The Modern Geometry of the Triangle and Circle with the Geometer's Sketchpad

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Contents

1 The Inscribed Angle Theorem

The Inscribed Angle Theorem is a basic result with many applications. The theorem is included in the first course in Euclidean geometry taken by high school students. At a glance, the theorem is surprising. Using the Geometer's Sketchpad, students can construct a model of the theorem and empirically observe what the theorem claims.

1.1 Inscribed Angle Theorem Stetement

Let A, B, C be three distinct points on a circle $\mathcal{C}$ in the plane. Let O be the center of $\mathcal{C}$. Since three points on $\mathcal{C}$ are non-collinear, then the line $l = \overleftrightarrow{BC}$ divides $\mathcal{C}$ into two disjoint sets called **intercepted arcs**. The two arcs $\overset{\frown}{BC}$ and $\overset{\frown}{CB}$ (B and C excluded) lie on opposite sides of l. Note that $\overset{\frown}{BC}$ is understood the counterclockwise arc from B to C.

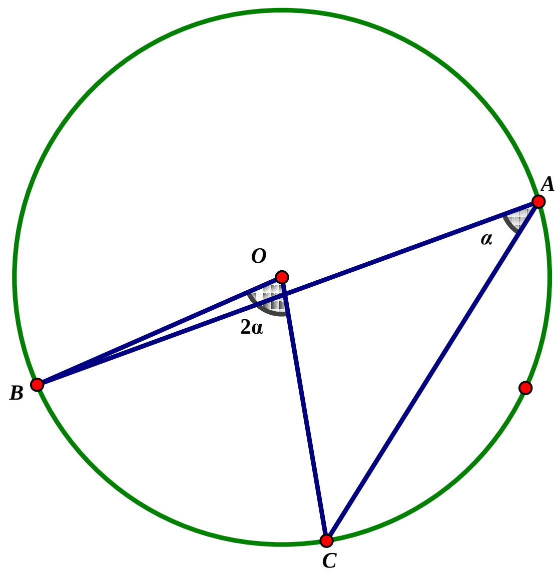
If the segment $\overline{BC}$ is a diameter, then the angle $\angle BOC$ has measure 180. If the segment $\overline{BC}$ is not a diameter, then the measures of the directed angles (i.e. counterclockwise) $\angle BOC$ and $\angle BOA$ satisfy

$$\begin{aligned} 0 &< \angle BOC < 180 \\ 180 &< \angle COB < 360 \\ 360 &= \angle BOC + \angle COB \,. \end{aligned}$$

The arc on the side of l that is opposite to A, denoted , is called the **intercepted arc** for $\angle BAC$. The angle $\angle BOC$ is called the **central angle** subtended by the arc $\overset{\frown}{BC}$.

Theorem 1. ***(The Inscribed Angle Theorem)*** *Given the information of the preceding paragraph,*

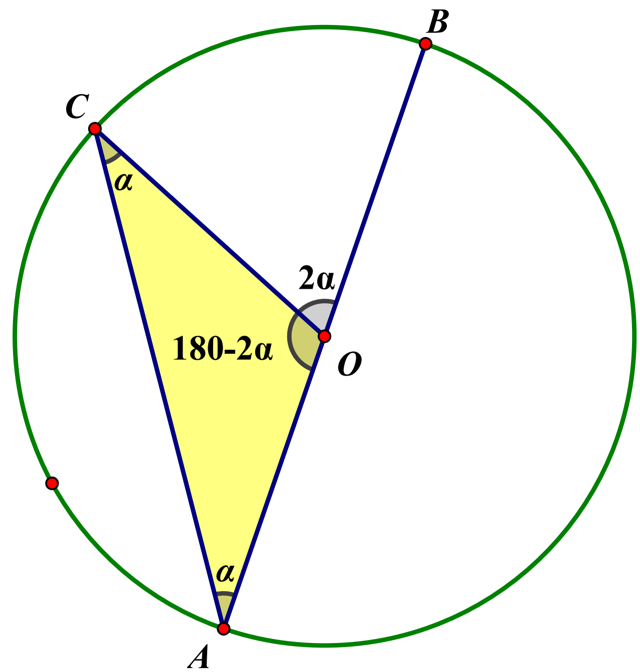
$$\angle BAC = \tfrac{1}{2}\angle BOC \,.$$



