Study Notes for Functional Analysis

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The objective of this note is to write a proof of the (*abelian*) spectral theorem.

- **Basics for topological vector spaces**
- **Finite-dimensional case**
- **Spectral theorem for compact operators**
- **Spectral theorem for bounded normal operators**

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1 Basics for topological vector spaces

Every topological vector space is a Hausdorff space

Types of topological vector spaces

- locally convex
- locally compact $\iff$ X has finite dimension
- metrizable $\iff$ has a countable local base
- metrizable and complete = F-space
- locally convex F-space = Fréchet space
- normable or normed $\iff$ locally bounded and locally convex
- Banach = complete normed space
- locally bounded with Heine-Borel property $\Longrightarrow$ finite-dimensional
  This implies that the infinite-dimensional spaces $C(\Omega)$, although it is a Fréchet space with Heine-Borel property, is not locally bounded, hence not normable.

1. Examples of Fréchet space with Heine-Borel property:
$$H(\Omega) = \{\text{holomorphic functions on an open set } \Omega\}$$
$$C_K^\infty = \{\text{smooth functions supported on a compact set } K\}$$
2. *Uniform boundedness* principle: An *equicontinuous* collection Γ of linear mappings is uniformly bounded; If a collection of linear mappings satisfy that the set of points with bounded orbits is of *second category*, then Γ is equicontinuous.
   As a consequence, pointwise boundedness in a space of second category implies uniform boundedness.

3. A linear map between F-spaces has a closed graph iff it is continuous.
   Corollary: Separately continuous bilinear map on two F-spaces is jointly continuous.

4. **The Hahn-Banach Theorems (for locally convex topological vector spaces)**
   - **Dominated extension theorem**: A linear functional on a subspace, dominated by a seminorm, extends to a linear functional on the whole space.
   - **Separation Theorem**:
     For A and B disjoint nonempty subsets of a topological vector space X:
     (a) If A is open, $\exists\, \Gamma \in X^*$ and $\gamma \in \mathbb{R}$ such that $\Re(\Gamma x) < \gamma \leq \Re(\Gamma y)$, $\forall x \in A$ and $\forall y \in B$
     (b) If A is compact and B is closed and X is locally convex, then $\exists\, \Gamma \in X^*$, $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\Re(\Gamma x) < \gamma_1 < \gamma_2 < \Re(\Gamma y)$, $\forall x \in A$ and $\forall y \in B$

     Corollary: If X is locally convex, then X^* separates points in X.
     Corollary: If X is locally convex and x is not in the closure of a subspace M, then $\exists\, \Gamma \in X^*$ vanishing on M but not on x.
   - **Continuous extension theorem**: Any *continuous* linear functional on a subspace of a locally convex space can be extended continuously to the whole space. (This depends only on the separation property in the above corollary.)

5. The *weak topology* on X is the weakest topology on X such that every element in X^* is continuous (the X^*-topology of X). X with its weak topology is a locally convex space with dual X^*.
   - A subset of X is weakly bounded iff X^* is bounded on this subset.
   - For a convex subset of a locally convex space X, the weak closure equals to the original closure, weakly dense equals originally dense.

6. The *weak *-topology* on X^* is the X-topology on X^*. Under the weak *-topology, the dual of X^* is X.
   Banach-Alaoglu Theorem
   If V is a neighborhood of 0 in X, then $K = \{\Lambda \in X^* : \ |\Lambda x| \leq 1 \ \ \forall x \in V\}$ is weak *-compact.□

   If X is separable, a weak *-compact subset of X^* is metrizable in the weak *-topology.
   In a locally convex space, weakly bounded equals originally bounded.

   A subset of a complete metric space (F-space) is compact iff it is closed and *totally bounded* (can be covered by any neighborhood of 0 plus a finite set).

   Vector-valued integrals are defined via scalar-valued ones through usage of the dual space. This uniquely characterizes the integral, but does not prove existence.

7. Ω a open set in $\mathbb{C}$ and X a complex topological vector space. A function $f : \ \Omega \to X$ is weakly holomorphic in if Λf is holomorphic. It is said to be strongly holomorphic if the limit of difference quotient exists at every point in Ω.

   If X is a Fréchet space (in fact, only locally convexity and quasi-completeness will suffice), and f is weakly holomorphic, then:

   (a) f is strongly continuous in Ω (uses only locally convexity and the crux of matter here).
   (b) f is strongly holomorphic in Ω (what we care about most).

8. For normed space X, the strong topology on X^* is stronger than the weak *-topology. With its natural norm $\mathscr{B}(X, Y)$ is a normed space. If Y is a Banach space, so is $\mathscr{B}(X, Y)$. X a normed space embeds isometrically into its second dual X^{**}. The image is the dual of weak *-topologized X^*. If it is onto, X is called *reflexive.*

9. Let X be a (complex) linear topological space and T a linear operator on X with domain $\mathscr{D}(T)$ and range $\mathscr{R}(T)$. The set of $\lambda \in \mathbb{C}$ such that $\mathscr{R}(\lambda I - T)$ is dense in X and $\lambda I - T$ has a continuous inverse is called the *resolvent set* $\rho(T)$ of T. $\sigma(T) = \mathbb{C} - \rho(T)$ is called the spectrum of T. $\sigma(T)$ consists of three disjoint parts:

   i. **Discrete spectrum:** the λ where $\lambda I - T$ does not have an inverse.

   ii. **Continuous spectrum:** the λ where $\lambda I - T$ has a discontinuous inverse.

   iii. **Residual spectrum:** the λ where $\lambda I - T$ has an inverse but $\mathscr{R}(\lambda I - T)$ is not dense in X.

