counterclockwise (CW)

$$\bar{R}(\theta) = \begin{Bmatrix} y(\theta)\ cos\ (-\theta) - t(\theta)\ sin\ (-\theta) \\ y(\theta)\ sin\ (-\theta) + t(\theta)\ cos\ (-\theta) \end{Bmatrix} = \begin{Bmatrix} y(\theta)\ cos\ (\theta) + t(\theta)\ sin\ \theta \\ -y(\theta)\ sin\ \theta + t(\theta)\ cos\ \theta \end{Bmatrix}$$

If cam rotate counterclockwise (CCW)

Note: $R(\theta)$ for most of the followers (except the radial knife-edged follower) does not correspond to (θ) (it is not located at θ). That is why we cannot do the polar plot for other types of followers because $R(\theta)$ is not located at θ. But $y(\theta)$ is located at θ because we create the displacement function in that way.

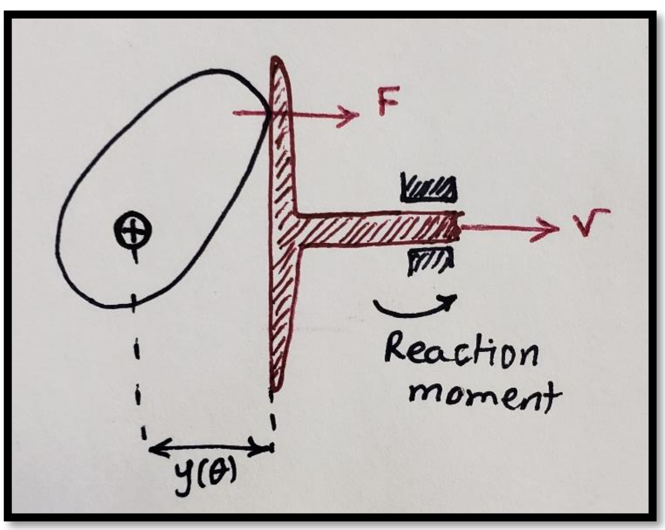
Note: $t(\theta)$ is showing the distance between the contact point and center of the follower and this distance will change during motion. We can prove that: $t(\theta) = y'(\theta) = \frac{dy(\theta)}{d\theta}$ (the proof in the book).

$$\bar{R}(\theta) = \begin{Bmatrix} y(\theta)\ cos\ \theta - y'(\theta)\ sin\ \theta \\ y(\theta)\ sin\ \theta + y'(\theta)\ cos\ \theta \end{Bmatrix}$$

You can use this vector and create points for different values of "θ" and plot your cam profile ($0 < \theta \leq 2\pi$).

Pressure Angle for Flat-Faced Follower:

Always, for flat-faced followers, we will have zero pressure angle ($\emptyset = 0$)! This is the main advantage of flat-faced followers. However, because of the offset of the contact point and line between the follower and center of the cam, we still have the moment reaction (reaction force).



Minimum base radius to avoid concave areas for flat-faced followers:

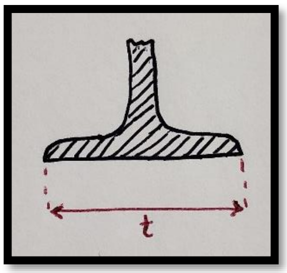
As it is mentioned before the flat-faced followers have problems with concave and we have to avoid them in our cam design. In that case, we need to have a positive value for the radius of curvature.

ρ is the function of $\bar{R}'(\theta)$ & $\bar{R}"(\theta)$ (for $\bar{R}(\theta) = \begin{Bmatrix} (r_0 + y(\theta))\ cos\ \theta - y'(\theta)\ sin\ \theta \\ (r_0 + y(\theta))\ sin\ \theta + y'(\theta)\ cos\ \theta \end{Bmatrix}$). We can write the equation for ρ and put it equal to zero to find the limit value for r_0. After a long calculation we will have:

$$\boxed{r_0 \geq -y(\theta) - y"(\theta)}$$

Face Length of the Follower:

The minimum & maximum face length of the follower $t(\theta)$ can be calculated from the following equations (they can be found from the vector):



$$t_{max} = y'(\theta)|_{max}$$

$$t_{min} = y'(\theta)|_{min}$$

Note: We don't have a formula for roller follower and that is a numerical process. So, in this class, we will not cover that type of follower.

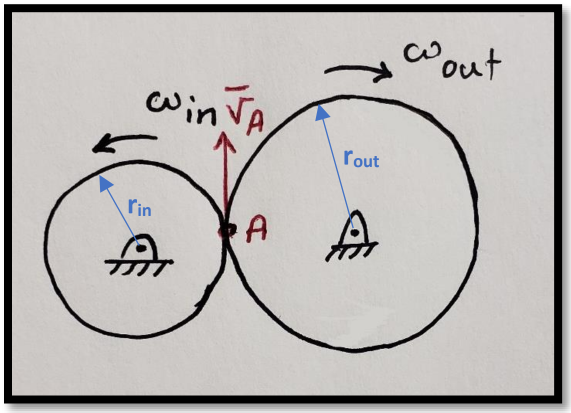
How to design Cam-profile in SolidWorks?

https://www.youtube.com/watch?v=Wn7CW9y42Pg&list=LL&index=10

https://www.youtube.com/watch?v=yhZ3N_cJLM0&list=LL&index=11

Gear Systems

In a gear system, we want a constant input/output velocity ratio. So, for a constant input angular velocity (ω_{in}), we want another different constant output angular velocity (ω_{out}). The best way to model that is using sticky wheels which in the contact point kind of stick to each other when one of them rotates in one direction the other one will rotate in opposite direction. The contact point (point A) will have a velocity $\bar{v}_A$.



Gear Ratio/Velocity Ratio ($\boldsymbol{R}$): It is the ratio of input angular velocity over output angular velocity and it would be a constant value.

$$R = \frac{\omega_{in}}{\omega_{out}}$$

From the above figure, we will have:

$$v_A = \omega_{in}.r_{in} = -\omega_{out}.r_{out}$$

$$\boxed{R = \frac{\omega_{in}}{\omega_{out}} = -\frac{r_{out}}{r_{in}}}$$

So by changing the radius of your wheels, you will have all possible velocity ratios. The radius of sticky circles (r_{in} & r_{out}) are called **pitch radius**.

Let's assume there is no friction between the wheels and these wheels are rotating with a constant angular velocity (no acceleration and no deceleration), and the wheels are rigid bodies. Based on this information, we can say power-in would be equal to power-out (the power loss is very little). This for most types of gears is true (we have an efficiency of around %98)

$$Power\ in = Power\ out$$

$$\omega_{in}.T_{in} = -\omega_{out}.T_{out}$$

Angular velocity

Torque

Note: This equation wrote as a scalar function because both angular velocity and torque are on the plane (moving around the z-axis) and they are going in the same direction.

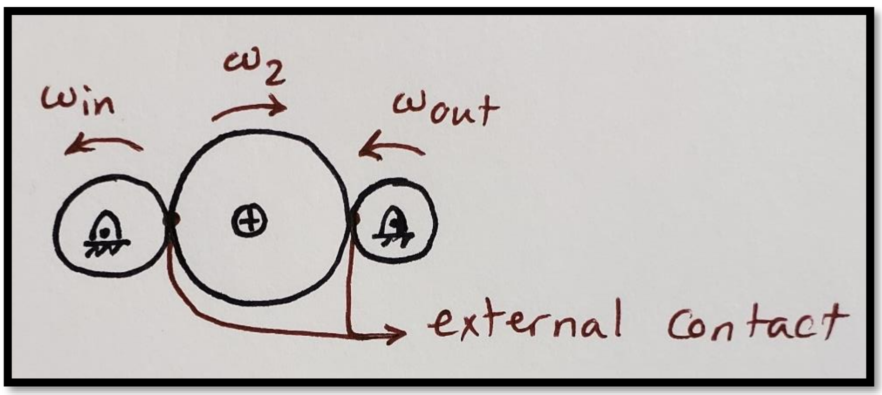
So, we can write the gear ratio as:

$$\boxed{R = \frac{\omega_{in}}{\omega_{out}} = -\frac{T_{out}}{T_{in}} = MA}$$

The gear ratio also is named mechanical advantage (MA) and tells you how much force you are gaining from your gear system.

Mainly the gear systems are used for these two purposes: 1) Increase/decrease speed 2) increase/decrease torque. And when you gain one, you will lose the other (when you gain torque, you will lose the angular velocity and if you increase the angular velocity, you will decrease the torque)

There are two ways we can create a system in which you can have the input and output rotating in the same direction. One way would be to put the third gear between input and output gears.

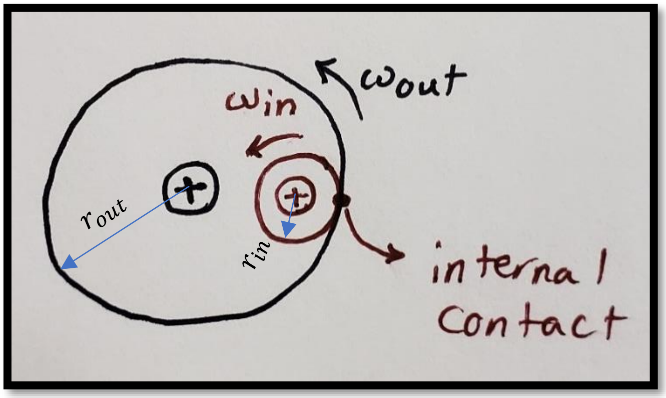


The total gear ratio would be:

$$R = \left(-\frac{\omega_{in}}{\cancel{\omega_2}}\right)\left(-\frac{\cancel{\omega_2}}{\omega_{out}}\right) = \frac{\omega_{in}}{\omega_{out}}$$

You can see the middle gear doesn't do anything except change the velocity direction.

In the second method, we don't need extra gear. In this method, two gears have to be inside of each other.



In general for gear ratio we will have:

$$\boxed{R = \frac{\omega_{in}}{\omega_{out}} = \pm \frac{r_{out}}{r_{in}}}$$

$-: for\ external\ contact\ for\ two\ gears$

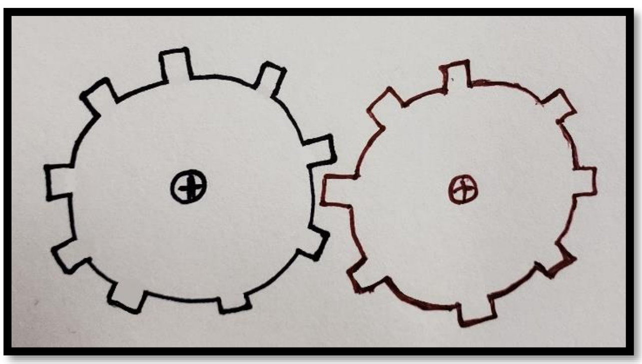
$+: for\ internal\ contact\ for\ two\ gears$

However, in reality, we cannot work with the sticky gear (there is not such a system that much sticky). So, we have to use different types of gear.

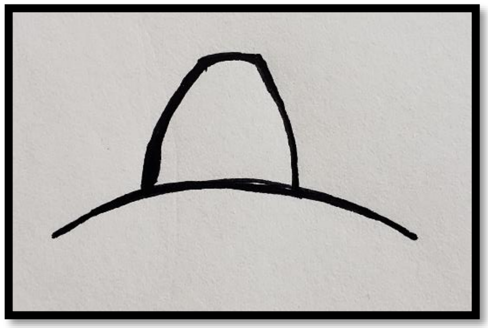
Some general information about gears:

- Gear teeth: There are many different profiles for teeth.

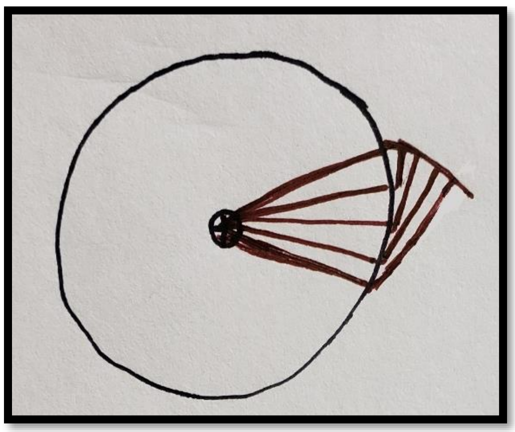
A) Square shape teeth: This type of teeth doesn't give a constant velocity ratio and depending on contacting point between the teeth and you will have a kind of force oscillation in your input/output velocity relation.

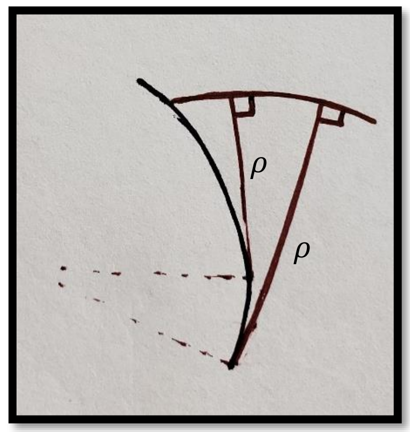


B) Involute profile: We will work with this type of tooth because this one has nice geometric properties that make it very good. There are some other types of teeth that we are not going to detail of them.



- Base circle: It is the circle which we create the teeth profile from. The normal to involute profile is tangent to the base circle.

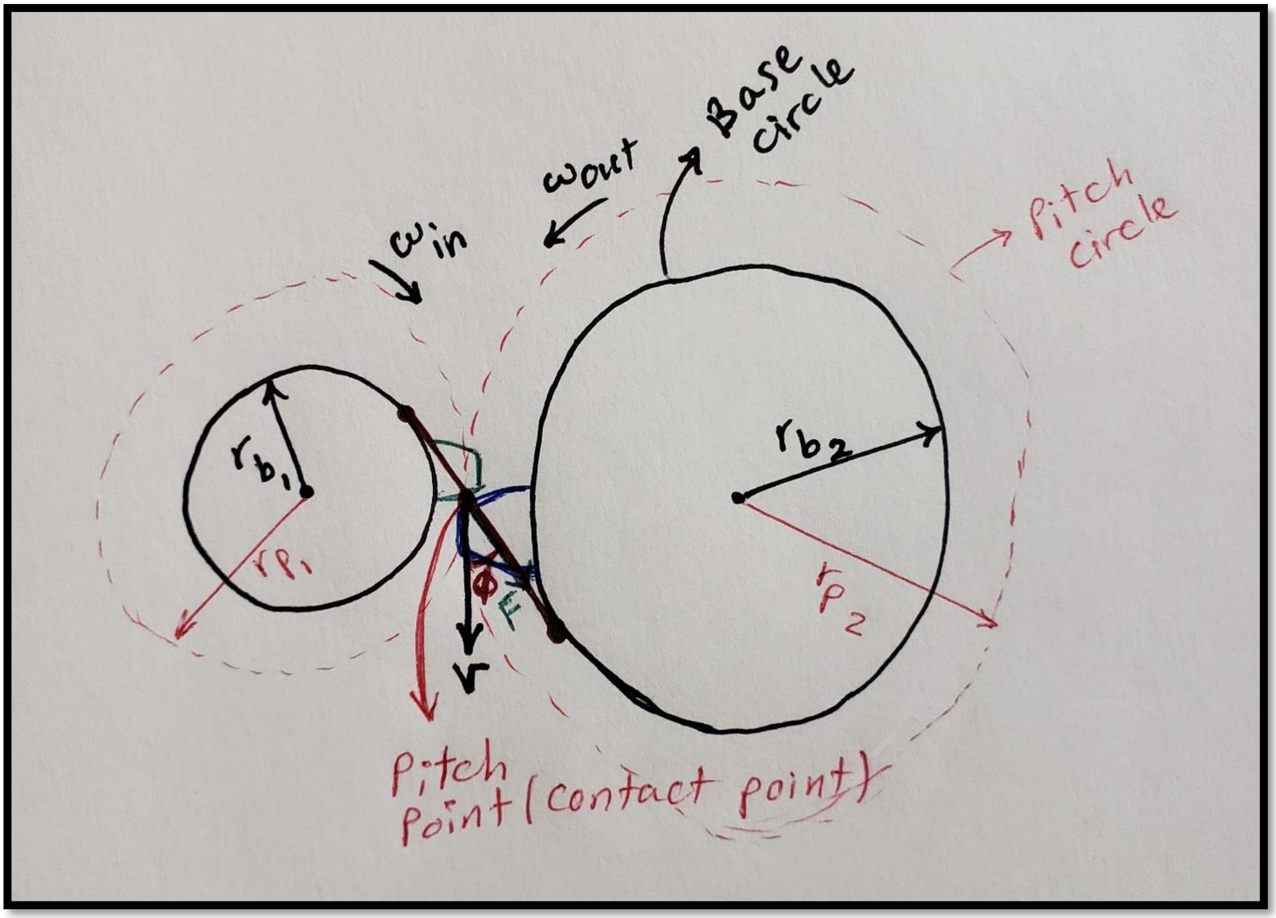




ρ: Radius of curvature of a tooth (this varies point by point because the profile is not circular)

- The contact line of meshing teeth is constant

The contact point can be changed but always remain on the tangent line between two circles (contact line). The force between two teeth always applies on this line and is normal to both teeth. The angle between force and velocity is the pressure angle (∅). This angle is constant for gears and there are some standard values for them which you can find in catalogs. Both wheels must have the same pressure angle and this angle is less than 30° to not waste a lot of force.