Proof. The proof of the Inscribed Angle Theorem is organized around the special case $\angle BAC$ where the segment $\overline{AB}$ is a diameter of the circle $\mathcal{C}$. The figure shown ahead illustrates the proof, to wit, the triangle $\triangle OAC$ is isosceles so that two base angles in figure labelled α are indeed equal. Hence $\angle AOC = 180 - 2\alpha$ so that $\angle BOC = 2\alpha$ since

$$\angle AOC + \angle BOC = 180\,.$$

The general case where neither $\overline{AB}$ nor $\overline{AC}$ can be proved using the special case. □



1.2 Thales Theorem

A beautiful consequence of the Inscribed Angle Theorem is Thales Theorem shown below. The theorem is proved as part of the 31st proposition in the third book of Euclid's Elements. The theorem is generally attributed to Thales of Miletus, who is said to have offered an ox, probably to the god Apollo, as a sacrifice of thanksgiving for the discovery.

Theorem 2. *Let A, B be two points in the plane. Then the circle with diameter $\overline{AB}$, denoted $\mathcal{C}(AB)$, consists of the points A and B along with all points P such that*

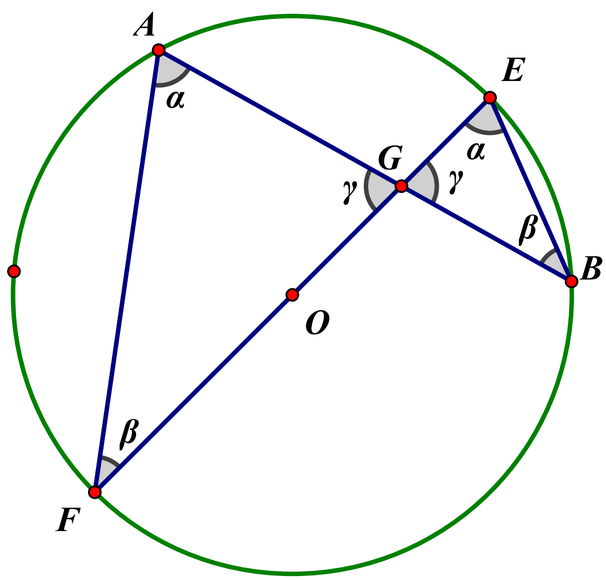
$$\angle APB = 90^0 .$$

1.3 Power of a Point with respect to a Circle

Let $\mathcal{C}$ be a circle with center O and radius r. Let G be a point lying inside the circle $\mathcal{C}$. So the distance $g = OG < r$. Given any chord $\overline{AB}$ of the circle $\mathcal{C}$ that passes through the point G, the product

$$AG \cdot BG = r^2 - g^2 . \tag{1.1}$$

This fact is an immediate consequence of the Inscribed Angle Theorem. It is striking because there are an infinite number of chords passing through the point G, yet the product $AG{\cdot}BG$ is constant.



The above figure illustrates why the equation (1.1) is true. Indeed, by the Inscribed Angle Theorem the angles labelled α are equal as well as the two labelled β are equal. Hence

$$\triangle AFG \sim \triangle EBG$$

by the AAA-Similarity Theorem. Part of the definition of triangle similarity is that the three ratios of corresponding side are equal. Hence

$$\frac{AG}{EG} = \frac{FG}{BG} = \frac{AF}{EB}$$

so that

$$AG \cdot BG = EG \cdot FG\,. \tag{1.2}$$

Since $E, F,$ and G do not vary, equation (1.2) says that the product $AG \cdot BG$ is constant. On the other hand, $EG = r + g$ and $FG = r - g$. Hence

