Representations of Compact Lie Groups

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The objective of this notes is to try to understand representation theory of the compact orthogonal and unitary groups.
•Complete reducibility and finite-dimensionality
•Theory of highest weight
•Representations of unitary groups
•Representations of orthogonal groups

Complete Reducibility and Finite-dimensionality

The existence of Harr measure and the compactness of G allow us to get a G-invariant Hermitian form on any Hilbert space representation of G through integration, thus making the representation *unitary*:
Suppose $\langle , \rangle_1$ is any inner product on a G-representation space H, define a new inner product $\langle , \rangle$ by

$$\langle u, v \rangle = \int_G \langle gu, gv \rangle_1 \, dg$$

It is easily checked that this new inner product is G-invariant.
We can prove the Peter-Weyl theorem, which states that the regular representation of G on $L^2(G)$ can be decomposed as a Hilbert space direct sum into irreducible finite-dimensional representations. After this, a trick that can be found in Lang's *Real Analysis* proves easily that all irreducible representations of G are finite-dimensional. And we can proceed to get *complete reducibility.* Notice here we needed only the fact that G is a compact group. If we require further that G be a Lie group, then we can prove that G is in fact *linear*.

Theory of Highest Weight

A real Lie algebra $\mathfrak{g}$ is called compact if the group Int $\mathfrak{g}$ is compact. The Killing form of a compact Lie algebra is negative semidefinite. On the other hand, if the Killing form of a Lie algebra $\mathfrak{g}$ is negative definite, then $\mathfrak{g}$ is a compact Lie algebra (without center).
For the moment let $\mathfrak{g}$ be any Lie algebra. We can decompose $\mathfrak{g}$ as $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is called the Borel subalgebra of $\mathfrak{g}$ and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ and $[\mathfrak{h}, \mathfrak{n}] = \mathfrak{n}$. Let $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ be the set of roots and $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{h})$ be the set of positive roots. We list the root system for the classical groups below. n is the *rank* of G $e_i(diag(a_1, \cdots, a_n)) = a_i$:

1. **Type** $A_n,\ n \geq 1$
   $G = SL(n+1, \mathbb{C})$ is connected. $K = SU(n+1)$ is a maximal compact subgroup.
   $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ is the Lie algebra of G.
   $\triangle = \{e_i - e_j\},\quad \triangle^+ = \{e_i - e_j \,|i < j\},\quad \Sigma = \{e_i - e_{i+1} | i = 1, \cdots, n\}$
2. **Type** $B_n,\ n \geq 2$
   For $G = SO(2n+1, \mathbb{C})$, $K = SO(2n+1)$, $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$
   $\triangle = \{\pm e_i \pm e_j,\ \pm e_i\},\quad \triangle^+ = \{e_i \pm e_j, e_i | i < j\},\quad \Sigma = \{e_i - e_{i+1},\ e_n | i = 1, \cdots, n-1\}$
3. **Type** $C_n,\ n \geq 3$
   $G = Sp(n, \mathbb{C})$, $K = Sp(n)$, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.
   $\triangle = \{\pm e_i \pm e_j,\ \pm 2e_i\},\quad \triangle^+ = \{e_i \pm e_j,\ 2e_i | i < j\},\quad \Sigma = \{e_i - e_{i+1},\ 2e_n | i = 1, \cdots, n-1\}$
4. **Type** $D_n,\ n \geq 4$
   $G = SO(2n, \mathbb{C})$, $K = SO(2n)$, $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$.
   $\triangle = \{\pm e_i \pm e_j, \},\quad \triangle^+ = \{e_i \pm e_j, | i < j\},\quad \Sigma = \{e_i - e_{i+1},\ e_{n-1} + e_n | i = 1, \cdots, n-1\}$