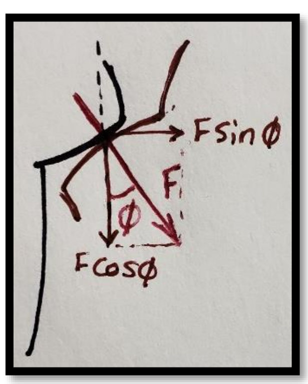


r_{b_1} & r_{b_2}: $Base\ radius$

r_{p_1} & r_{p_2}: $Pitch\ radius$

$\emptyset$: $Pressure\ angle$

Pitch circle: It is an imaginary sticky circle that can make the same velocity ratio. You can't measure it but it geometrically exists.



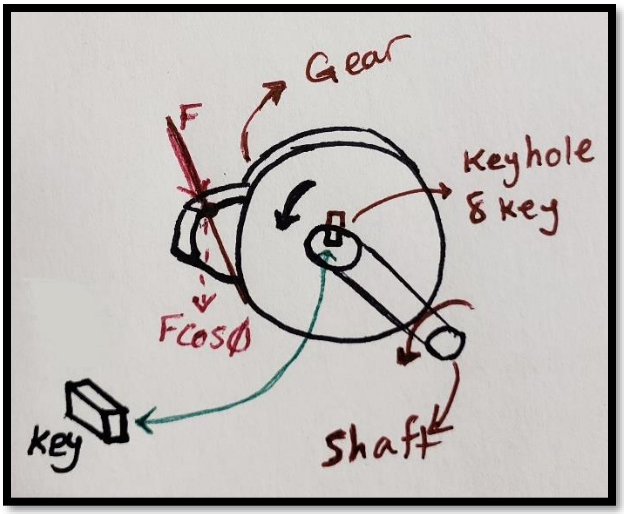
$F\ cos\ \emptyset\ .r_{p_2} = T_{out}$

$F\ sin\ \emptyset$: A radial force on the output shaft

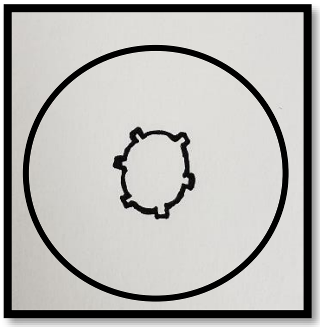
So, the involute profile is great, has really good properties, and gives you a constant velocity ratio, but creates some reaction forces.

How do we transmit this force from the gear to the shaft? There are different ways:

Using keyhole on shaft and gear and key for connection between them. So, we transmit the force from the gear to the key and from the key to the shaft.



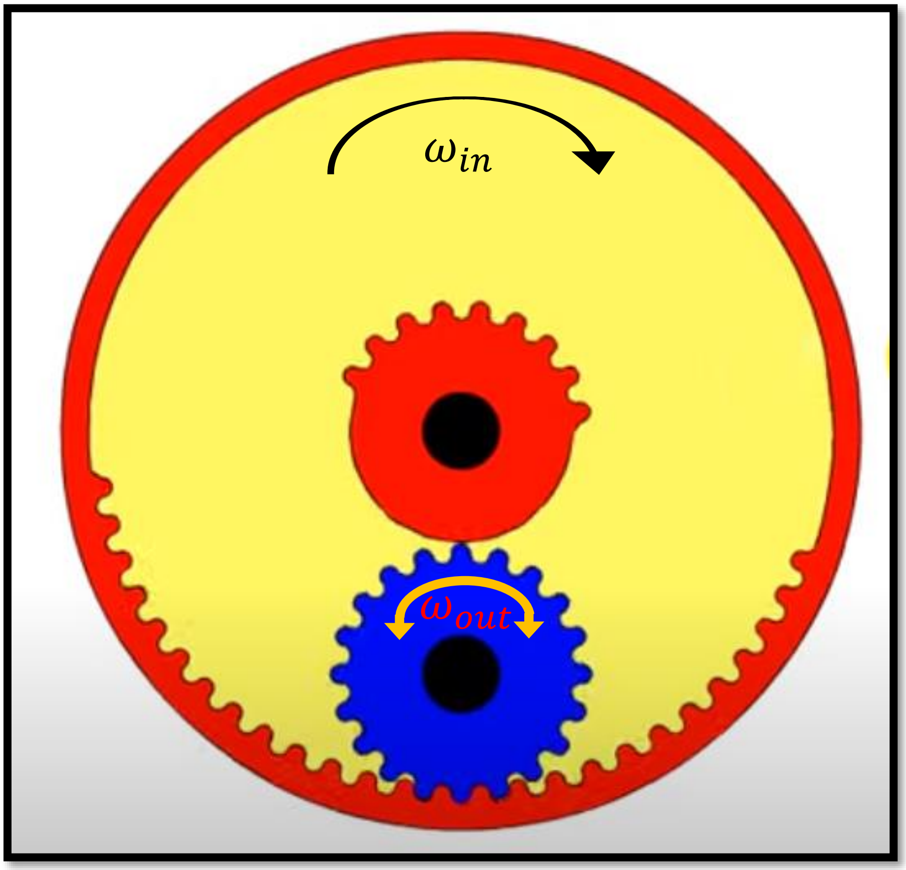
Or other keyhole shapes (something like multiple keyholes)



Also, you can use press-fit them depending on how big is the torque in the output.

Usually, the gear systems are designed to transmit the constant velocity ratio but you can create gear systems to do many other things (create dwells, moving in the part of the cycle and not moving in part of the cycle, etc.)

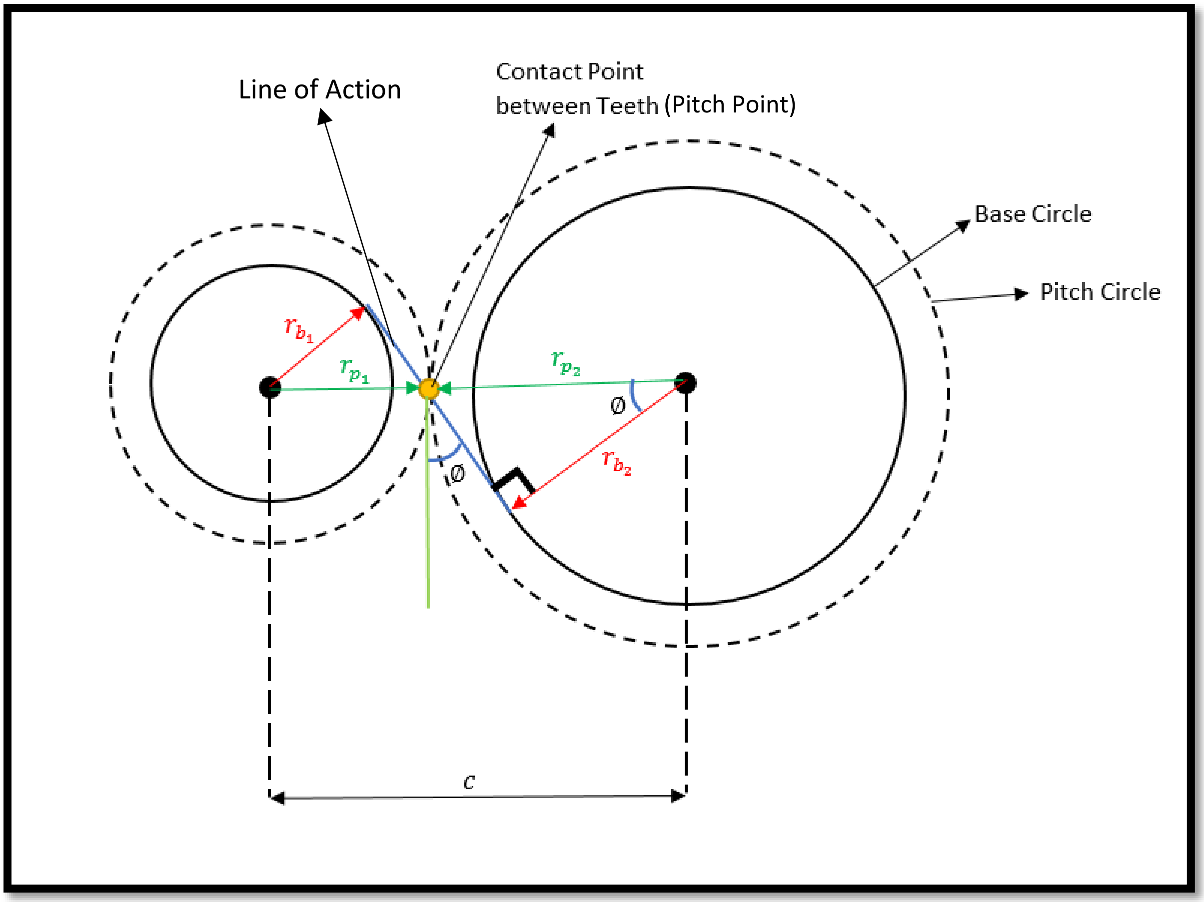
Example: Create a gear system that moves in the first half in one direction (forward) and in the second half in the other direction (backward).



https://www.youtube.com/watch?v=fmA9Vnu33FY

Gear Systems (Continue)

Geometry of Gears



Line of Action: https://www.youtube.com/watch?v=_p6npjPSIbI

r_{b_1}, r_{b_2}: Base Radius

r_{p_1}, r_{p_2}: Pitch Radius

c: Center Distance

$\emptyset$: Phase Angle

$$\boxed{\cos\emptyset = \frac{r_b}{r_p}} \Rightarrow$$ Relation between base radius and pitch radius

- Two meshing gears have the same pressure angle

Gear Ratio

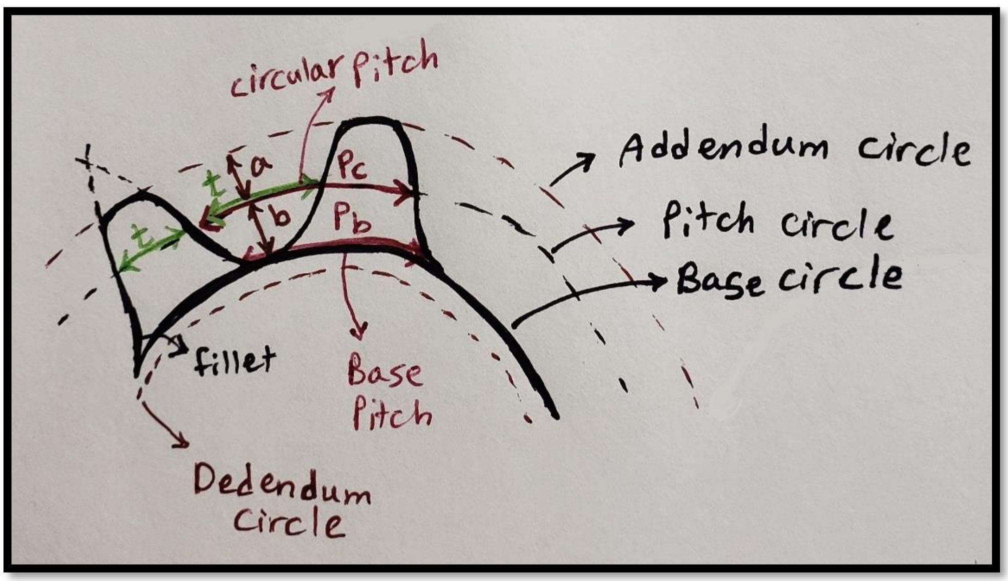
$$R = \frac{\omega_{in}}{\omega_{out}} = \pm\frac{r_{p_{out}}}{r_{p_{in}}} \Rightarrow R = \pm\frac{\frac{r_{b_{out}}}{\cancel{\cos\emptyset}}}{\frac{r_{b_{in}}}{\cancel{\cos\emptyset}}} = \pm\frac{r_{b_{out}}}{r_{b_{in}}}$$

$$\boxed{R = \frac{\omega_{in}}{\omega_{out}} = \pm \frac{r_{p_{out}}}{r_{p_{in}}} = \pm \frac{r_{b_{out}}}{r_{b_{in}}}}$$

So, we can calculate the gear ratio as the ratio of the base radius which is unchanged (they are geometry construction of our gears). So, if the distance between the gears is a little bit changed (if the gears are not completely fit each other), we still have the same gear ratio!

Addendum circle: The teeth of gear theoretically can continue till reaching each other at one point but then they become weak. So, based on design parameters, we decide the teeth of the gears cut at some point. The cutting circle is called the addendum circle.

Also, at the connecting point between teeth and circle, we are using fillet to increase the strength.



a: $Addedum$

b: $Dedendum$

p: $Pitch$ (Circular distance between two teeth)

p_c: $Circular\ Pitch$ (Measure at pitch circle)

p_b: $Base\ Pitch$ (Measure at base circle)

N: $Number\ of\ teeth$

t: $thickness\ of\ teeth$ (Usually it is equal to the distance between two teeth)

Dedendum Circle: Sometimes we develop the involute profile a little bit below the base circle to make thicker teeth (and sometimes a little on top of the base circle). In that case, you cannot see the base circle.

p_c & p_b can be found from the following equations:

$$\boxed{p_c = \frac{2\pi r_p}{N} \qquad , \qquad p_b = \frac{2\pi r_b}{N}}$$

For two gears to mesh, we need to have:

1) Same pressure angle (∅)

2) Same pitch (p_c & p_b) because the gears contact each other at the pitch circle (the tooth from one gear have to place between two teeth of the other gear).

In reality, in the manufacturing process, it is a little bit hard to find p_c from the above equation. In catalogs instead of circular pitch (p_c) you will find something called diametral pitch.

Pitch diameter

US: Diametral Pitch $\quad P_d = \frac{N}{d_p} \left(\frac{teeth}{inch}\right) \qquad d_p = 2r_p \qquad$ (It is also equal to $P_d = \frac{\pi}{p_c}$)

Or sometimes in catalogs $(D.P.)$

International System: Module $\quad m = \frac{d_p}{N} \ (mm)$

In the process of selecting gear for a specific task, first, you have to pick a pressure angle (∅), then diametral pitch, and finally the number of teeth from the catalog. The diametral pitch shows you how big the teeth are (for more torque, we need bigger teeth).