$$EG \cdot FG = r^2 - g^2\,.$$

In terms of the dot product (see §2.3 ahead), the ratio

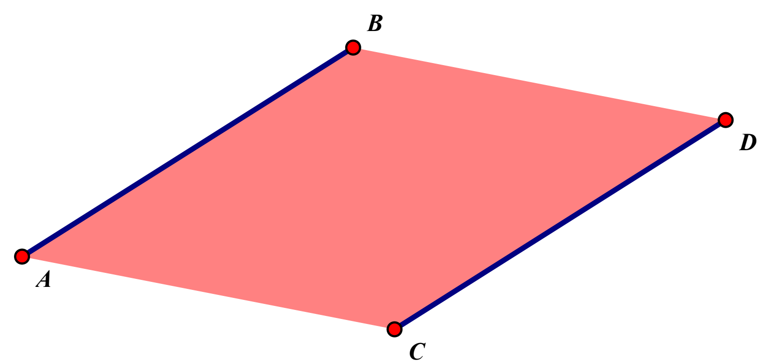
$$\begin{aligned}\frac{AG}{BG} &= \frac{AG^2}{AG \cdot BG} \\ &= \frac{AG^2}{r^2 - g^2} \\ &= \frac{OA^2 + OG^2 - 2(\overrightarrow{OA} \cdot \overrightarrow{OG})}{r^2 - g^2}.\end{aligned}$$

This ratio is very useful in applying the power of a point concept.

2 Geometric Vectors

The vector concept is central to understanding the concept of force. The vector concept does not require any calculus concepts. Moreover, basic high geometry could easily include the vector concept.

In 2 and 3-space a vector is informally described as an "arrow" with the understanding that two arrows describe the same vector if they have the same magnitude and direction.



2.1 Geometric Vector Definition

Formally, an **arrow** or **directed line segment** is an ordered pair $[A, B]$ of points in the plane $\mathcal{E}$; the first point A is called the **initial point** and the second point B is called the **terminal point**. Two directed line segments (i.e. arrows) $[A, B]$ and $[C, D]$ are said to be equivalent, written $[A, B] \sim [C, D]$, iff the following occurs:

1. If any three of the points are non-collinear, then the quadrilateral $\square ABDC$ is a parallelogram.

2. If the four points are collinear and $[E, F]$ is a directed line segment such that both $\square ABFE$ and $\square CDFE$ are convex quadrilaterals, then $\square ABFE$ is a parallelogram if and only if $\square CDFE$ is a parallelogram.

We know a lot about parallelograms. In particular, a convex quadrilateral is a parallelogram iff opposite sides have equal length. In the above figure,

$$[A, B] \sim [C, D] \qquad [A, C] \sim [B, D].$$

Formally, the $\sim$ relation is an **equivalence relation** on the set $\mathcal{E} \times \mathcal{E}$ of all ordered pairs of directed line segments. The equivalence class of a directed line segment $[A, B]$ is what one means by a vector, that is, the vector is all arrows with the same magnitude and direction. It is tradition to write $\overrightarrow{AB}$ rather than $[A, B]$. It is also tradition to write $\mathbf{v}$ for a vector when the use of a particular directed line segment is not needed.

A rational number $\frac{m}{n}$ is the set of all ratios $\frac{p}{q}$ of integers equivalent to $\frac{m}{n}$, that is,

$$\frac{m}{n} = \frac{p}{q} \qquad \Longleftrightarrow \qquad np = mq.$$

By analogy, the equality

$$\overrightarrow{AB} = \overrightarrow{CD} \qquad \Longleftrightarrow \qquad [A, B] \sim [C, D].$$

