On Certain Automorphic Forms Associated to Quadratic Forms

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- Oda[1977] proved that there are linear maps

$$\mathcal{S}_k(X;\Gamma) \longrightarrow S_{k-\frac{n-2}{2}}(\Gamma_0(N),\chi)$$

- The maps are constructed by integration against θ-kernel.
- Weil[1964] constructed a unitary representation for $\widetilde{Sp}$ giving group-theoretical interpretation and generalization to the theory of θ-functions.
- $O(2,n)$ and $SL(2,\mathbb{R})$ form a dual reductive pair in the sense of Howe, and the Oda correspondence can be reinterpreted in terms of representation theory.

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- **Stone-von Neumann theorem**
  An irreducible unitary representation of $\mathbb{H}$ is completely determined by its central character ψ. More precisely, it is isomorphic to (the L^2-induction) $\pi_\psi :=\text{Ind}_{\mathbb{M}}^{\mathbb{H}}\psi$.

The Schrodinger Model and Weil Representation

$Sp(\mathbb{W})$ acts trivially on $Z \Longrightarrow Sp(\mathbb{W})$ preserves ψ
Schur's lemma gives a representation of $Sp(\mathbb{W})$ on π_ψ, called the Weil representation ρ_ψ.
Realize ρ_ψ^∞ on the Schrodinger model $\mathcal{S}(\mathbb{Y})$

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} f(y) &= \chi(\det A) f(A^t y) \\ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} f(y) &= \psi(\frac{x^t B x}{2}) f(y) \\ \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} f(y) &= \gamma \hat{f}(y) \end{aligned}$$

($\hat{f}$ is the Fourier transform and χ and γ determined by ψ)

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- The correspondence is called Howe duality correspondence.

Consider ρ_ψ over $\mathbb{A} = \mathbb{A}_k$ (!)

Theorem (Poisson summation on $\mathbb{Y}_{\mathbb{A}}$)

There exist a unique H_k-invariant distribution, on the Schrodinger model it is realized as

$$\Theta(\Phi) = \sum_{y \in Y_k} \Phi(y)$$

This distribution is also $\mathbb{G}_k$-invariant.

(G, G') a dual reductive pair in $\mathbb{G}$
φ a cusp form on G'

$$\theta(\Phi, \varphi)(g) = \int_{G'_k \backslash G'_{\mathbb{A}}} \Theta(hg \cdot \Phi)\varphi(h) \, dh$$

the theta lift of φ gives an automorphic form on G.

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- Casselman's subrepresentation theorem allows us to look at the principle series only.
- On the Schrodinger model the action of $\mathfrak{sl}(2)$ is

$$\begin{aligned}
\hbar &= x_1\frac{\partial}{\partial x_1} + \cdots + x_{n+2}\frac{\partial}{\partial x_{n+2}} + \frac{n+2}{2} \\
e &= \frac{i}{2}(x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2) \\
f &= \frac{i}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_{n+2}^2}\right)
\end{aligned}$$

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- C^* spanned by Dirac delta and its derivatives

$$C^*=\bigoplus_{\alpha}\mathbb{C}\cdot\delta^{(\alpha)}$$

Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C^* \arrow[r] & \mathcal{S}^* \arrow[r] & A^* \arrow[r] & 0 \\
0 \arrow[r] & C^* \arrow[r] \arrow[u, "e_1"] & \mathcal{S}^* \arrow[r] \arrow[u, "e_2"] & A^* \arrow[r] \arrow[u, "e_3"] & 0
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By the snake lemma we have a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker e_1 \arrow[r] & \ker e_2 \arrow[r] & \ker e_3 \arrow[r] & \operatorname{coker} e_1 \arrow[r] & \cdots
\end{tikzcd}$$

e_1 is surjective (take Fourier transform...) so $\operatorname{coker} e_1 \simeq 0$ and we have a short exact sequence!