Example for a gear catalog:

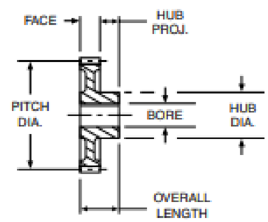
Spur Gears

16 and 12 Diametral Pitch (Steel & Cast Iron)

1 **20° Pressure Angle (will not operate with 14-1/2° spurs)**

ALL DIMENSIONS IN INCHES
ORDER BY CATALOG NUMBER OR ITEM CODE

No. of Teeth	Pitch Dia.	Bore	Hub Dia.	Hub Proj.	Style See Page 323	Without Keyway or Setscrew Catalog Number	Without Keyway or Setscrew Item Code	With Keyway and Setscrew† Catalog Number	With Keyway and Setscrew† Item Code
16 DIAMETRAL PITCH						Face = .750" Outside Dia. = Pitch Dia. + .125" Overall Length = .750" + Hub Proj.			
STEEL									
12	.750	.375	.56	.50		YB12	09916	YB12-3/8†	46141
14	.875		.69			YB14	09918	YB14-3/8†	46142
15	.938	.375	.75	.50		YB15	09920	YB15-3/8†	45991
		.500				–	–	YB15-1/2	46143
16	1.000	.500	.81	.50		YB16	09922	YB16-1/2	46144
18	1.125		.94			YB18	09924	YB18-1/2	46145
20	1.250	.625	1.05	.50		YB20	09926	YB20-5/8	46146
24	1.500	.625	1.20	.50		YB24	09928	YB24-5/8	46147
		.750				–	–	YB24-3/4	46148
28	1.750	.625	1.45	.50		YB28	09930	YB28-5/8	46149
		.750				–	–	YB28-3/4	46150
30	1.875	.625	1.58	.50		YB30	09932	YB30-5/8	46151
		.750				–	–	YB30-3/4	46152
		.875			A	–	–	YB30-7/8	46153
32	2.000	.625	1.70	.50		YB32	09934	YB32-5/8	46154
		.750				–	–	YB32-3/4	46155
		.875				–	–	YB32-7/8	46156
		1.000				–	–	YB32-1	46157
36	2.250	.625	1.95	.50		YB36	09936	–	–
40	2.500		2.20	.62		YB40	09938	–	–
48	3.000		2.00			YB48A	10572	–	–
56	3.500		2.50			YB56A	10574	–	–
60	3.750		2.75			YB60A	10576	–	–
64	4.000	.750	2.88	.75		YB64A	10578	–	–
72	4.500		3.38			YB72A	10580	–	–
80	5.000		3.88			YB80A	10582	–	–
CAST IRON									
96	6.000	.750	1.75	.75		YB96	10584	–	–
128	8.000		2.00			YB128	10588	–	–
144	9.000		2.00		D	YB144	10590	–	–
160	10.000	.875	2.00	.75		YB160	10592	–	–
192	12.000			1.00		YB192	10594	–	–
12 DIAMETRAL PITCH						Face = 1.000" Outside Dia. = Pitch Dia. + .167" Overall Length = 1.000" + Hub Proj.			
STEEL									
12	1.000	.500	.75	.62		YD12	09940	YD12-1/2*	46158
13	1.083	.625	.83	.62		YD13	09942	YD13-5/8‡	46159
14	1.167		.92			YD14	09944	YD14-5/8	46160
15	1.250		.99			YD15	09946	YD15-5/8	46161
16	1.333		1.07			YD16	09948	YD16-5/8	46162
18	1.500	.750	1.24	.62		YD18	09950	YD18-3/4	46163
20	1.667		1.32			YD20	09952	YD20-3/4	46164
21	1.750	.750	1.40	.62		YD21	09954	YD21-3/4	46165
		.875				–	–	YD21-7/8	46166
24	2.000	.750	1.65	.62		YD24	09956	YD24-3/4	46167
		.875			A	–	–	YD24-7/8	46168
		1.000				–	–	YD24-1	46169
28	2.333	.750	1.99	.62		YD28	09958	YD28-3/4	46170
		.875				–	–	YD28-7/8	46171
		1.000				–	–	YD28-1	46172
30	2.500	.750	2.15	.62		YD30	09960	–	–
36	3.000		1.94	.88		YD36A	10596	–	–
42	3.500		2.44			YD42A	10598	–	–
48	4.000	.875	2.88	.88		YD48A	10600	–	–
54	4.500		3.38			YD54A	10602	–	–



STANDARD TOLERANCES

DIMENSION		TOLERANCE
BORE	All	±.0005

16 D.P.

12 D.P.

REFERENCE PAGES

Alterations — 322
Horsepower Ratings — 58, 59
Lubrication — 322
Materials — 323
Selection Procedure — 49

†3/8" bore have one setscrew. No keyway.

YB15-1/2 and larger have standard keyway at 90° to setscrew. See page 323.

*YD12-1/2 has one setscrew. No keyway.

‡YD13-5/8 has one setscrew. No keyway.

YD14-5/8 bore and larger have standard keyway at 90° to setscrew.

There is a possibility that two or more teeth of the gears be in contact at the same time. In that case, all contact points (pitch points) will be on the “Line of Action” and the motion would be smoother but with more friction. The number of teeth in contact at the same time is one of the design parameters.

Note: The pressure angle always remains the same because it is related to the gears circle.

Note: The formula sheet for a different type of gear will be posted in Moodle.

Depending on your application, sometimes you need more load to apply to the gear so you can design gear with thicker teeth at the connection point to the gear surface (a) and sometimes you need more space between two teeth on the gear (because one of the gears is very small compared to the other one) and you can get closer to the center of gear (b). In these two examples, we have the same gear ratio, same pressure angle, and the same number of teeth, but they have different performances when they are meshing to gather.

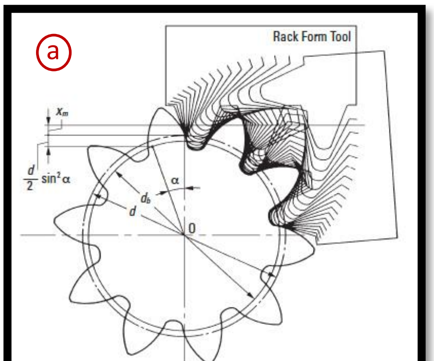


Fig. 4-6 Generating of Positive Shifted Spur Gear
($\alpha = 20°$, $z = 10$, $x = +0.5$)

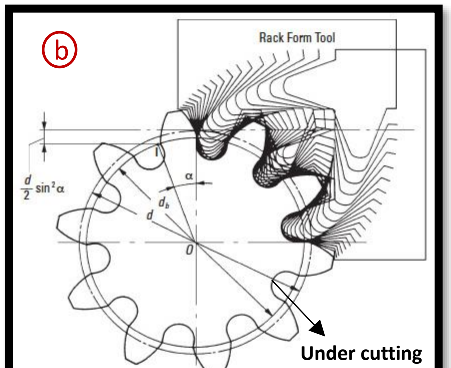


Fig. 4-2 The Generating of a Standard Spur Gear
($\alpha = 20°$, $z = 10$, $x = 0$)

Undercutting: cutting under the base circle. It is not always a good decision because it makes your gear weaker.

For Two Gears to Mesh:

- Same diametral pitch
- Same pressure angle
- Same addendum & dedendum (to make sure they don't hit each other)

We had:

$$r_p = \frac{r_b}{\cos \emptyset}$$

$$p_c = \frac{2\pi r_p}{N}$$

$$P_d = D.P. = \frac{N}{d_p} = \frac{N}{2r_p}$$

$$R = \frac{\omega_{in}}{\omega_{out}} = \pm\frac{r_{p_{out}}}{r_{p_{in}}} = \pm\frac{r_{b_{out}}}{r_{b_{in}}}$$

But we cannot measure r_p & r_b! So, for Gear Ratio, we can write it based on the number of teeth:

$$R = \frac{\frac{\cancel{p_{c_{out}}}N_{out}}{\cancel{2\pi}}}{\frac{\cancel{p_{c_{in}}}N_{in}}{\cancel{2\pi}}} = \pm\frac{N_{out}}{N_{in}} \Leftarrow (p_{c_{in}} = p_{c_{out}}) \text{ same diametral pitch}$$

$$\boxed{R = \pm\frac{N_{out}}{N_{in}}}$$

So, the ratio of the velocity or the ratio of the torque is the same as the inverted ratio of the number of teeth.

Example: Select two spur gears to have an output angular velocity that is three times the input angular velocity. These gears have a pressure angle of $\emptyset = 20°$ and a diametral pitch of $P_d = 16$.

In this problem, it is not mentioned about internal or external contact, so we assume two gears have an external contact.

$$\omega_{out} = -3\,\omega_{in}$$

$$\boxed{R = \frac{\omega_{in}}{\omega_{out}} = -\frac{\omega_{in}}{3\,\omega_{in}} = -\frac{1}{3}}$$

The previous table is for external spur gears, pressure angle of $\emptyset = 20°$, the diametral pitch of $P_d = 16$, and we can find two gears with this ratio from it (there are many tables for different gears and different specifications). From this table, we have several options:

(36 & 12), (60 & 20), (72 & 24), etc.

Let's pick (36 & 12). So,

$$N_{in} = 36\ teeth \qquad \& \qquad N_{out} = 12\ teeth$$

This gear system:

$$\left.\begin{aligned} \omega_{out} &= -3\,\omega_{in} \\ T_{out} &= -\tfrac{1}{3}\,T_{in} \end{aligned}\right\} \qquad \text{Increase angular velocity } (\omega) \text{ \& decrease torque } (T)$$

If we have **standard gears**, we can calculate other parameters:

1) Addendum: $a = \frac{1.0}{P_d} = 0.0625\ in$
2) Dedendum: $b = \frac{1.25}{P_d} = 0.078\ in$
3) Pitch Radius:

$$P_d = \frac{N}{2r_p}$$

$$r_{p_{in}} = \frac{N}{2P_d} = \frac{36}{2\times 16} = 1.125$$

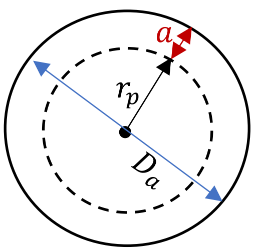
$$r_{p_{out}} = \frac{N}{2P_d} = \frac{12}{2\times 16} = 0.375$$

4) Outside diameter:

$$D_a = 2r_p + 2a$$

$$D_{a_{in}} = 2r_{p_{in}} + 2a_{in} = 2\times 1.125 + 2\times 0.0625 = 2.375$$

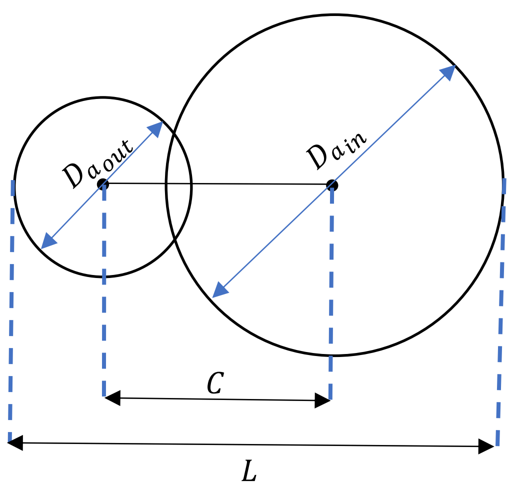
$$D_{a_{out}} = 2r_{p_{out}} + 2a_{out} = 2\times 0.375 + 2\times 0.0625 = 0.875$$



5) Center Distance

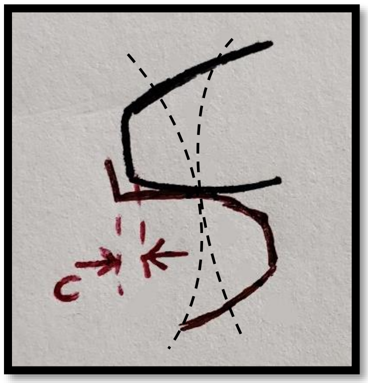
$$C = r_{p_{in}} + r_{p_{out}} = 1.125 + 0.375 = 1.5$$

$L = 2r_{p_{in}} + 2r_{p_{out}} + 2a = 2\times(1.125 + 0.375 + 0.0625) = 3.125$ (Overall dimension of the system)



6) Clearance (at the pitch point)

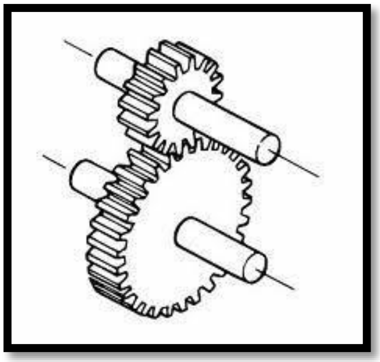
$$c = b - a = 0.078 - 0.0625 = 0.0155$$

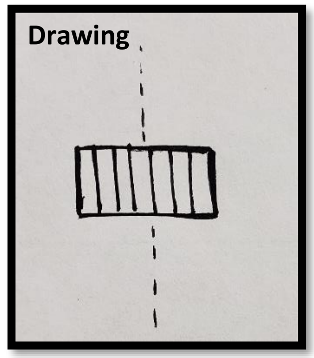


Gear Systems (Continue)

Different types of gears:

1) **Spur Gears:** Shafts are parallel to each other. Also, the axes of the shafts and teeth are parallel



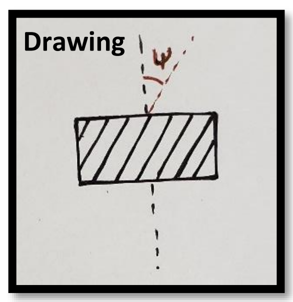


$$R = \frac{\omega_{in}}{\omega_{out}} = \pm \frac{N_{out}}{N_{in}}$$

2) **Helical Gears: Parallel Axes or Non-parallel and Non-intersecting Axes.** The teeth are at an angle Ψ (Helical Angle) with respect to the shaft (shaft is in direction of angular velocity).

Parallel
Helical Gears

Nonparallel &
Nonintersecting
Helical Gears



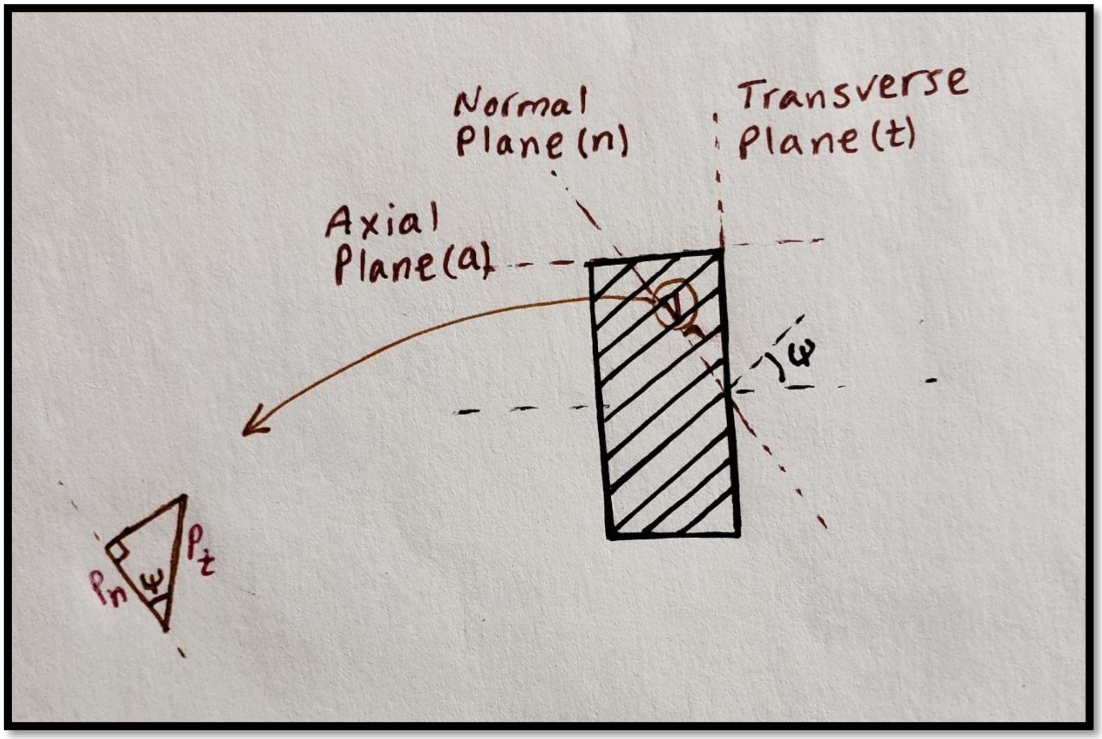
Advantages & disadvantages: Run smoother because contact between teeth of the gears changes smoothly, they are stronger because of longer and thicker teeth and if you need the axes of gears to be in an angle you have to use helical gears and it is better for higher torque and better for smooth operation but it has more friction (axial forces) and slightly less efficient than spur gears.

Helical Gears, Parallel Axes:

All the magnitudes and geometry of the helix gear depending on which plane you are looking at.

We have three different planes: Normal plane, Transverse plane, and Axial plane. The normal plane is the same plane we used for the Spur gears and when you manufacture your gear using this plane (cut the teeth in the perpendicular direction to the normal plane).

All the magnitude we calculated before, now can be written in form of axial (a), normal (n), or transverse (t). For instance for circular pitch, we will have: axial pitch (P_a), normal pitch (P_n), and transverse pitch (P_t).



For circular pitch:

$cos\,\Psi = \frac{p_n}{p_t}$ $\Rightarrow$ $\boxed{p_n = p_t\, cos\,\Psi}$ & $\boxed{p_a = \frac{p_n}{sin\,\Psi}}$

But in the catalogs, we are using the diametral pitch $(P_d = \frac{\pi}{p_c})$.

Diametral Pitch ← $\boxed{P_n = \frac{\pi}{p_n}}$ → Circular Pitch

$\boxed{P_n = P_t\, cos\,\Psi}$

For the rest of them is the same method. In catalogs, they are using the transverse magnitudes.

Helical Gears

8 and 6 Transverse Diametral Pitch (Bronze & Steel – Hardened)

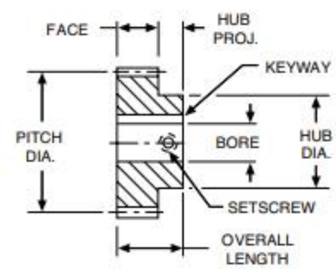
14-1/2° Normal Pressure Angle 45° Helix Angle

All gears with hubs have setscrew at 90° to keyway. Steel gears have teeth only hardened, except as noted. Teeth on all steel gears are polished.