2.2 Vector Arithmetic

In defining vector arithmetic (addition, subtraction, and scaling), the equivalence of directed line segments is central. Given two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, the vector sum

$$\overrightarrow{AB} + \overrightarrow{CD}$$

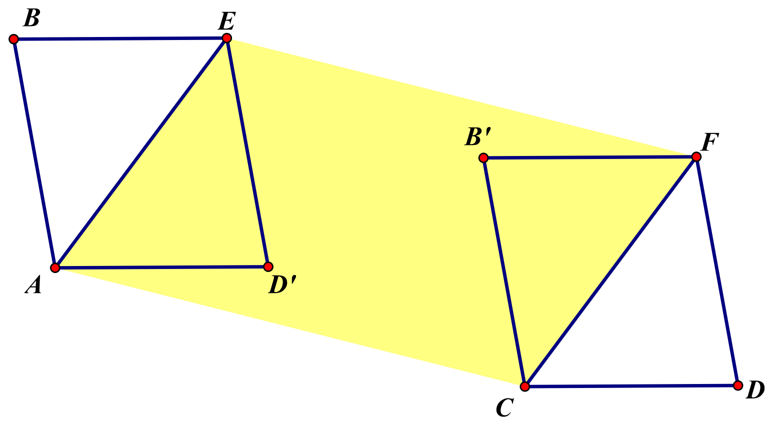
requires a common initial point. For example, if we use the point A as the initial point, then there is a unique point D' such that

$$\overrightarrow{CD} = \overrightarrow{AD'}.$$

Then the vector sum

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AE}$$

where the segment $\overline{AE}$ is the diagonal of the parallelogram $\square ABED'$ beginning at A.



Since $\overrightarrow{AE}$ means the vector represented by the directed segment $[A, E]$, the above definition cannot depend upon the choice initial point. On the right side of the above figure, the point C is chosen as the initial point. The figure illustrates that $[A, E] \sim [C, F]$, i.e., the quadrilateral $\square AEFC$ is a parallelogram. This means that vector definition is well defined.

In a similar way, both vector subtraction and the scaling of a vector by a number are defined.

2.3 The Dot Product Geometrically

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane, the angle θ between is well defined. Indeed, if $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$, then

$$\theta = \angle AOB.$$

More importantly, if $\mathbf{u} = \overrightarrow{PC}$ and $\mathbf{v} = \overrightarrow{PD}$, then

$$\angle CPD = \angle AOB,$$

that is, the two angles have the same measure.

The length of a vector $\mathbf{u} = \overrightarrow{OA}$ is defined as

$$|\mathbf{u}| = OA,$$

the length of the segment $\overline{OA}$. Since opposite sides of a parallelogram are equal in length, this definition is well-defined.

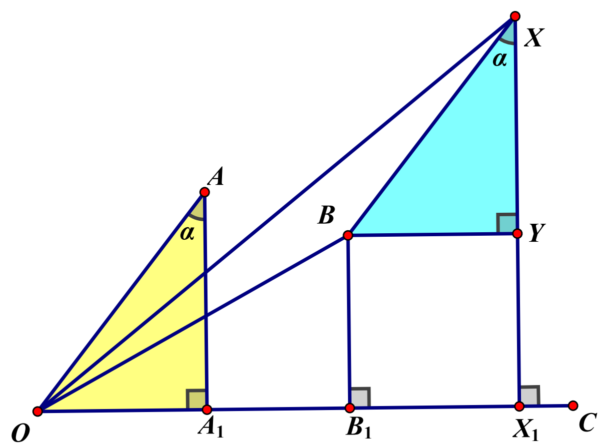
There is a natural way to multiply two vectors to produce a number. This binary operation on vectors is called the **dot product**. Geometrically, if $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$, then the dot product of $\mathbf{u}$ with $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta\,. \tag{2.1}$$

In words, the dot product $\mathbf{u} \cdot \mathbf{v}$ is the product of three numbers: the lengths $|\mathbf{u}|$, $|\mathbf{v}|$, and the cosine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

The dot product has three basic properties: for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars λ,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\lambda \mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$



The first two properties are self-evident. The third property requires verification. The figure shown above illustrates why the third property is true. In the figure $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OB}$, and $\mathbf{w} = \overrightarrow{OC}$. Next, the vectors

$$\overrightarrow{OA_1} = \tfrac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}, \qquad \overrightarrow{OB_1} = \tfrac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}, \qquad \overrightarrow{OX_1} = \tfrac{(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}.$$