Consider finite-dimensional representations of $\mathfrak{g}$. Any such a representation V admits a weight space decomposition: $V = \bigoplus_{\lambda} V_\lambda$. We can fix an order on Σ, or equivalently, fix a positive Weyl chamber, this gives an order on all possible weights. The fact that V is of finite dimension implies that there is only a finite number of weights, so we have a *highest weight* with a chosen ordering. Such a highest weight μ is characterized by the following properties:

1. $\forall\, v_0 \in V_\mu$, we have $\mathfrak{n}v_0 = 0$. So $V_\mu \subset V^{\mathfrak{n}} = \{v \in V : \mathfrak{n}v = 0\}$.
2. μ is dominant and algebraically integral
3. If V is an irreducible representation, then $V_\mu = V^{\mathfrak{n}}$ and is of dimension 1. Moreover, V is uniquely determined by μ.

There is also an *existence* statement, which provide us with an irreducible finite-dimensional representation with highest weight μ for any dominant and algebraically integral linear functionals on $\mathfrak{h}$. Thus, dominant and algebraically integral linear functionals on $\mathfrak{h}$ stand in one-to-one correspondence with the irreducible finite-dimensional representations of $\mathfrak{g}$. Weyl's *unitary trick* then gives description of all irreducible representations of a compact *connected* Lie group.

Representations of Unitary Groups

Let us start by giving some examples of representations of $G = U(n)$ (and thus of $SU(n)$).

1. Consider the natural action of G on $S^k\mathbb{C}^n$, identified with the space of homogeneous polynomials of degree k in n complex variables. For any $\alpha \in Z^n$, $|\alpha| = k$, z^α is a weight vector with weight α. The action of E_{ij} is $z_i \dfrac{\partial}{\partial z_j}$. Thus the only highest weight vector is z_1^k. So this representation is irreducible.
2. Consider the natural action of G on $\bigvee^k \mathbb{C}^n$. Let $u_1, \cdots, u_n$ be the standard normal basis of $\mathbb{C}^n$, then the weight vectors are $u_\gamma = u_{j_1} \wedge \cdots \wedge u_{j_k}$ where $j_1 < \cdots < j_k$. The weight being $\gamma = (\gamma_1, \cdots, \gamma_n)$ where $\gamma_j = 1$ if j appears in u_γ and 0 otherwise. The action of E_{ij} is that it changes the u_j into u_i if j appears in u_γ, and 0 otherwise. It is clear that the only highest weight vector is $u_1 \wedge \cdots \wedge u_k$ and the highest weight is $e_1 + \cdots + e_k$. These representations are therefore irreducible, and we will see later that they are fundemental.

If λ and μ are highest weights for irreducible representations of G, so is $\lambda + \mu$. Then there exists *fundamental weights* $\omega_1, \cdots, \omega_k$ with k being the rank of G such that each dominant integral weight λ (in other words highest weight for an irreducible) can be written uniquely as $\lambda = \sum_{j=1}^k n_j \omega_j$.

In the case of $SU(n)$, the $\bigvee^j \mathbb{C}^n$'s for $j = 1, \cdots, n-1$ are irreducible representations with highest weight ω_j. So every irreducible representation of $SU(n)$ is a subspace of tensor products of the fundamental representations $\bigvee^j \mathbb{C}^n$. (Thus they are in 1-to-1 correspondence with the *Young tableau* with $n-1$ rows.) Let $\alpha_i = e_i - e_{i+1}$ be the simple roots of $\mathfrak{g}_{\mathbb{C}}$, and H_i be the dual basis for the α_i, then the highest weight ω_j of the $\bigvee^j \mathbb{C}^n$ is 1 on H_j and 0 on the others. More explicitly,

$$\omega_i = e_1 + \cdots + e_i$$

Remark: Note here that the last term with a minus sign is identically zero on $\mathfrak{h}_{\mathbb{C}}$, so we could remove this term.

Theorem: The irreducible representations of $SU(n)$ and hence holomorphic finite-dimensional representations of $SL(n, \mathbb{C})$ are parameterized by their highest weights $\lambda = \sum_{j=1}^{n-1} n_j \omega_j$, where the ω_j's are described above and the n_j's are nonnegative integers.