ALL DIMENSIONS IN INCHES
ORDER BY CATALOG NUMBER OR ITEM CODE

No. of Teeth	Pitch Dia.	Bore	Hub Dia.	Hub Proj.	Keyway	Style See Page 323	RIGHT HAND Catalog Number	RIGHT HAND Item Code	LEFT HAND Catalog Number	LEFT HAND Item Code
8 TRANSVERSE DIAMETRAL PITCH							**Face without Hubs = 1.000" –with Hubs = .750" Overall Length = Face + Hub Proj.**			
STEEL–HARDENED										
8	1.000	.500	–	–	1/8 x 1/16	A	H808R*	18066	H808L*	18064
10	1.250	.625	–	–			H810R*	18070	H810L*	18068
12	1.500	.750	–	–	3/16 x 3/32		H812R	18074	H812L	18072
16	2.000	.875	–	–			H816R	18078	H816L	18076
20	2.500						H820R	18082	H820L	18080
24	3.000						H824R	18086	H824L	18084
32	4.000						H832R	18090	H832L	18088
8	1.000	.500	.75	.50	1/8 x 1/16	A	HS808R*	18092	HS808L*	18094
10	1.250	.625	1.00	.50			HS810R*	18096	HS810L*	18098
12	1.500	.750	1.25	.50	3/16 x 3/32		HS812R*	18100	HS812L*	18102
16	2.000	1.000	1.62	.50	1/4 x 1/8		HS816R	18104	HS816L	18106
20	2.500		2.00				HS820R	18108	HS820L	18110
24	3.000						HS824R	18112	HS824L	18114
32	4.000						HS832R	18116	HS832L	18118
40	5.000						HS840R	18120	HS840L	18122
48	6.000		2.25				HS848R	18124	HS848L	18126
BRONZE										
8	1.000	.500	.75	.50	1/8 x 1/16	A	HB808R	18356	HB808L	18358
10	1.250	.625	1.00	.50			HB810R	18360	HB810L	18362
12	1.500	.750	1.24	.50	3/16 x 3/32		HB812R	18364	HB812L	18366
16	2.000	1.000	1.62	.50	1/4 x 1/8		HB816R	18368	HB816L	18370
20	2.500		2.00				HB820R	18372	HB820L	18374
24	3.000						HB824R	18376	HB824L	18378
32	4.000					B	HB832R	18380	HB832L	18382
40	5.000						HB840R	18384	HB840L	18386
48	6.000		2.25				HB848R	18388	HB848L	18390
6 TRANSVERSE DIAMETRAL PITCH							**Face without Hubs = 1.250" –with Hubs = 1.000" Overall Length = Face + Hub Proj.**			
STEEL–HARDENED										
8	1.333	.625	–	–	1/8 x 1/16	A	H608R	18000	H608L	18002
10	1.667	.750	–	–	3/16 x 3/32		H610R	18004	H610L	18006
12	2.000	1.000	–	–	1/4 x 1/8		H612R	18010	H612L	18008
15	2.500						H615R	18014	H615L	18012
18	3.000						H618R	18018	H618L	18016
24	4.000						H624R	18022	H624L	18020
8	1.333	.625	1.00	.75	1/8 x 1/16	A	HS608R	18024	HS608L	18026
9	1.500	.750	1.18	.75	3/16 x 3/32		HS609R	18028	HS609L	18030
10	1.667		1.34				HS610R	18032	HS610L	18034
12	2.000	1.000	1.62	.75	1/4 x 1/8		HS612R	18036	HS612L	18038
15	2.500	1.250	2.00	.75	5/16 x 5/32		HS615R	18040	HS615L	18042
18	3.000		2.25				HS618R	18044	HS618L	18046
20	3.333		2.50				HS620R	18048	HS620L	18050
24	4.000						HS624R	18052	HS624L	18054
30	5.000						HS630R	18056	HS630L	18058
36	6.000						HS636R	18060	HS636L	18062
BRONZE										
12	2.000	1.000	1.62	.75	1/4 x 1/8	A	HB612R	18328	HB612L	18330
15	2.500	1.250	2.00	.75	5/16 x 5/32		HB615R	18332	HB615L	18334
18	3.000		2.25				HB618R	18336	HB618L	18338
20	3.333		2.50				HB620R	18340	HB620L	18342
24	4.000					B	HB624R	18344	HB624L	18346
30	5.000						HB630R	18348	HB630L	18350
36	6.000						HB636R	18352	HB636L	18354

*Hardened all over.



STANDARD TOLERANCES

DIMENSION		TOLERANCE
BORE	All	±.0005

REFERENCE PAGES

Alterations — 322
Horsepower Ratings — 67, 68
Lubrication — 322
Materials — 323
Selection Procedure — 66

NOTE: Normal Diametral Pitch is equal to the Transverse Diametral Pitch divided by the cosine of the Helix Angle.

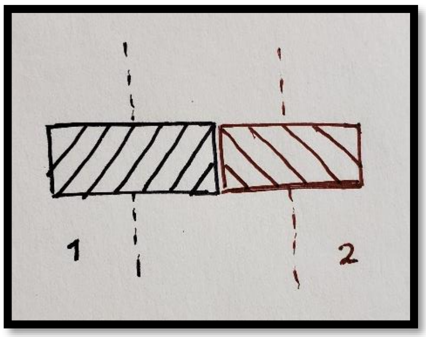
Also, the pressure angle ($\emptyset$) will be changed with the plane we are using (Note: the pressure angle depends on the profile of the teeth and defines the direction of the force).

$\emptyset_t$: Transverse pressure angle

$\emptyset_n$: Normal pressure angle

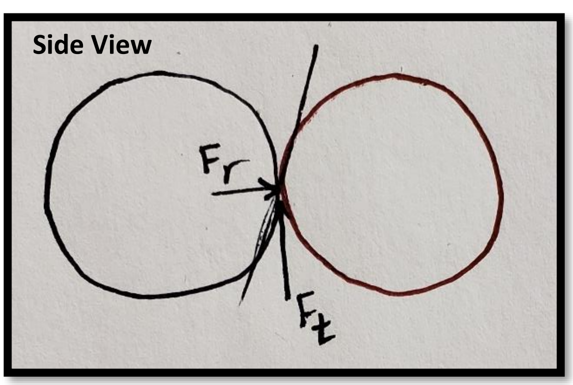
$$\boxed{\tan \emptyset_t = \frac{\tan \emptyset_n}{\cos \Psi}}$$

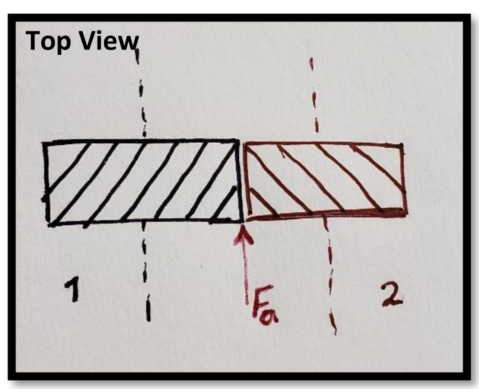
We can use these formulas for parallel helical gears:



For meshing, we need:

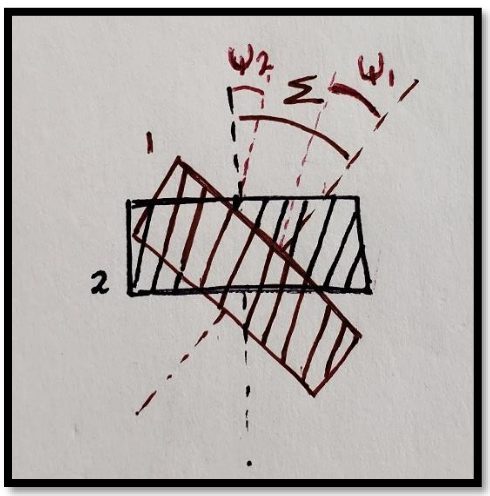
- $\Psi_1 = -\Psi_2$
- Same normal pitch, $P_{n_1} = P_{n_2}$ (in this case, because the angles are the same so we will have the same transverse pitch)
- Same pressure angle in the normal plane for both gears ($\emptyset_{n_1} = \emptyset_{n_2}$)
- In this type of gear, in addition to radial and tangential force, we will have axial force.





That means, we have to put a specific type of bearings to keep the gears inside of the shaft and prevent the gears to move along the shafts. Also, we can use two gears of opposite hands together (or one gear with teeth in two opposite directions for each half which is known as **Herringbone** gears). So, in this case, you have axial force in one direction and another axial force in the other direction and they cancel each other.

Helical Gears, Nonparallel, and Nonintersecting Axes: This is not a very common helical gear system and is only used for particular tasks (parallel one is more common).



Σ (Sigma): Angle between shafts axes

$\boldsymbol{\Psi_1}$, $\boldsymbol{\Psi_2}$: Helical Angles

$$\boxed{\Sigma = \boldsymbol{\Psi_1} \pm \boldsymbol{\Psi_2}}$$

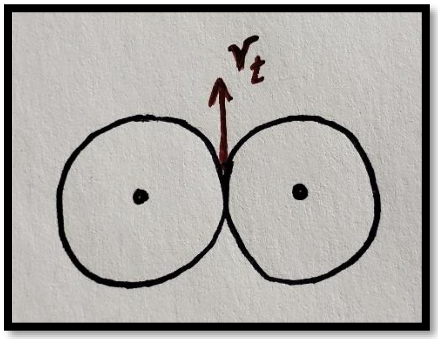
For meshing, we need:

- Same normal pitch ($P_{n_1} = P_{n_2}$)
- Compatible helix angles, ($\Sigma = \Psi_1 \pm \Psi_2$)
- Same pressure angle in the normal plane for both gears ($\emptyset_{n_1} = \emptyset_{n_2}$)

Gear ratio: Can we still use the same formula?

The number of teeth of the output over the number of teeth of the input gives the gear ratio regardless of the geometry.

Gear ratio for parallel axes:



The velocity of the contact point is the same in the transverse plane:

$v_{t_1} = v_{t_2}$

$\omega_1 r_{t_1} = \omega_2 r_{t_2}$ (r_t is the radius in the transverse plane)

Then gear ratio would be:

$$R = \frac{\omega_1}{\omega_2} = \pm \frac{r_{t_2}}{r_{t_1}}$$

Also, we know $p_t = \frac{2\pi r_t}{N}$ (p_t: circular pitch and r_t pitch radius)

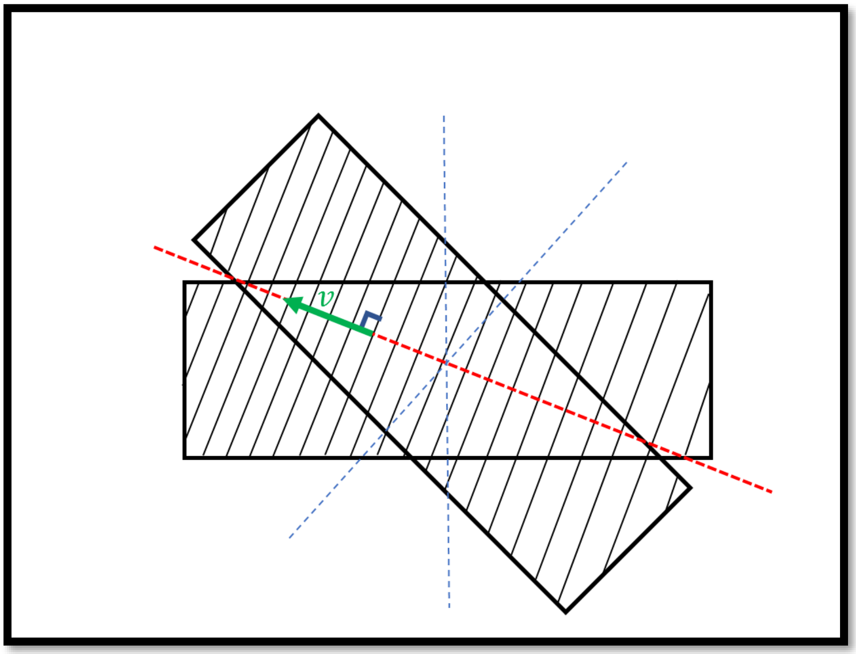
$$\left.\begin{array}{l} \dfrac{r_{t_2}}{r_{t_1}} = \dfrac{(p_{t_2}N_2)/2\pi}{(p_{t_1}N_1)/2\pi} = \dfrac{p_{t_2}N_2}{p_{t_1}N_1} \\ p_t = p_n/\cos\Psi \end{array}\right\} \Rightarrow \frac{r_{t_2}}{r_{t_1}} = \frac{p_{n_2}N_2 \cos\Psi_1}{p_{n_1}N_1 \cos\Psi_2}$$

For meshing, we know: $\boldsymbol{p_{n_1}} = \boldsymbol{p_{n_2}}$ & $\Psi_1 = -\Psi_2$, so $\cos\Psi_1 = \cos\Psi_2$

$$\frac{r_{t_2}}{r_{t_1}} = \frac{N_2}{N_1} \Rightarrow \boxed{R = \pm\frac{r_{t_2}}{r_{t_1}} = \pm\frac{N_2}{N_1}} \Rightarrow$$ Proof for the parallel case

Gear ratio for nonparallel axes:

In this case, we cannot define that the tangent velocity will be the same because the axes are at an angle with each other but the normal component of the velocity would be the same ($v_{n_1} = v_{n_2}$).



$$R = \frac{\omega_1}{\omega_2}$$

$$v_{n_1} = v_{n_2} \Rightarrow v_{t_1} cos\,\Psi_1 = v_{t_2} cos\,\Psi_2 \Rightarrow \omega_{t_1} r_{t_1} cos\,\Psi_1 = \omega_{t_2} r_{t_2} cos\,\Psi_2$$

$$(\omega_{t_1} = \omega_1\,,\, \omega_{t_2} = \omega_2) \Rightarrow \omega_1 r_{t_1} cos\,\Psi_1 = \omega_2 r_{t_2} cos\,\Psi_2$$

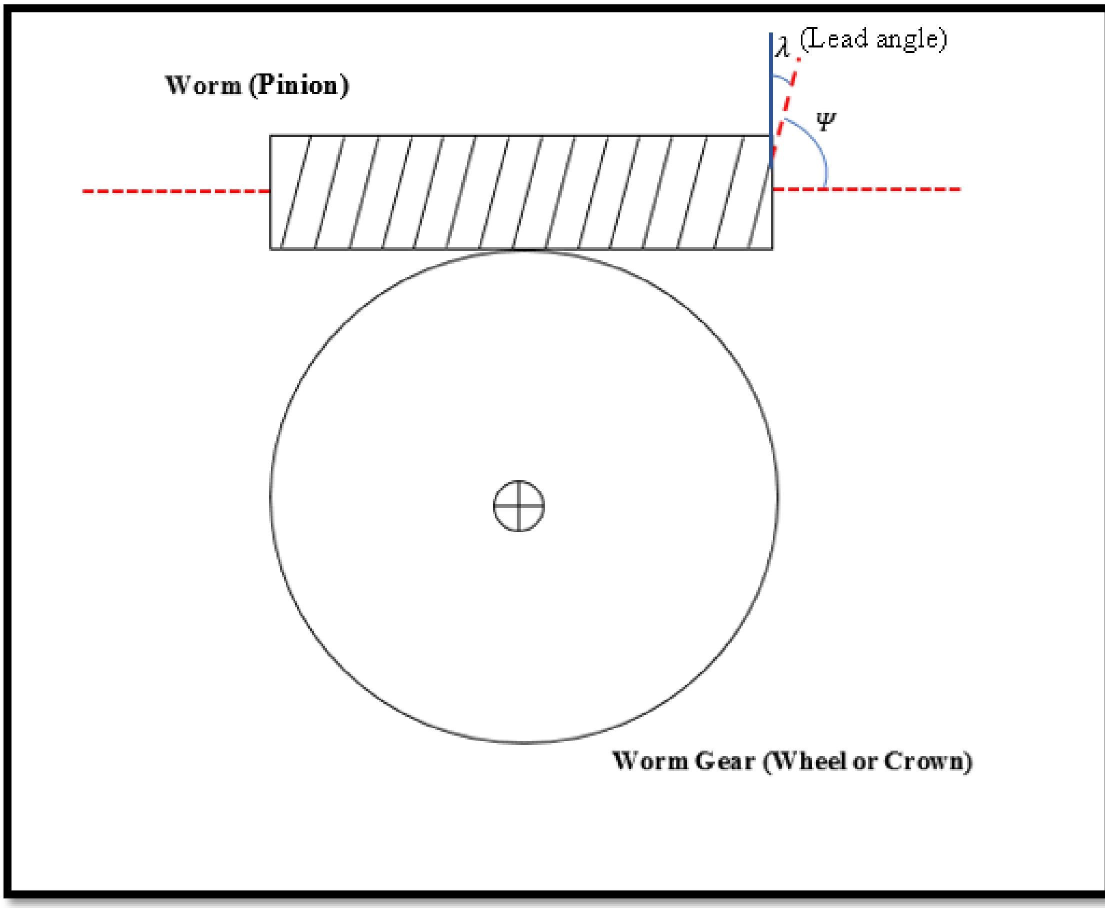
$$\left.\begin{array}{c} R = \dfrac{r_{t_2} cos\,\Psi_2}{r_{t_1} cos\,\Psi_1} = \dfrac{\left({(p_{t_2} N_2)}/{2\pi} \right) cos\,\Psi_2}{\left({(p_{t_1} N_1)}/{2\pi} \right) cos\,\Psi_1} \\ p_t = {p_n}/{cos\,\Psi} \end{array}\right\} \Rightarrow R = \frac{p_{n_2} N_2}{p_{n_1} N_1}$$