On the other hand, the two triangles highlighted in the figure are congruent:

$$\triangle OAA_1 \cong \triangle BXY\,.$$

Hence, the vector sum $\overrightarrow{OX_1} = \overrightarrow{OA_1} + \overrightarrow{OB_1}$, that is,

$$\tfrac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \left(\tfrac{\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right)\mathbf{w}$$

so that

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}.$$

3 The Center of Mass

The concept of **center of mass** in the form of the center of gravity was first introduced by the ancient Greek physicist, mathematician, and engineer Archimedes of Syracuse. He worked with simplified assumptions about gravity that amount to a uniform field, thus arriving at the mathematical properties of what we now call the center of mass. Archimedes showed that the torque exerted on a lever by weights resting at various points along the lever is the same as what it would be if all of the weights were moved to a single point—their center of mass. In his work on floating bodies, Archimedes demonstrated that the orientation of a floating object is the one that makes its center of mass as low as possible. He developed mathematical techniques for finding the centers of mass of objects of uniform density of various well-defined shapes.

Later mathematicians who developed the theory of the center of mass include Pappus of Alexandria, Guido Ubaldi, Francesco Maurolico, Federico Commandino, Simon Stevin, Luca Valerio, Jean-Charles de la Faille, Paul Guldin, John Wallis, Louis Carré, Pierre Varignon, Alexis Clairaut, Leonhard Euler, and Jakob Steiner.

Newton's second law of motion is reformulated with respect to the center of mass in Euler's first law of motion. In classical mechanics, Euler's Laws of Motion extend Newton's Laws of Motion for point particle to rigid body motion.

The center of mass concept can be introduced into plane geometry using the vector concept.

3.1 Center of Mass Theorem

Theorem 3. *Given n points $A_1, \ldots, A_n$ in the plane, there is a unique point G such that for any point O , the vector equation*

$$n\overrightarrow{OG} = \sum_{j=1}^{n} \overrightarrow{OA_j}\,.$$

Said differently, if the directed line segment $[O, X]$ *represents the vector sum* $\sum_{j=1}^{n} \overrightarrow{OA_j}$ *, then* G *is the point between* O *and* X *such that*

$$\frac{OX}{OG} = n\,.$$

The point G *is the* ***center of mass*** *(or* ***centroid****) of the points* $A_1, \ldots, A_n$ *.*

The proof of Theorem 3 is organized around the concept of vector addition, i.e. parallelograms. The following basic lemma is central to the proof of Theorem 3.

Lemma 3.1. *1. Given three points* O, P *, and* A *, the vector sum*

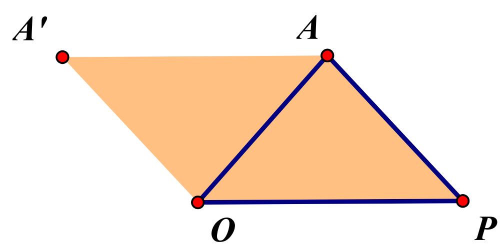
$$\overrightarrow{OP} + \overrightarrow{PA} = \overrightarrow{OA}\,.$$

2. Let $O, P, X,$ *and* Y *be four points satisfying the vector equation*

$$\overrightarrow{OX} = \overrightarrow{OP_n} + \overrightarrow{PY}$$

where $\overrightarrow{OP_n} = n\,\overrightarrow{OP}$ *for some positive integer* $n \geq 2$ *. Then the line segments* $\overline{OX}$ *and* $\overline{PY}$ *intersect at a unique point* G *. Moreover,*

$$\frac{XO}{GO} = \frac{YP}{GP} = n\,.$$



$\overrightarrow{\mathbf{OA}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PA}}$

Proof. In terms of a parallelogram, the first equation means the following:

Let $A' = \tau_{\overrightarrow{PA}}(O)$ denote the translation of the point O by the vector $\overrightarrow{PA}$. By definition, A' is the unique point such that the quadrilateral $\square OPAA'$ is a parallelogram. Then the above vector equation $\overrightarrow{OP} + \overrightarrow{PA} = \overrightarrow{OA}$ means that $\overline{OA}$ is a diagonal of $\square OPAA'$. The vector equation also includes that $\overrightarrow{OA'} = \overrightarrow{PA}$ and $\overrightarrow{OP} = \overrightarrow{A'A}$. See the figure.