We rewrite it as

$$\begin{tikzcd} 0 \arrow[r] & C_e^* \arrow[r] & \mathcal{S}_e^* \arrow[r] & A_e^* \arrow[r] & 0 \end{tikzcd}$$

$\mathcal{S}_e^*$ (archimedean) Jacquet module of ρ_ψ^*

A_e^* (tempered) distributions supported on the light cone

C_e^* reduces to classical invariant theory

$T_s = \hbar - s - \frac{2+n}{2}$, a holomorphic family of operators

$$\begin{tikzcd}
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\end{tikzcd}$$

Classical invariant theory implies the first arrow is a bijection unless $s \in \mathbb{Z}_{\leq 0}$

Proposition

The discrete series of $O(2,n)$ occurs when its highest K-type has highest weight $(k,0,\cdots,0)$ where $k \geq \frac{n}{2}+1$ is an integer and the corresponding discrete series of $SL(2,\mathbb{R})$ is $V_{k-\frac{n-2}{2}}$.

$$\mathcal{V}_k \longleftrightarrow V_{k-\frac{n-2}{2}}$$

Remark

- The method is distribution-theoretic
- Generalizes to nondiscrete spectrum as well as other groups

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- $B_{m,n}$ is generic
- $B_{m,n}$ the local components of liftings of Hilbert modular forms.

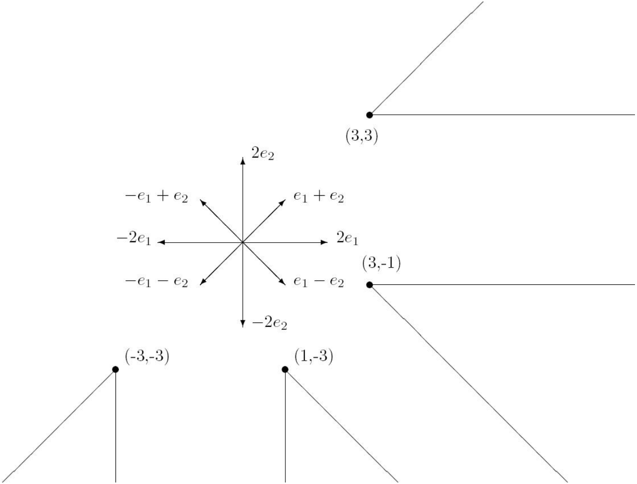

Figure: Root System for $Sp(2,\mathbb{R})$ and Range for Blattner Parameter

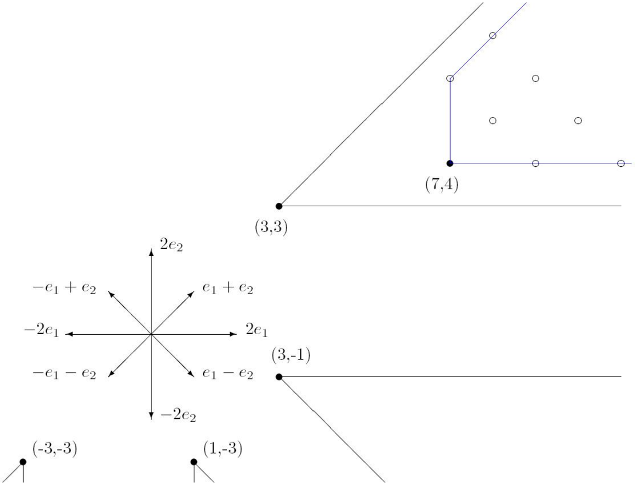

Figure: K-types That Occur for Holomorphic Discrete Series

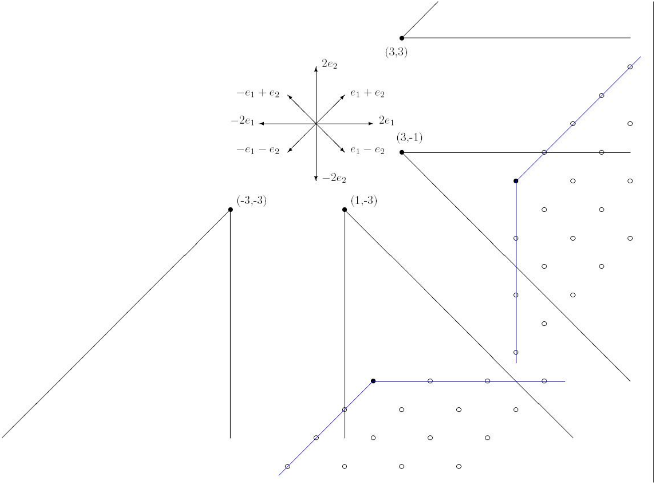

Figure: *K*-types That Occur for the Big Discrete Series