(The gears are not parallel so there is no +/- in this

For meshing, we know: $\boldsymbol{p_{n_1} = p_{n_2}} \Rightarrow \boxed{R = \frac{N_2}{N_1}} \Rightarrow$ Proof for nonparallel case

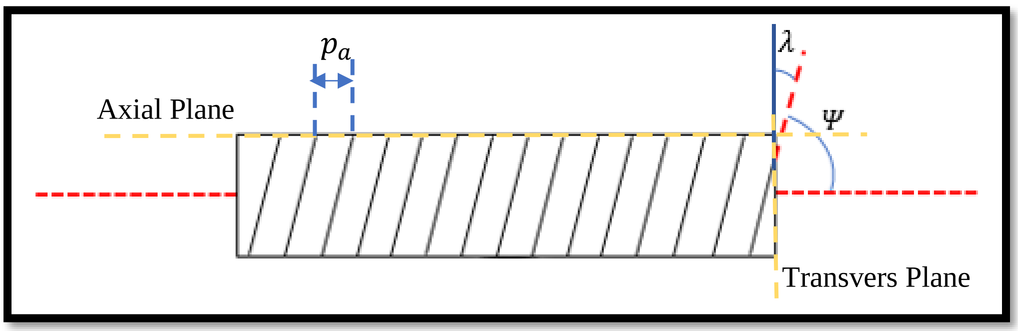
Gear Systems (Continue)

3) **Worm & Worm Gear:** This is a special case of helix gear when the helix angle is so big and the teeth curls on the wheel.



Some of the properties of this type of gear:

- They are at perpendicular angles (the transverse pitch of gear will equal to the axial pitch of the worm)
- Usually 1 to 3 teeth in the worm
- We will have high gear ratios (if we have 1-3 teeth on worm and a high number of teeth on gear)
- High friction (sliding velocity along the teeth is higher)
- Not very efficient (efficiency between 40% to 85%, $efficiency = \frac{power\ out}{power\ in} \times 100$)
- Motion from worm to worm gear only (They are not reversible and work in one direction)



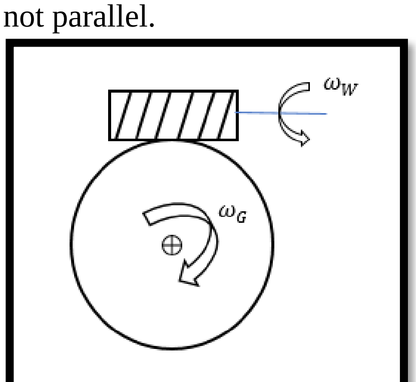
Lead: Motion of worm per revolution

$L = N.p_a \quad (N = number\ of\ teeth\ of\ the\ worm\ \&\ p_a = pitch\ in\ the\ axial\ plane)$

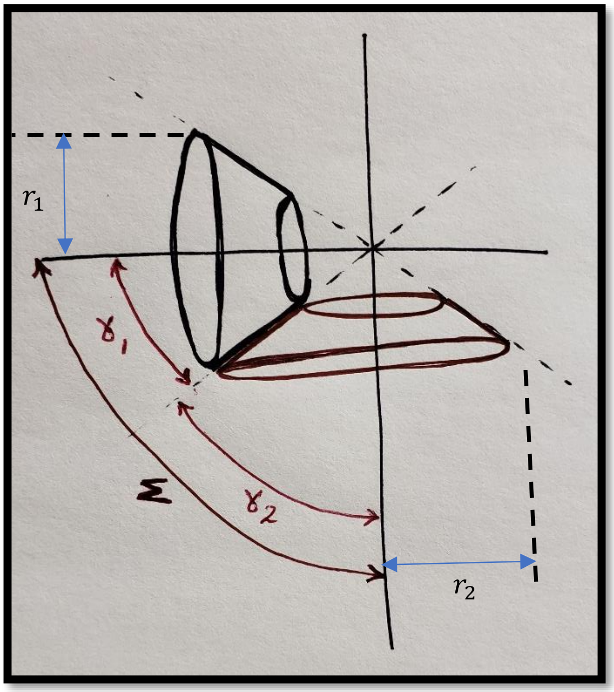
For worm and worm gear to mesh:

- The axial pitch of the worm is equal to the transverse pitch of the gear ($P_{a_W} = P_{t_G}$)
- The helix angle of the gear has to be the lead angle of the worm ($\lambda_W = \Psi_G$)
- The axial velocity of the point on the worm will be equal to the tangent velocity of one point on the gear

Exercise: Proof the gear ratio would be: $R = \frac{\omega_W}{\omega_G} = \frac{N_G}{N_W}$ (Use $v_{a_W} = v_{t_G}$). In this case, we don't need $\pm$ because the axes are not parallel.



4) Bevel Gears: Using for Nonparallel and Intersecting Axes



Σ $(sigma)$: Angle between axes. In this case, is 90° but it can be any angle that you want

$\gamma(gamma)$: Angle for each gear cone

- $\Sigma = \gamma_1 + \gamma_2$
- We can have different teeth for these gears (helical teeth or straight teeth or other different shapes of teeth) and all the formulas apply here.
- Magnitudes measured at the outside circle of the cones
- Gear ratio: $R = \frac{\omega_1}{\omega_2} = \frac{N_2}{N_1}$ .In this case, we don't need $\pm$ because the axes are not parallel.

Note: So, you can always use the number of teeth of the gears to find the gear ratio (no matter what is the type of the gears).

https://www.youtube.com/watch?v=W3D1IFwMuYc

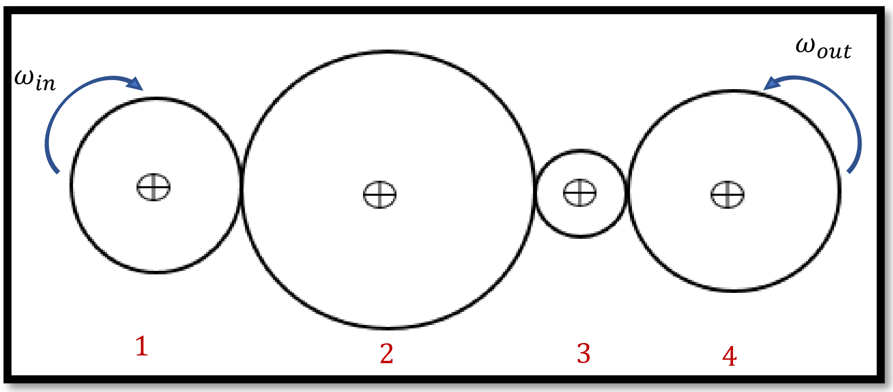
Gear Ratio for Gear Trains

We call gear trains when we have several gears arranged either in series or in parallel for purpose of creating a single gear ratio.

In general, for gear ratio, we have:

$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{T_{out}}{T_{in}} = \frac{N_{out}}{N_{in}}$$

a) Simple gear train: Gears mounted on fixed axes and just one gear on each axis. The gears can be any type of gear (not only spur gear) and their axis can be at 90° with each other, as long as there is only one gear per axes.



$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_1}{\omega_4} = \left(\frac{\omega_1}{\omega_2}\right).\left(\frac{\omega_2}{\omega_3}\right).\left(\frac{\omega_3}{\omega_4}\right) = \left(-\frac{\cancel{N_2}}{N_1}\right).\left(-\frac{\cancel{N_3}}{\cancel{N_2}}\right).\left(-\frac{N_4}{\cancel{N_3}}\right) = \boxed{-\frac{N_4}{N_1}}$$

So, only 1st and last gears count for gear ratio. That means the intermediate gears don't affect the value of the gear ratio for a gear train and they only affect the sign of it!

Then why do we use these gear trains? There are different reasons, it can be because we want to change the direction of motion or maybe there is a big gap between input and output shafts or if the size of input and output gears are very different (one of them very small and the other one is very big) which is not good, with adding gears with medium size between them, the speed can change gradually.

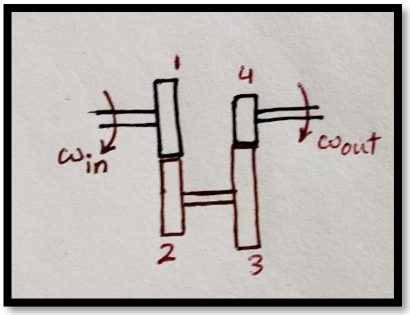
Note: All of the gears in the above example have to have the same diametral pitch and pressure angle, so we can make them mesh two by two.

The general gear ratio formula for simple gear trains:

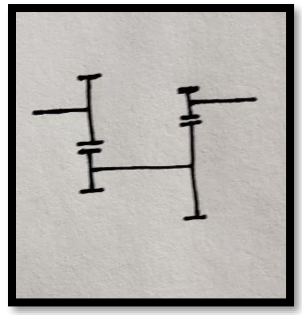
$$R = (-1)^m \frac{N_{out}}{N_{in}} \;, \quad m = number\ of\ meshing\ stages$$

b) **Compound gear train:** Gears are mounted on fixed axes, but we can have more than one gear on each axis (per shaft).

Note: The gears have to rotate with the shaft in general (there are some exceptions). So, if two gears are mounted to a shaft that means they have the **same angular velocity**.



Kinematic Sketch



$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_1}{\omega_4} = \left(\frac{\omega_1}{\omega_2}\right).\left(\frac{\omega_2}{\omega_3}\right).\left(\frac{\omega_3}{\omega_4}\right)$$

Remember that we don't want to cancel the similar ω in the above equation, because we only know about the relation of the gears two by two (we don't know what are ω_1& ω_4)!

$$R = \left(-\frac{N_2}{N_1}\right).(1).\left(-\frac{N_4}{N_3}\right) = \boxed{\frac{N_2 N_4}{N_1 N_3}}$$

Note: In the above example, gear 1 and 2 (same for gear 3 and 4) have to have the same diametral pitch and pressure angle, so we can make them mesh but gears 2 and 3 can be completely different (there is no relation between them)!

The general gear ratio formula for compound gear trains:

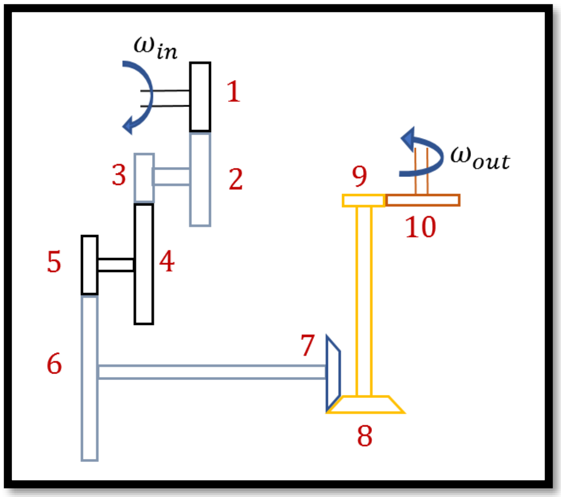
$$R = (-1)^m \frac{\Pi\, N_{driven}}{\Pi\, N_{driving}} \;, \quad m = number\ of\ meshing\ stages$$

Note: Π means the product of a series of values.

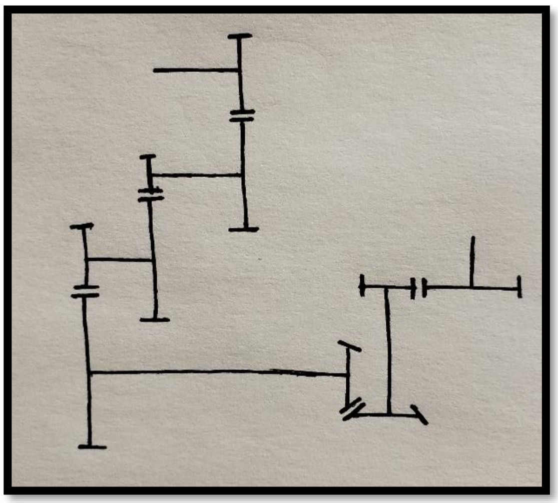
If this equation is confusing, you can use step by step process between every two gears same as the previous example.

Example: The following gear system is a combination of some helical and bevel gears. A) Find the gear ratio of the system. B) Find the output angular velocity when $\omega_{in} = 1800\ rpm$.

$N_1 = 21$
$N_2 = 35$
$N_3 = 23$
$N_4 = 48$
$N_5 = 39$
$N_6 = 40$
$N_7 = 23$
$N_8 = 34$
$N_9 = 40$
$N_{10} = 46$



A)
Kinematic sketch:



$$R = \frac{N_2 N_4 N_6 N_8 N_{10}}{N_1 N_3 N_5 N_7 N_9} = \frac{35 \times 48 \times 40 \times 34 \times 46}{21 \times 23 \times 39 \times 23 \times 40} = \boxed{6.06}$$

Note: In this case, we are not putting +/- because the axes are not parallel (there is no meaning for + or -). We only use +/- for gears with a parallel axis! However, we have to keep track of rotation from input to output.

B)
$R = \frac{\omega_{in}}{\omega_{out}}$ $\qquad \omega_{out} = \frac{\omega_{in}}{R} = \frac{1800}{6.06} = \boxed{296.8\ rpm}$

So, this system reduces the angular velocity and increases the output torque!

Gear Systems (Continue)

Guidelines for Design of Compound Gear Trains:

For a given value of R,

- Iterative process
- Approximate the number of stages needed. This estimation can be done by the difference between the input and output size of gears and the number of teeth for each of them. We have some limitations for the number of teeth for two gears in connection and we cannot jump from a very small gear to a very big gear (or reverse). And this is one of the reasons that in the catalog we have a limitation for the number of teeth for the gears (N_{min}, N_{max}). So, the maximum gear ratio at a single stage is limited (always $R < 10$).

$$R_i^{\,n} = R$$

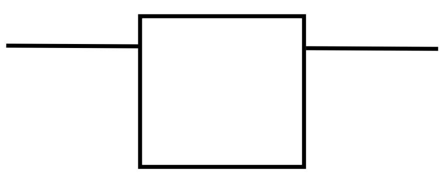
  For a given R and R_i is your assumption (< 10), and you can calculate n (n is the 1st approximation to find the number of stages)

- Space requirements: Limit gear size (limit in pitch diameter d_p and that means a limit in gear ratio at each stage R_i)
- N_i have to be an integer number

Example: Design of compound gear train, based on following requirements.

We want:

- $R = 12{:}1$
- Input/output axes collinear
- $Box\ size\ < 5\ in$



For several reasons this system cannot be a single stage, first, the axes have to be collinear and R is too big for a single stage!

Step 1: # of stages:
This number is small enough that we can assume that we will be able to reach this R value with two stages. We can use the formula in both ways. We can assume this is the number of stages and then find the gear ratio for each stage or we can take some gear ratio for each stage and find the number of stages.

Let's start with two stages and find the gear ratio for each stage.