To argue the second equation, let $Y' = \tau_{\overrightarrow{PY}}(O)$, i.e., Y' is the unique point such that $\overrightarrow{OY'} = \overrightarrow{PY}$, that is, $\square OPYY'$ is a parallelogram. Then the vector equation

$$\overrightarrow{OX} = \overrightarrow{OP_n} + \overrightarrow{PY}$$

means that the line segment $\overline{OX}$ is a diagonal of the parallelogram $\square OP_nXY'$. Next, since $\overrightarrow{PY} = \overrightarrow{OY'} = \overrightarrow{P_nX}$ and $P \in \overline{OP_n}$, $Y \in \overline{Y'X}$. Hence, $\overline{PY}$ intersects $\overline{OX}$ at a unique point G.

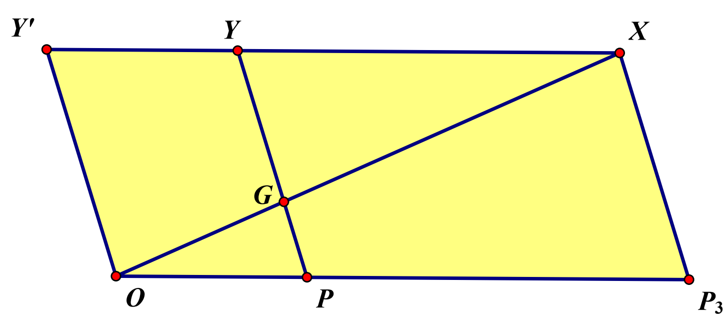
To prove the last equation, observe that $\triangle OPG \sim \triangle OP_nX$ since $\overline{PG} \parallel \overline{P_nX}$. Hence

$$\frac{XP_n}{GP} = \frac{OX}{OG} = \frac{OP_n}{OP} = n$$

by definition of P_n. On the other hand, because $\overrightarrow{PY} = \overrightarrow{P_nX}$, $YP = XP_n$ so that

$$\frac{YP}{GP} = \frac{XP_n}{GP} = n\,.$$

The figure shown below illustrates the $n = 3$ case. $\square$



Proof of Theorem 3:

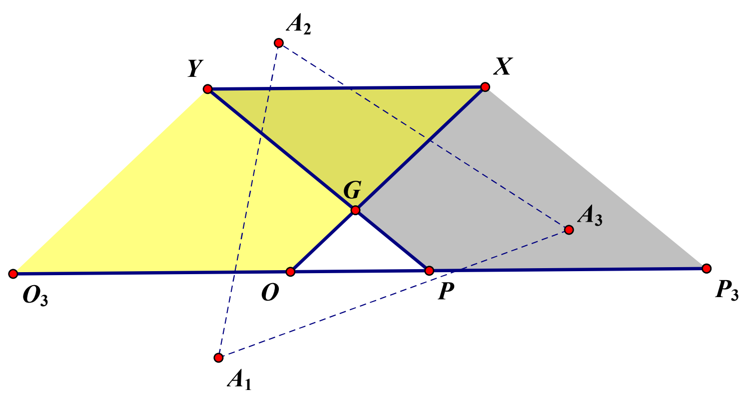
Proof. By the first equation of Lemma 3.1,

$$\sum_{j=1}^{n}\overrightarrow{OA_j} = n\,\overrightarrow{OP} + \sum_{j=1}^{n}\overrightarrow{PA_j}$$

Let X and Y be the points such that

$$\overrightarrow{OX} = \sum_{j=1}^{n}\overrightarrow{OA_j}\,, \qquad \overrightarrow{PY} = \sum_{j=1}^{n}\overrightarrow{PA_j}\,.$$