$U(n)$ is a direct product of its center and $SU(n)$. So on any irreducible representation the center acts via a character, and the restriction to $SU(n)$ is also irreducible. So the irreducibles are indexed by (λ, n) where λ is a dominant integral weight given above, and n an integer giving the central character.

Representations of Orthogonal Groups

Let us consider the special orthogonal groups first. We use the form defined by the matrix that have 1's on the anti-diagonal entries and zeros else where. There are two cases depending on the size of matrix being odd or even.

Case 1. $G = SO(2n+1)$. The positive roots are $\Sigma = \{e_i - e_{i+1},\ e_n | i = 1, \cdots, n-1\}$, the *coroots* are the dual basis consisting of $H_i - H_{i+1} - H_{n+i+1} + H_{n+i+2}$ for $e_i - e_{i+1}$ and $2H_n - 2H_{n+2}$ for e_n. Thus the fundamental weights defined by the relations $\omega_i(H_{\alpha_j}) = \delta_{ij}$ are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, n-1$ and $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. Similar to the $U(n)$ case, the natural representation on $\wedge^k \mathbb{C}^{2n+1}$ have highest weight $e_1 + \cdots + e_k$. Thus they have $\omega_1, \cdots, \omega_{n-1}, 2\omega_n$ as highest weight. For $k = 1, \cdots, n-1$ the $\wedge^k \mathbb{C}^{2n+1}$ are fundamental representations.

Case 2. $G = SO(2n)$. The positive roots are $\Sigma = \{e_i - e_{i+1},\ e_{n-1} + e_n | i = 1, \cdots, n-1\}$, the *coroots* consists of $H_i - H_{i+1} - H_{n+i+1} + H_{n+i+2}$ for $e_i - e_{i+1}$ and $H_{n-1} + H_n - H_{n+1} - H_{n+2}$ for $e_{n-1} + e_n$. Thus the fundamental weights are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, n-2$ and $\omega_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n)$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. The natural representation on $\wedge^k \mathbb{C}^{2n+1}$ have highest weight $e_1 + \cdots + e_k$. Thus they have $\omega_1, \cdots, \omega_{n-2}, \omega_{n-1} + \omega_n$ as highest weight. For $k = 1, \cdots, n-2$ the $\wedge^k \mathbb{C}^{2n+1}$ are fundamental representations. While $\wedge^n \mathbb{C}^{2n+1}$ is an irreducible representation of $O(2n)$, it decomposes into two representations of $SO(2n)$ with highest weight $2\omega_{n-1}$ and $2\omega_n$ respectively.

The full orthogonal group $O(2n+1)$ is a direct product of its center $\mathbf{Z}_2$ and $SO(2n+1)$, so irreducibles of $O(n)$ are indexed by (λ, ϵ), where $\lambda = l_1\omega_1 + \cdots + 2l_n\omega_n$ with integers l_j identifying the restriction to $SO(2n+1)$ and $\epsilon = \pm 1$ gives the central character.

For $O(2n)$ we need to decide whether the representation of $SO(2n)$ is type I or type II.

Consider $g_0 = \begin{pmatrix} I_{2n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$, so that $O(2n) = SO(2n) \bigcup g_0 SO(2n)$.For $\lambda = l_1\omega_1 + \cdots + l_n\omega_n = \lambda_1 e_1 + \cdots + \lambda_n e_n$

1. If $g_0 \cdot \lambda \neq \lambda$, then $Ind_{SO(2n)}^{O(2n)} \pi_\lambda$ is an irreducible representation of $O(2n)$. The condition is equivalent to $\lambda_n \neq 0$, or $l_{n-1} \neq l_n$.

2. If $g_0 \cdot \lambda = \lambda$, then π_λ can be extended to an irreducible representation of $O(2n)$ in two ways: π_λ on which g_0 acts trivially, and $\det \otimes \pi_\lambda$ on which g_0 acts by -1.