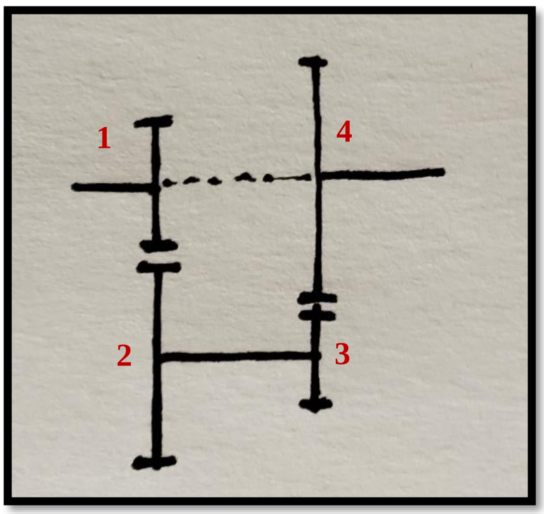
$$R_i^{\,2} = 12$$

R_i: *The gear ratio of each stage*

$$R_i = 3.5 \; (between\ 3\ and\ 4)$$

$$R_1 = \frac{N_2}{N_1} \cong 3.5 \qquad R_2 = \frac{N_4}{N_3} \cong 3.5$$

Step 2: Estimate kinematic sketch



Step 3: Use the equation for R and the number of teeth for the gears

$$R = \frac{N_2 N_4}{N_1 N_3} = 12$$

The axes are parallel, so we can apply +/- to the equation. We have two stages, so it will be +.

Step 4: For using catalogs, first we have to select the type of gear (helical gear or spur gear)

Step 5: For Helix gear, select the helix angle, transverse diametral pitch, and normal pressure angle.

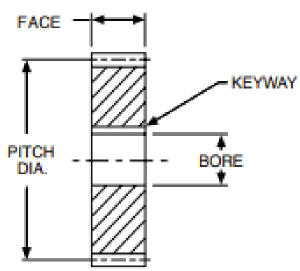
Step 6: If we don't have more conditions, we can go to the catalog and select the number of teeth for each gear in the way the value for R becomes 12 (The gear ratio for each stage should be between 3 to 4). This would be the last step and the problem is solved.

Helical Gears

24 through 10 Transverse Diametral Pitch (Steel – Hardened)

14-1/2° Normal Pressure Angle – 45° Helix Angle

B



These gears are hardened all over, except as noted. Teeth on all steel gears are polished.

ALL DIMENSIONS IN INCHES
ORDER BY CATALOG NUMBER OR ITEM CODE

No. of Teeth	Pitch Dia.	Bore	Keyway	Style See Page 323	RIGHT HAND Catalog Number	RIGHT HAND Item Code	LEFT HAND Catalog Number	LEFT HAND Item Code
24 TRANSVERSE DIAMETRAL PITCH					**Face: 8-15 Teeth = .375" 18-72 Teeth = .250"**			
8	.333	.1875	*	A	H2408R	18268	H2408L	18270
10	.417	.250	**		H2410R	18272	H2410L	18274
12	.500				H2412R	18276	H2412L	18278
15	.625	.375			H2415R	18280	H2415L	18282
18	.750		1/8 x 1/16		H2418R	18284	H2418L	18286
20	.833	.500			H2420R	18288	H2420L	18290
24	1.000				H2424R	18292	H2424L	18294
30	1.250	.625			H2430R	18296	H2430L	18298
36	1.500				H2436R†	18300	H2436L†	18302
48	2.000				H2448R†	18304	H2448L†	18306
60	2.500				H2460R†	18308	H2460L†	18310
72	3.000				H2472R†	18312	H2472L†	18314
20 TRANSVERSE DIAMETRAL PITCH					**Face:8-15 Teeth = .563" 18-72 Teeth = .375"**			
8	.400	.250	**	A	H2008R	18228	H2008L	18230
10	.500	.3125			H2010R	18232	H2010L	18234
12	.600	.375			H2012R	18236	H2012L	18238
15	.750	.4375	1/8 x 1/16		H2015R	18240	H2015L	18242
20	1.000	.500			H2020R	18244	H2020L	18246
25	1.250	.625			H2025R	18248	H2025L	18250
30	1.500	.750	3/16 x 3/32		H2030R†	18252	H2030L†	18254
40	2.000				H2040R†	18256	H2040L†	18258
50	2.500				H2050R†	18260	H2050L†	18262
60	3.000				H2060R†	18264	H2060L†	18266
16 TRANSVERSE DIAMETRAL PITCH					**Face = .500"**			
12	.750	.375	1/16 x 1/32	A	H1612R	18200	H1612L	18202
16	1.000	.500	1/8 x 1/16		H1616R	18204	H1616L	18206
20	1.250				H1620R	18208	H1620L	18210
24	1.500				H1624R†	18212	H1624L†	18214
32	2.000				H1632R†	18216	H1632L†	18218
40	2.500				H1640R†	18220	H1640L†	18222
48	3.000				H1648R†	18224	H1648L†	18226
12 TRANSVERSE DIAMETRAL PITCH					**Face = .750"**			
12	1.000	.625	1/8 x 1/16	A	H1212R	18170	H1212L	18168
15	1.250				H1215R	18174	H1215L	18172
18	1.500				H1218R†	18178	H1218L†	18176
24	2.000				H1224R†	18182	H1224L†	18180
30	2.500				H1230R†	18186	H1230L†	18184
36	3.000				H1236R†	18190	H1236L†	18188
10 TRANSVERSE DIAMETRAL PITCH					**Face = .875"**			
8	.800	.375	1/16 x 1/32	A	H1008R	18130	H1008L	18128
10	1.000	.500	1/8 x 1/16		H1010R	18134	H1010L	18132
12	1.200	.625			H1012R	18138	H1012L	18136
15	1.500	.750	3/16 x 3/32		H1015R†	18142	H1015L†	18140
20	2.000				H1020R†	18146	H1020L†	18144
25	2.500				H1025R†	18148	H1025L†	18150
30	3.000				H1030R†	18154	H1030L†	18152
40	4.000				H1040R†	18158	H1040L†	18156

*1/16" wide x .04" deep slot cut on end of gear for drive pin, not key.
**3/32" wide x .06" deep slot cut on end of gear for drive pin, not key.
†Teeth only hardened.

STANDARD TOLERANCES

DIMENSION		TOLERANCE
BORE	All	±.0005

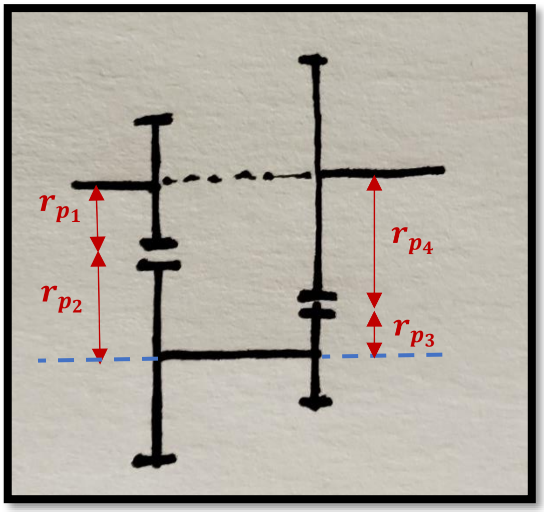
REFERENCE PAGES

Alterations — 322
Horsepower Ratings — 67
Lubrication — 322
Materials — 323
Selection Procedure — 66

NOTE: Normal Diametral Pitch is equal to the Transverse Diametral Pitch divided by the cosine of the Helix Angle.

Step 7: In this case, we have more conditions. Based on axes collinear:

$$r_{p_1} + r_{p_2} = r_{p_3} + r_{p_4}$$



Because we are using helical gears, for every two connected gears:

$$P_{t_1} = P_{t_2} \qquad \& \qquad P_{t_3} = P_{t_4} \qquad \text{(transvers diametral pitch)}$$

$$\emptyset_{n_1} = \emptyset_{n_2} \qquad \& \qquad \emptyset_{n_3} = \emptyset_{n_4}$$

(The pressure angle can be normal or transverse any of them is in the catalog)

Use relation:

$$\frac{\boldsymbol{N_i}}{\boldsymbol{2r_{p_i}}} = P_{t_i}$$

Step 8: Based on size limitation for the box:

Let's assume gear 2 is bigger than gear 1 and gear 4 is bigger than gear 3 (reduction gear train):

$$r_{p_1} + r_{p_2} + (r_{p_2} + a_2) + (r_{p_4} + a_4) < 5\ in$$

"a" is an addendum

Select the gears and check the conditions:
Now we have all equations we need. Let's start with satisfying the gear ratio condition:

Steps 1-3:
Let's preselect the number of teeth for pinions (small gears) and check if we can find the number for the big gears.

$$N_1 = 12$$
$$N_3 = 12$$

$$R = \frac{N_2 N_4}{144} = 12$$

$$N_2 N_4 = 1728$$

Now, you can decompose 1200 in prime factors, and try to find the match number in the catalog from a combination of these numbers.

$$1728 = 2^6 \times 3^3$$

For instance 48 and 36:

$$\boxed{\begin{array}{c} 2^4 \times 3 = 48 \\ 2^2 \times 3^2 = 36 \end{array}}$$

So, we can check the gear ratios

$$R_1 = \frac{N_2}{N_1} = \frac{48}{12} = 4 \qquad R_2 = \frac{N_4}{N_3} = \frac{36}{12} = 3$$

$$R = R_1 \times R_2 = 4 \times 3 = 12$$

Step 4:

Let's choose helical gear

Step 5:

Based on the available catalog, let's select $\Psi = 45°$ (it is a very common helix angle)
$P_t = 24$
$\emptyset_n = 14 - 1/2°$

We have to check if such gears are available in the catalog. In this case, we have both 36 and 48 teeth gears.

Step 6: It is not applied to this case.

Step 7: For collinear axes condition, we need to find the pitch radius of the gears (r_{p_i}) and check the following relation between them.

$$r_{p_1} + r_{p_2} = r_{p_3} + r_{p_4}$$

We can use a different formula for finding r_{p_i}, but in this case, we have the number of teeth and transverse diametral pitch so we can use:

$$\frac{\boldsymbol{N_i}}{\boldsymbol{2r_{p_i}}} = P_{t_i} \Rightarrow r_{p_i} = \frac{\boldsymbol{N_i}}{\boldsymbol{2P_{t_i}}}$$

$$r_{p_1} = \frac{N_1}{2P_{t_1}} = \frac{12}{2 \times 24} = 0.25\ in$$
$$r_{p_2} = \frac{N_2}{2P_{t_2}} = \frac{48}{2 \times 24} = 1\ in$$
$$r_{p_3} = \frac{N_3}{2P_{t_3}} = \frac{12}{2 \times 24} = 0.25\ in$$
$$r_{p_4} = \frac{N_4}{2P_{t_4}} = \frac{36}{2 \times 24} = 0.75\ in$$

$$r_{p_1} + r_{p_2} = 0.25 + 1 = 1.25\ in$$
$$r_{p_3} + r_{p_4} = 0.25 + 0.75 = 1\ in$$

$r_{p_1} + r_{p_2} \neq r_{p_3} + r_{p_4}$ $\Rightarrow$ They are not collinear (it is not a good design!)

One option is changing the diametral pitch. We can do is selecting different diametral pitches for gears 2 and 3. Remember the gears 1 &2 and 3 & 4 two by two have to have the same diametral pitch but 2 & 3 can have different diametral pitches!

Let's change the diametral pitch for gears 3 & 4 (keep $P_{t_1} = P_{t_2} = 24$)
:
So, we want $r_{p_3} + r_{p_4} = 1.25$. Also, we know the gear ratio between gears 3 & 4 is equal to 3.

$$r_{p_3} + r_{p_4} = r_{p_3} + 3r_{p_3} = 4r_{p_3} = 1.25$$
$$r_{p_3} = 0.31 \qquad \& \qquad r_{p_4} = 0.94$$

Note: We don't use these values of r_p. These just give us an estimation for selecting the correct diametral pitch.

$P_{t_4} = \frac{N_4}{2r_{p_4}} = \frac{36}{2\times 0.94} = 19.2$ Let's take $P_{t_4} = 20$ and check the error

In the selected catalog, for $P_{t_4} = 20$, unfortunately, we don't have $N_4 = 36$ but let's imagine we have it. Then, we will have:

$$r_{p_1} = \frac{N_1}{2P_{t_1}} = \frac{12}{2 \times 24} = 0.25\ in$$
$$r_{p_2} = \frac{N_2}{2P_{t_2}} = \frac{48}{2 \times 24} = 1\ in$$
$$r_{p_3} = \frac{N_3}{2P_{t_3}} = \frac{12}{2 \times 20} = 0.3\ in$$
$$r_{p_4} = \frac{N_4}{2P_{t_4}} = \frac{36}{2 \times 20} = 0.9\ in$$

$$r_{p_1} + r_{p_2} = 0.25 + 1 = 1.25\ in$$
$$r_{p_3} + r_{p_4} = 0.3 + 0.9 = 1.2\ in$$

$r_{p_1} + r_{p_2} \cong r_{p_3} + r_{p_4}$ The numbers do not exactly match but are close enough.

This was one strategy. Also, we can repeat the process with different numbers from the beginning.

Next Iteration:

Let's take these values:

$P_{t_{1,2}} = 10$ $P_{t_{3,4}} = 12$

$N_1 = 8$
$N_3 = 10$

Therefore, we will have:

$$N_2 N_4 = 12 \times 8 \times 10 = 960 = 2^6 \times 3 \times 5$$

Select closest numbers from the catalog:
$N_2 = 28$
$N_4 = 34$

$R = \frac{N_2 N_4}{N_1 N_3} = 11.9 \Longrightarrow$ It is not exactly 12

Then, we can calculate the error:

$$error = \frac{\|R_{real} - R_{desired}\|}{R_{desired}} \times 100 = \frac{\|11.9 - 12\|}{12} \times 100 = 0.83\%\ error$$

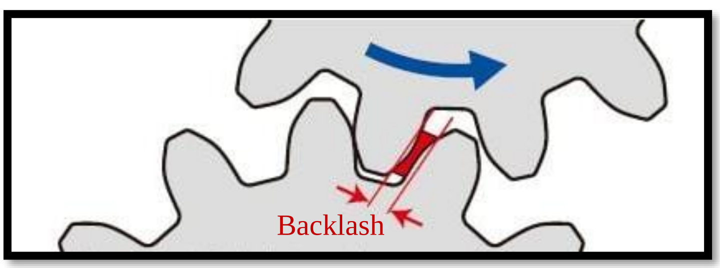
That is a very small error and negligible.

$$r_{p_1} = \frac{N_1}{2P_{t_1}} = \frac{8}{2 \times 10} = 0.4\ in$$
$$r_{p_2} = \frac{N_2}{2P_{t_2}} = \frac{28}{2 \times 10} = 1.4\ in$$
$$r_{p_3} = \frac{N_3}{2P_{t_3}} = \frac{10}{2 \times 12} = 0.416\ in$$
$$r_{p_4} = \frac{N_4}{2P_{t_4}} = \frac{34}{2 \times 12} = 1.416\ in$$

$$r_{p_1} + r_{p_2} = 0.4 + 1.4 = 1.8\ in$$
$$r_{p_3} + r_{p_4} = 0.416 + 1.416 = 1.83\ in$$

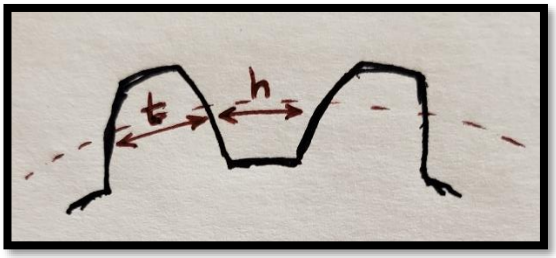
$r_{p_1} + r_{p_2} \cong r_{p_3} + r_{p_4}$ The numbers do not exactly match but are close enough.

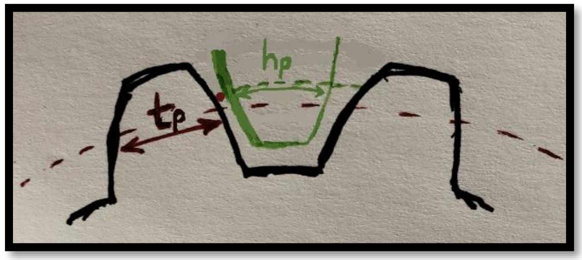
In this case, if we want the gears to be completely collinear, we can change slightly the center distance and the gears will still mesh with the same gear ratio (R) and pressure angle ($\emptyset$). Then there will be a small gap between the teeth of one of the gear sets (for instance gear 3 & 4). That cause an increase in the backlash and more space between the gears' teeth.



Note: We only want contact in the direction of the force. So, we always want a little bit of backlash between gears. That would be good for lubrication of the gears too. But we don’t want too much backlash! For instance, if you driving in both directions when you want to change the direction of the motion, then teeth hit each other, making noise, vibration, impact, etc.