In other words, X and Y are the unique points such that the directed segment $[O,X]$ represents the vector $\sum_{j=1}^{n}\overrightarrow{OA_j}$, while $[O,Y]$ represents $\sum_{j=1}^{n}\overrightarrow{PA_j}$. Hence $\overrightarrow{OX} = n\overrightarrow{OP} + \overrightarrow{PY}$. Next, let P_n be the unique point such that $\overrightarrow{OP_n} = n\overrightarrow{OP}$. Then by the second and third equations of Lemma 3.1, the proof is complete. The figure below illustrates the $n=3$ case. □



Corollary 1. *Let G be the center of mass of n points $A_1,\dots,A_n$. Then the vector sum $\sum_{j=1}^{n}\overrightarrow{GA_j} = \mathbf{0}$, that is, the vector sum is the zero vector.*

3.2 Center of Mass Theorem for Conics

The center of mass has a natural numerical property described as follows: given a point O and n points $A_i, i = 1,\dots,n$, the center of mass G of the n points satisfies

$$n\,OG^2 = \sum_{i=1}^{n}(OA_i^2 - GA_i^2)\,. \tag{3.1}$$

The equation (3.1) can be justified using the dot product. In geometric terms,

$$p_i = OA_i^2 - GA_i^2$$

is the power of G with respect to the circle with center A_i and radius $r_i = OA_i$; see (1.1). From this perspective, if $g = OG$, then

$$g^2 = \tfrac{1}{n} \sum_{i=1}^{n} p_i,$$

the average of the point powers.

Suppose next that the points $A_1, A_2, \ldots, A_n$ all lie on a circle $\mathcal{C}$ with radius r and center O. Then $OA_i = r$ for all i. Let $\overline{EF}$ be the diameter of $\mathcal{C}$ passing through the center of mass G. Also let $\overline{A_iB_i}$ be the chord of $\mathcal{C}$ through G having A_i as one the endpoints. Then from §1.3

$$A_iG \cdot B_iG = EG \cdot FG = r^2 - g^2.$$

On the other hand,

$$\begin{aligned} \frac{A_iG}{B_iG} &= \frac{A_iG^2}{A_iG \cdot B_iG} \\ &= \frac{A_iG^2}{r^2 - g^2}. \end{aligned}$$

Add up these ratios:

$$\sum_{i=1}^{n} \frac{A_iG}{B_iG} = \frac{\sum_i A_iG^2}{r^2 - g^2}.$$

The number A_iG^2 is the square of the length of the vector $\overrightarrow{GA_i}$. On the other hand,

$$\overrightarrow{GA_i} = \overrightarrow{OA_i} - \overrightarrow{OG}.$$

Hence the length of $\overrightarrow{GA_i}$ is the dot product

$$\begin{aligned} \overrightarrow{GA_i} \cdot \overrightarrow{GA_i} &= (\overrightarrow{OA_i} - \overrightarrow{OG}) \cdot (\overrightarrow{OA_i} - \overrightarrow{OG}) \\ &= (\overrightarrow{OA_i}) \cdot (\overrightarrow{OA_i}) + (\overrightarrow{OG}) \cdot (\overrightarrow{OG}) - 2(\overrightarrow{OA_i}) \cdot (\overrightarrow{OG}) \\ &= OA_i^2 + OG^2 - 2(\overrightarrow{OA_i}) \cdot (\overrightarrow{OG}) \\ &= r^2 + g^2 - 2(\overrightarrow{OA_i}) \cdot (\overrightarrow{OG}). \end{aligned}$$

Consequently, the sum

$$\sum_{i=1}^{n} \frac{A_iG}{B_iG} = \frac{nr^2 + ng^2 - 2\sum_i (\overrightarrow{OA_i}) \cdot (\overrightarrow{OG})}{r^2 - g^2}. \qquad (3.2)$$

However, by the third property of the dot product,

$$\begin{aligned}\sum_i(\overrightarrow{OA_i})\cdot(\overrightarrow{OG}) &= \big(\sum_i \overrightarrow{OA_i}\big)\cdot\overrightarrow{OG}\\ &= n\overrightarrow{OG}\cdot\overrightarrow{OG}\\ &= ng^2\,.\end{aligned}$$