The formula for backlash:



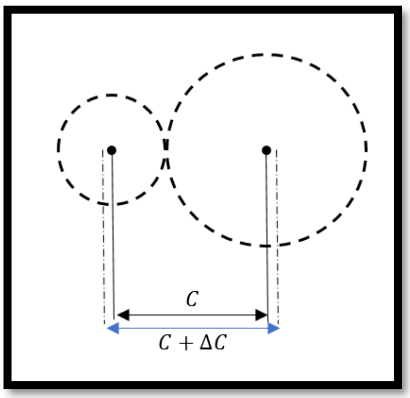


$t = h$ At the pitch circle (for one gear)
$h_p \neq t_p$ At the pitch circle (for two gears in contact)

$\boxed{B = h_p - t_p}$

To create backlash (B):

- Decrease the thickness of the tooth (t_p), so ($t < h$)
- Increase the distance between centers (Increase from C to $C + \Delta C$)



For this case, the backlash would be:

$$\boxed{B = 2\,\Delta C\ tan\ \emptyset}$$

Gear Systems (Continue)

Some other parameters of the gears: These are more analysis parameters (not design parameters)

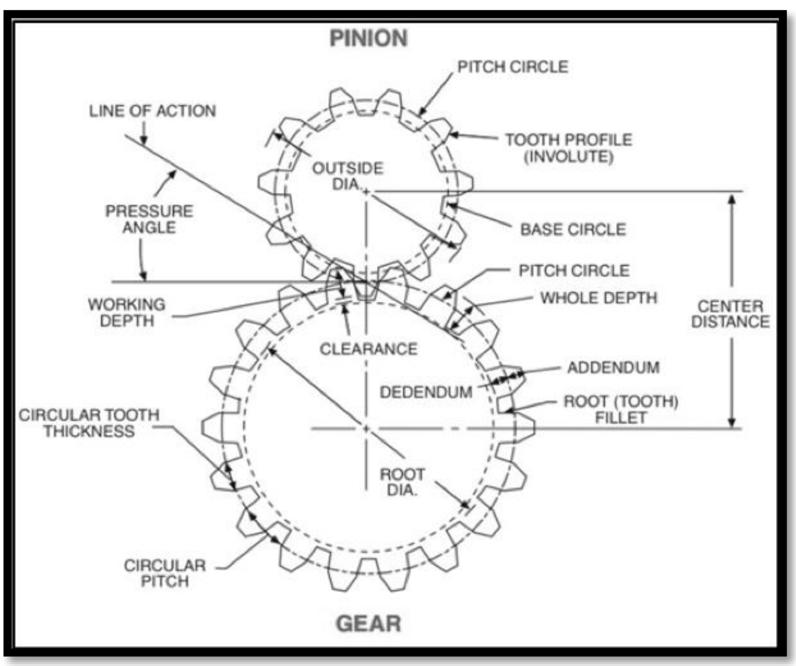
Contact ratio: The number of teeth in contact at the same time. Because this keeps changing (sometimes is one or two, depending on which part of the contact along the teeth you are, so we use an average per revolution)

$$Z = \sqrt{(r_{p_1} + a_1)^2 - {r_{b_1}}^2} + \sqrt{(r_{p_2} + a_2)^2 - {r_{b_2}}^2} - C \sin \emptyset$$

Contact ratio (number of teeth in contact): $\boxed{m_c = \frac{Z}{p_b}}$

p_b: Base pitch (at the base circle)
a: Addendum
r_p: Pitch radius
r_b: Base radius
C: Center distance
$\emptyset$: Pressure angle

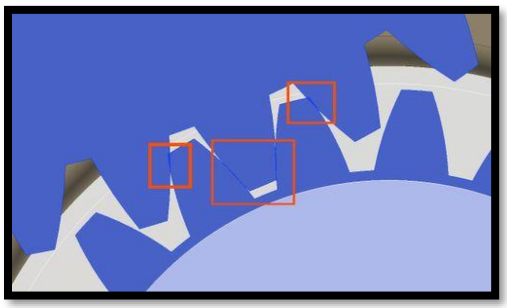
$m_c > 1$, usually $m_c \cong 1.2$



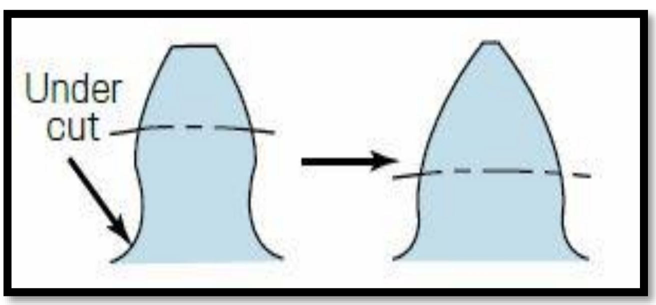
Note: If you find the $m_c < 1$that is not good. It means on average there are some moments in which no teeth from the gears are in contact! For $m_c > 1$, we will have a smoother motion because that means before one tooth completely loses contact, another tooth is in contact (but not too much because then you will increase the friction).

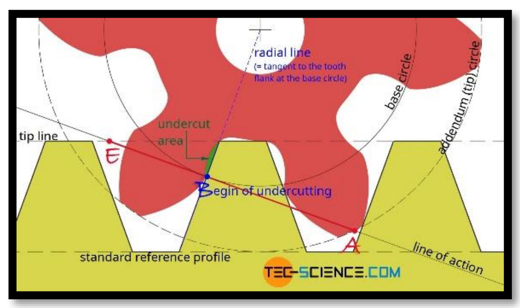
The minimum number of teeth for avoiding interference/undercutting:

Interference: When the gears contact outside of the involute profile. Either the center distance has been changed or the teeth are too tall because the addendum is too big. Also, sometimes happen when the number of teeth between the pinion and the other gear is very different.



Undercutting: It is similar to interference but it's happened during manufacturing. It is cutting a gear inside the base circle.





Sometimes we are doing that because we need a small pinion and a big gear but this makes the teeth very weak at the base. One way to avoid this is not selecting the gears with very different numbers of teeth.

$$N_p = \frac{2K}{(1+2R)\, sin^2\emptyset}\left(R + \sqrt{R^2 + (1+2R)\, sin^2\, \emptyset}\right)$$

Minimum number of teeth of pinion

N_p: Number of teeth of the pinion (small gear)

$R = \frac{N_G}{N_P}$ Gear ratio

$\emptyset$: Pressure angle

$K = 1$ for standard gears, $K = 0.8$ for stub teeth ($a = \frac{0.8}{P_d}$, $b = \frac{1}{P_d}$)

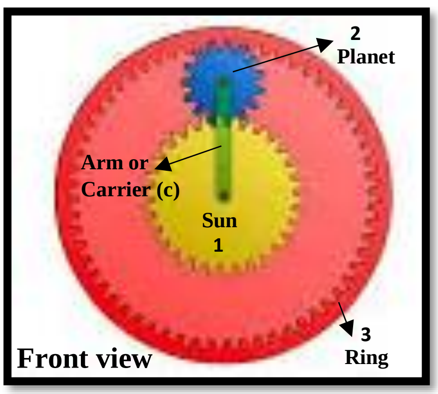
So, for the minimum number of teeth, when you do gear selection just check that you are not selecting too small gear (if not may you have interference)

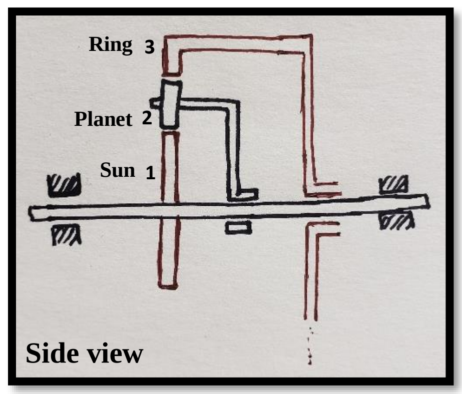
a) **Planetary gear train:**

This type of gear train can be used in any application that needs a big gear reduction in a small space. This system has two degrees of freedom and unlike the common systems with the mobility of two that we need two inputs (two motors) and we get one output, in this system we can do something different. We can put just one input and split it into two outputs or one input and use another input to control the output and many other applications.

https://www.youtube.com/watch?v=ARd-Om2VyiE

Planetary gear train: Some of the gears (They may be simple gear trains or compound gear trains) are mounted on the rotating shaft.





In this particular setup, the Sun gear rotates with the shaft when the Ring and Planet gears and carrier are rotating collinearly to the shaft but independent.

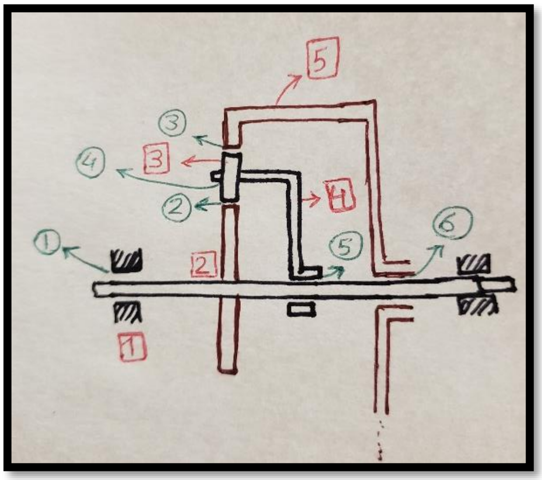
We can have more than one planet, but it doesn't change kinematic of system or motion and just it is for force distribution purposes.

Note: The planet is free to rotate on the carrier. The planet is displaced by the rotation of the carrier but they don't have the same angular velocity.

Question: What is the total angular velocity of the planet if the planet rotating with the angular velocity of ω_2 respect to carrier and carrier has the angular velocity of ω_c?

$$\omega_{total\_2} = \omega_2 + \omega_c$$

Question: What is the mobility of this system?



n = 5 , j = 6 (1, 4, 5, 6: revolute , 2 & 3: gear) , $f_1 = f_4 = f_5 = f_6 = 1$ $f_2 = f_3 = 2$

$$M = 3(n-1) - \sum_{i=1}^{j}(3-f_i) = 3 \times (5-1) - 4 \times (3-1) - 2 \times (3-2)$$

$$\boxed{M = 12 - 8 - 2 = 2}$$

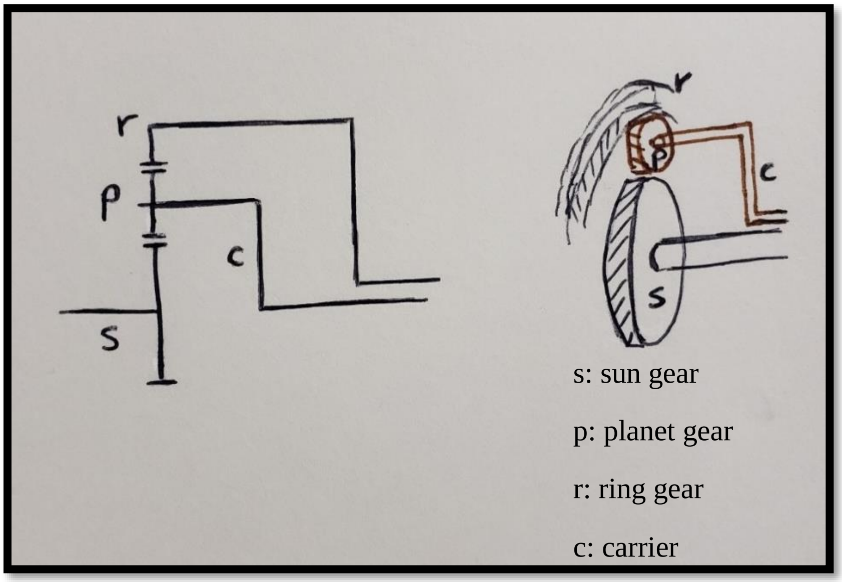
It is 2 degree of freedom system. In this case, we can do two things.

1) Having 2 inputs and 1 output:
In this system, we will have four angular velocities: ω_1(sun), ω_2 (planet), ω_3 (ring), ω_c (carrier). Between these velocities ω_2 is almost useless because the planet is freely rotating and it is hard to control and calculate. So, we can assume that as one of our dependent variables.

So, between ω_1, ω_3, and ω_c, we can select two of them as input and find the third one as output.

2) Having 1 input and 2 outputs (This is the way the planetary gear trains are used for differentials):
Speed is distributed between two outputs according to the force/torque of each output.

Compute the gear ratio of the planetary:



We only know how to compute the one type of gear ratio, when we have two gears, meshing together and they are on the fixed axes. So, to compute the gear ratio of this system:

1) We will "fix" the carrier (for our calculations). Then that would change to a simple gear train case (sun meshing with the planet and planet meshing with ring).
2) Select input(s) & output(s): Doesn't matter which one you take as the input/output and this is for computing the gear ratio of the whole planetary. let's take $\omega_{in} = \omega s_{/c}$ & $\omega_{out} = \omega r_{/c}$. The $\omega s_{/c}$ is the angular velocity of the sun with respect to the carrier and $\omega r_{/c}$ is the angular velocity of the ring with respect to the carrier (writing them with respect to the carrier because we assumed the carrier is fixed)
3) Compute R: $R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega s_{/c}}{\omega r_{/c}}$ $\quad (\omega s_{/c} = \omega_s - \omega_c \ , \ \ \omega r_{/c} = \omega_r - \omega_c)$

$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega s_{/c}}{\omega r_{/c}} = \left(\frac{\omega s_{/c}}{\omega p_{/c}}\right)\left(\frac{\omega p_{/c}}{\omega r_{/c}}\right) = \left(-\frac{N_p}{N_s}\right)\left(\frac{N_r}{N_p}\right) = \boxed{-\frac{N_r}{N_s}} \quad \text{(In this case)}$$

Note: That was just one example that showed the methodology ("fix" the carrier) for calculating *R* for either single or compound gear train. If we have a different setup of gear, then we will have a different formula for *R*.

In the general case, we can say the main equation is: $\frac{\omega s_{/c}}{\omega r_{/c}} = \frac{\omega_s - \omega_c}{\omega_r - \omega_c}$. This is one equation for a **two-degree of freedom** system (three variables and two of them are independent). So, we will use the input/output given in the problem and solve it for the new *R* based on this *R*.

Example: The carrier is the input, "fix" the ring gear (so we will have one input-output system), and the sun gear is the output.

$$R_{new} = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_c}{\omega_s}$$

$$\omega_r = 0$$

$R = \frac{\omega_s - \omega_c}{\omega_r - \omega_c} = \frac{\omega_s - \omega_c}{-\omega_c} \Longrightarrow -R\omega_c = \omega_s - \omega_c \Longrightarrow \omega_s = (1-R)\omega_c$

$$\boxed{R_{new} = \frac{\omega_c}{(1-R)\omega_c} = \frac{1}{(1-R)}}$$

Example: The sun is the input, "fix" the ring gear (we will have one input-output system), and the carrier is the output.

$$R_{new} = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_s}{\omega_c}$$

$\omega_r = 0$ (for "fix" ring gear)

$R = \frac{\omega_s - \omega_c}{\omega_r - \omega_c} = \frac{\omega_s - \omega_c}{-\omega_c} \Longrightarrow -R\omega_c = \omega_s - \omega_c \Longrightarrow \omega_s = (1-R)\omega_c$

$$\boxed{R_{new} = \frac{(1-R)\omega_c}{\omega_c} = (1-R)}$$

If we **don't** "fix" the ring gear,

$R = \frac{\omega_s - \omega_c}{\omega_r - \omega_c} \Longrightarrow R(\omega_r - \omega_c) = \omega_s - \omega_c$

$\omega_s = \omega_r R + (1-R)\omega_c \Longrightarrow$ We will have **one** input (sun gear) & **two** outputs and distribute the power and velocity between them.