Thus,

$$\sum_{i=1}^{n}\frac{A_iG}{B_iG}=\frac{\sum_i A_iG^2}{r^2-g^2}=\frac{n(r^2-g^2)}{(r^2-g^2)}=n\,. \tag{3.3}$$

Equation (3.3) is a special case of a theorem proven by myself, Leonid Hanin, and his son Boris in 2007 for quadratic surfaces in $\mathbb{R}^n$. For circles, the theorem is stated formally as follows:

Theorem 4. *Given n points A_i on a circle $\mathcal{C}$ with center O and radius r, let G be the center of mass of the A_i. For each point P inside the circle $\mathcal{C}$, let B_i be the second point of the unique chord $\overline{A_iB_i}$ passing through P. Then the sum*

$$\sum_{i=1}^{n}\frac{A_iP}{B_iP}=n$$

for a point P lying inside $\mathcal{C}$ if and only if P lies on the circle $\mathcal{C}(OP)$ with diameter $\overline{OG}$. Moreover, the inverse of $\mathcal{C}(OG)$ in the circle is the line of points lying outside the circle $\mathcal{C}$ where

$$\sum_{i=1}^{n}\frac{A_iP}{B_iP}=n.$$

Proof. First of all, since $A_iO=B_iO=r$, the sum

$$\sum_{i=1}^{n}\frac{A_iO}{B_iO}=n.$$

Next, let $p=OP$. Replacing G with P in (3.2), we get

$$\sum_{i=1}^{n}\frac{A_iP}{B_iP}=\frac{n\Big(r^2+p^2-2\overrightarrow{OP}\cdot\overrightarrow{OG}\Big)}{r^2-p^2}.$$

Hence

$$\sum_{i=1}^{n}\frac{A_iP}{B_iP}=n \qquad\Longleftrightarrow\qquad \overrightarrow{OP}\cdot\overrightarrow{OG}=p^2.$$

On the other hand, if θ is the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OG}$, then

$$\overrightarrow{OP}\cdot\overrightarrow{OG} = p^2 \qquad \Longleftrightarrow \qquad \cos\theta = \frac{p}{g}\,.$$

But,

$$\cos\theta = \frac{p}{g} \qquad \Longleftrightarrow \qquad \angle OPG = 90\,.$$

Hence by Theorem 2

$$P \in \mathcal{C}(OG) \qquad \Longleftrightarrow \qquad \sum_{i=1}^{n} \frac{A_iP}{B_iP} = n.$$

The inverse of the circle $\mathcal{C}(OG)$ in the circle $\mathcal{C}$ is the line l that passes through the inverse G' of G in $\mathcal{C}$ that is perpendicular to the diameter of $\mathcal{C}$ through the point G.

Let P be a point lying outside $\mathcal{C}$. Then the inverse P' of P lies inside the circle $\mathcal{C}$. In particular, $P \in l \iff P' \in \mathcal{C}(OP)$. By analogy, the sum

$$\sum_{i=1}^{n} \frac{A_iP}{B_iP} = \frac{n\left(r^2 + p^2 - 2\overrightarrow{OP}\cdot\overrightarrow{OG}\right)}{p^2 - r^2}; \tag{3.4}$$

the denominator $p^2 - r^2$ comes from P lying outside $\mathcal{C}$. Next, $\overrightarrow{OP'} \perp \overrightarrow{P'G}$ by Theorem 2. On other hand, since $O, P,$ and P' are collinear,

$$\overrightarrow{OP} \perp \overrightarrow{P'G}$$

as well. Finally, since $\overrightarrow{OG} = \overrightarrow{OP'} + \overrightarrow{P'G}$,

$$\begin{aligned}
\overrightarrow{OP}\cdot\overrightarrow{OG} &= \overrightarrow{OP}\cdot\left(\overrightarrow{OP'} + \overrightarrow{P'G}\right)\\
&= \overrightarrow{OP}\cdot\overrightarrow{OP'} + \overrightarrow{OP}\cdot\overrightarrow{P'G}\\
&= \overrightarrow{OP}\cdot\overrightarrow{OP'}\\
&= r^2\,.
\end{aligned}$$

Hence, (3.4) simplifies to n. □