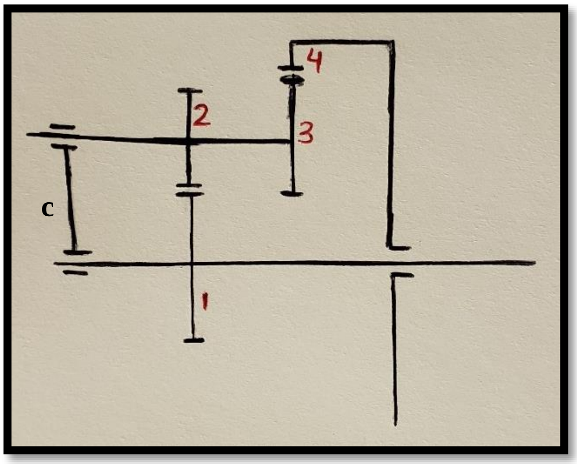
Or

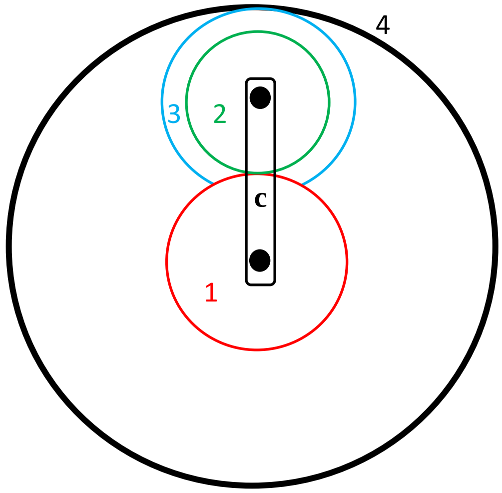
$\omega_c = \frac{1}{(1-R)}\omega_s + \frac{-R}{1-R}\omega_r \Longrightarrow$ We will have **two** inputs & **one** output

So, in the first case, for the same input, we can have different torques and different velocities for each output (for example on the wheels)

Gear Systems (Continue)

Example: Compound planetary gear train





$N_1 = 60$
$N_2 = 16$
$N_3 = 24$
$N_4 = 100$

a) What is the Gear Ratio of this planetary?
b) If Ring (4) is fixed and $\omega_1 = 100\ rpm$, compute the angular velocity of the carrier (ω_c).

a)

Step1: "Fix" the carrier (if the carrier rotates, we don't know how to calculate other things). After that, we only have a compound gear train. Gears 2 and 3 are connected (on the same shaft), so they have the same angular velocity.

Step2: Select your input and your output. The best choices can be sun gear (1) and ring gear (4). Either of them can be input or output. Let's take sun gear (1) as input and ring gear (4) as the output.

$$\omega_{in} = \omega_{1/c}$$

$$\omega_{out} = \omega_{4/c}$$

Step3: Compute the gear ratio

$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_{1/c}}{\omega_{4/c}} = \frac{\omega_1 - \omega_c}{\omega_4 - \omega_c} = \left(\frac{\omega_{1/c}}{\omega_{2/c}}\right)\left(\frac{\omega_{2/c}}{\omega_{3/c}}\right)\left(\frac{\omega_{3/c}}{\omega_{4/c}}\right) = \left(-\frac{N_2}{N_1}\right)(1)\left(\frac{N_4}{N_3}\right) = -\frac{N_2 N_4}{N_1 N_3}$$

$$\boxed{R = \frac{\omega_1 - \omega_c}{\omega_4 - \omega_c} = -\frac{N_2 N_4}{N_1 N_3} = -\frac{16 \times 100}{60 \times 24} = -1.11}$$

Gear ratio of whole planetary

b)

$\omega_4 = 0$ (Ring fixed)

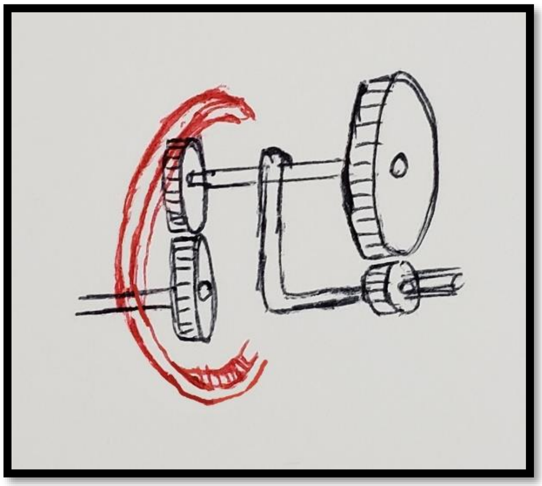
$\omega_1 = 100\ rpm$ (input)

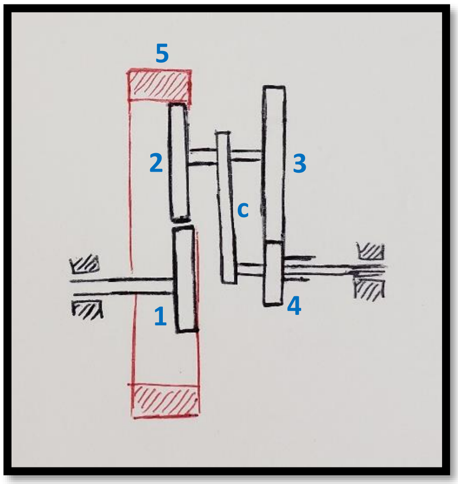
$\omega_c = ?$ (output)

$R = \frac{\omega_1 - \omega_c}{-\omega_c}$ $\quad\quad$ $\omega_1 = (1 - R)\omega_c$ $\quad\quad$ $\omega_c = \frac{1}{(1-R)}\omega_1 = \frac{1}{(1+1.11)} \times 100 = \boxed{47.37\ rpm}$

Note: We can have planetary gear trains with more than one stage

Example: Two stages of planetary gears train





How to find the stages of a planetary gear train? One way for that is: to start from a point as the input and see where the flow of power goes and check if it goes to two different places. For instance in this case, if we start from 1 we have two paths. We can go through 1-2-3-4 or 1-2-5.

Stage 1: 1-2-5 $\quad\quad$ Both are on same
Stage 2: 1-2-3-4 $\quad\quad$ carrier

Note: We have to compute one gear ratio for each stage!

For each gear ratio, we will do the same process (fix the carrier, select input, select output, and compute the gear train).

For stage 1: I will get 1 as the input and 5 as the output:

$$R_1 = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_{1/c}}{\omega_{5/c}} = \left(\frac{\omega_{1/c}}{\omega_{2/c}}\right)\left(\frac{\omega_{2/c}}{\omega_{5/c}}\right) = \left(-\frac{\cancel{N_2}}{N_1}\right)\left(\frac{N_5}{\cancel{N_2}}\right) = \boxed{-\frac{N_5}{N_1}}$$

For stage 2: I will get 1 as the input and 4 as the output:

$$R_2 = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_{1/c}}{\omega_{4/c}} = \left(\frac{\omega_{1/c}}{\omega_{2/c}}\right)\left(\frac{\omega_{2/c}}{\omega_{3/c}}\right)\left(\frac{\omega_{3/c}}{\omega_{4/c}}\right) = \left(-\frac{N_2}{N_1}\right)1\left(-\frac{N_4}{N_3}\right) = \boxed{\frac{N_2 N_4}{N_1 N_3}}$$

In this problem, the angular velocities that we can control are: ω_1, ω_4, ω_5, ω_c (we don't care about planet gear as we discussed before). Also, we have two equations:

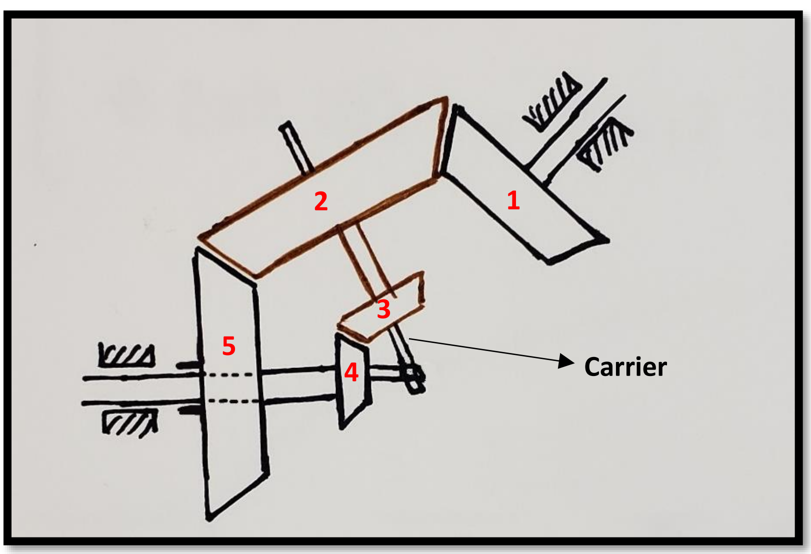
$$\frac{\omega_1 - \omega_c}{\omega_5 - \omega_c} = -\frac{N_5}{N_1}$$

$$\frac{\omega_1 - \omega_c}{\omega_4 - \omega_c} = \frac{N_2 N_4}{N_1 N_3}$$

Four unknowns & two equations, so two degrees of freedom system (same as the single stage).

Planetary Gear Trains with Bevel Gears

We can use the planetary gear train for any type of gear. Here we are explaining the Bevel gear. We can use the same equations for any type of gear (Spur, bevel, helical, or worm).

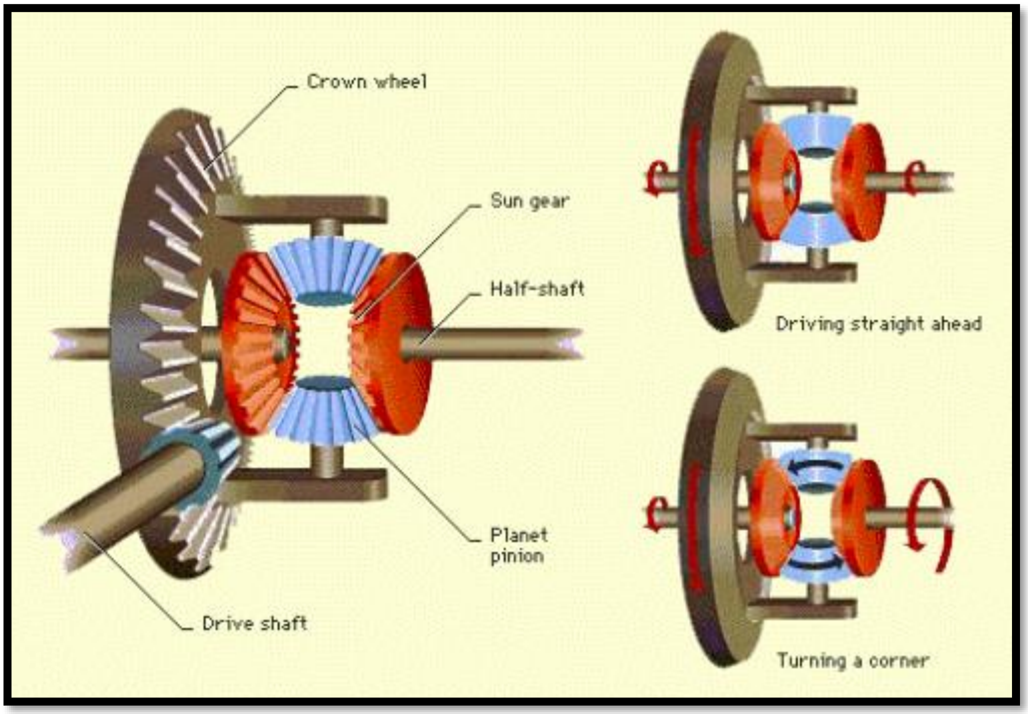


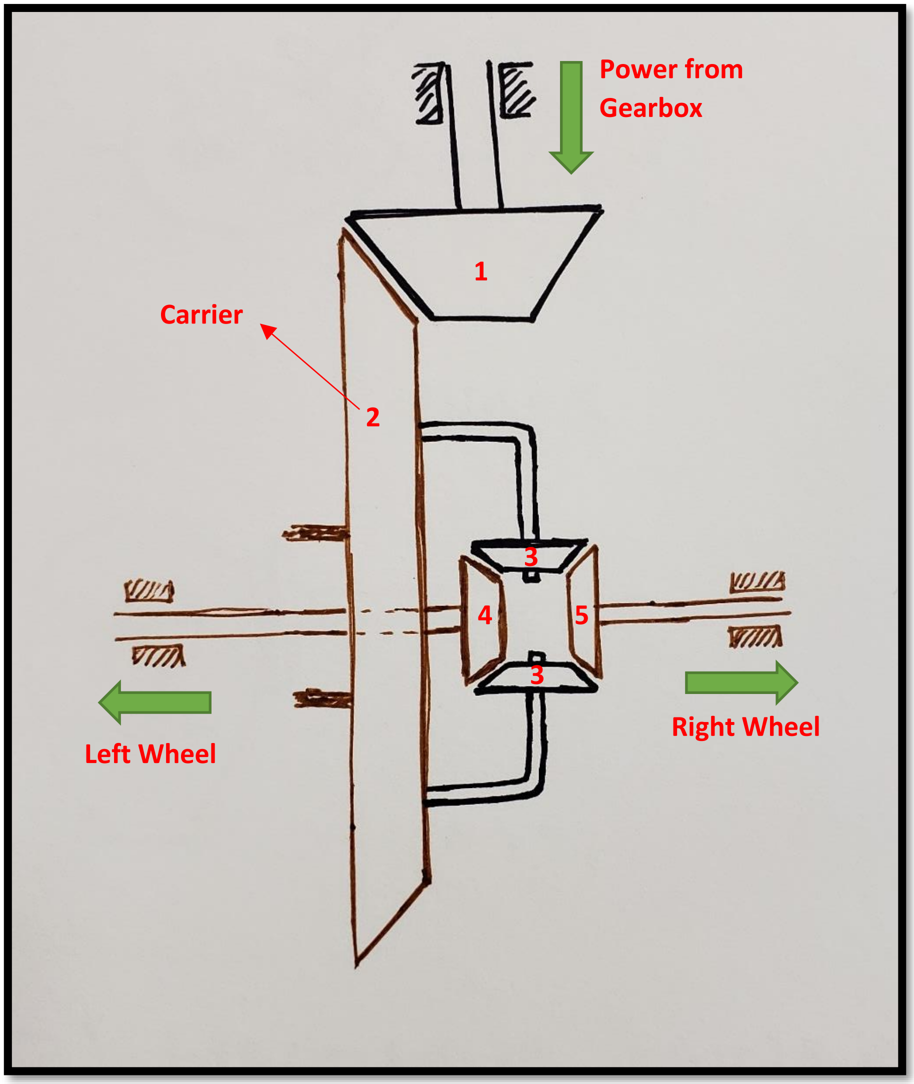
How many stages do we have? Two stages
Stage 1: 1-2-5
Stage 2: 1-2-3-4

Similar to the previous example and we can use the same methodology.

Differential

It is planetary with bevel gears (with very specific geometry)





Simple gear train between 1 and 2 (a little bit reduction before going to differential)

$$R_p = \frac{N_2}{N_1}$$

For differential, we have carrier (ω_2), input (ω_4), output (ω_5)

Note: To calculate the gear ratio, you take the right wheel or left wheel as input or output.

Note: Same as before, for calculating the gear ratio we assume the carrier is fixed (in reality the real input is the carrier but for computing the gear ratio we assumed it is fixed).

Compute the planetary gear ratio:

$$R = \frac{\omega_{in}}{\omega_{out}} = \frac{\omega_4/_2}{\omega_5/_2} = \frac{\omega_4 - \omega_2}{\omega_5 - \omega_2} = \left(\frac{\omega_4/_2}{\omega_3/_2}\right)\left(\frac{\omega_3/_2}{\omega_5/_2}\right) = \left(\frac{\cancel{N_3}}{N_4}\right)\left(-\frac{N_5}{\cancel{N_3}}\right) = -\frac{N_5}{N_4}$$

That means the gear ratio is equal to the ratio of the number of teeth for the gear of the right wheel to the left wheel. For the differential always $N_5 = N_4$, because we want to transmit the same angular velocity to both wheels.

So, for differential we will have:

$R = -1$ (The wheels are rotating in opposite directions)

$$\boxed{\frac{\omega_4 - \omega_2}{\omega_5 - \omega_2} = -1}$$

$\omega_4 - \omega_2 = -\omega_5 + \omega_2 \Rightarrow \omega_4 + \omega_5 = 2\omega_2 \Rightarrow \frac{\omega_4 + \omega_5}{2} = \omega_2$

Note: So, when the car going in a straight way, ω_4 & ω_5 are equal but when the car turns, one of them becomes bigger than the other one based on $\frac{\omega_4+\omega_5}{2} = \omega_2$.

Summary of formulas:

Number of teeth: N
Pressure angle: $\emptyset$
Pitch diameter/radius: d_p, r_p
Diametral pitch (used in some catalogs): $P_d = \frac{N}{d_p}$ [teeth/in]
Module (used in some other catalogs): $m = \frac{d_p}{N}$ [mm]
Circular pitch: $p_c = \frac{\pi d_p}{N}$
Base pitch: $p_b = \frac{\pi d_b}{N} = p_c cos\,\emptyset$
Base diameter: $d_b = d_p\ cos\ \emptyset$
Center distance: $C = \frac{d_{p_1}+d_{p_2}}{2}$
For Standard Gears:
Standard addendum: $a = \frac{1}{P_d}$
Standard dedendum: $b = \frac{1.25}{p_d}$
Outside diameter: $D_o = d_p + 2a$
Clearance: $c = b - a$
Backlash: $B = h_p - t_p$