   Theorem: A bounded linear operator on a Hilbert space H has a compact spectrum.
   Proof: It is easy to see that if $\|T\| \leq 1$ then

$$(I - T)^{-1} = \sum_{j=1}^{\infty} T^j \tag{1}$$

   converges to the two-sided inverse of $I - T$. So for any $\lambda \in \mathbb{C}$ such that $|\lambda| \geq \|T\|$, we have

$$\lambda I - T = \lambda(I - \lambda^{-1}T)$$

   where both factor on the right have an inverse. Thus $\sigma(T)$ is a *bounded* set. To see that $\rho(T)$ is open, suppose $\lambda_0 \in \rho(T)$, that is, $\lambda_0 I - T$ is invertible

$$\lambda - T = -(\lambda_0 - \lambda)I + (\lambda_0 I - T) = (\lambda_0 I - T)(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}) \tag{2}$$

   The second factor on the right is invertible whenever $|\lambda - \lambda_0| \leq \|(\lambda_0 I - T)^{-1}\|^{-1}$. So $\rho(T)$ is open and $\sigma(T)$ is *closed.* □

2 Spaces of Finite Dimension

If dim$X = n < \infty$, the spectral theorem is a simple and beautiful one.

10. Every linear operator is bounded/continuous. After choosing a basis, X can be identified with $\mathbb{C}^n$ and every linear operator can be represented by an n-by-n matrix. We are essentially trying to diagonalize this matrix.

    Theorem: Let T be a normal operator on X, there is an orthonormal basis of X consisting of eigenvectors of T.

    In terms of matrix, this theorem can be stated as follows:
    Theorem: Let M be a normal matrix, there is a unitary matrix U such that UMU^* is a

diagonal matrix.

Proof: $P(x) = \det(M - xI)$ is a polynomial of degree n with complex coefficients. By the fundemantal theorem of algebra, it has a zero λ. The fact that $\det(M - \lambda I) = 0$ implies that $T - \lambda I$ is not rank n, so as a linear operator it has a nonzero kernel V_λ. Pick any vector $v' \in V_\lambda$, let $v = \frac{v'}{\|v'\|}$ be the unit eigenvector, extend to an orthonormal basis of X. Then the matrix for T becomes $\begin{bmatrix} \lambda & * \\ 0 & N \end{bmatrix}$, where N is now an $(n-1)$-by-$(n-1)$ matrix. Since T is normal, $* = 0$, $\lambda \in \mathbb{R}$ and N is normal. We proceed by induction. □

Proof: Consider M as a member of the n^2-dimensional vector space of n-by-n matrices, then the set $\{M^k\}_{k=0}^{n^2}$ with $n^2 + 1$ elements is linearly *dependent*. This gives a polynomial $f(X)$ such that $f(M) = 0$. Now consider the set P of all such polynomials, it is a nonempty filtered partially ordered set with order given by division. By Zorn's lemma, there exists a minimal elements $p(X)$. The fact that P is an ideal of the PID $\mathbb{C}[X]$ gives that $P = (p(X))$. Let λ be a root of $p(X)$, then $p(X) = (X - \lambda)q(X)$ with $q(X) \in \mathbb{C}[X]$. We know that $A = M - \lambda I$ is *not* invertible, otherwise $p(M) = Aq(M) = 0$ gives $q(M) = 0$ and $q(X)$ divides $p(X)$, contradicting the minimality of $p(X)$. So A is not one-to-one, let $v \neq 0$ be a vector in $\ker A$, we proceed as in the first proof. □

Example of non-normal operators
In the finite-dimensional case *normal* means exactly *diagonalizable*. So one example of non-normal operator would be $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3 Spectral Theorem for Compact Operators

11. From now on H is a Hilbert space.
Theorem: *(Characterization of Compact Operators) Suppose $T \in \mathscr{B}(H)$, Then the following are equivalent:*

(1) The image of unit ball under T has compact closure.

(2) The image under T of every bounded sequence has a convergent subsequence.

(3) Weakly convergent sequence is mapped by T to strongly convergent sequence.

(4) T is the limit of a sequence of finite-rank operators.

If T satisfies any of the above, T is called a *compact* operater.

Theorem: *Every nonzero element of the spectrum of a compact operator is an eigenvalue with a finite-dimensional eigenspace. The spectrum is compact, at most countable and 0 is the only possible limit point.*

Proof: Suppose that $0 \neq \lambda$ is *not* an eigenvalue.

(1) $\mathscr{R}(\lambda I - T) = H$.
If not, let $H_n = \mathscr{R}(\lambda I - T)^n$ and $H_0 = H$, we have a descending chain of closed subspaces, invariant under T:

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

The compactness of T guarantees the finiteness of this chain: suppose the containment is always proper, we can find $x_n \in H_n$ s.t. $x_n \perp H_{n+1}$, $|x_n| = 1$. Then for $m > n$

$$T(x_m) - T(x_n) = T(x_m) + (\lambda I - T)(x_n) + \lambda x_n = z + \lambda x_n$$

where $z = T(x_m) + (\lambda I - T)(x_n) \in H_{n+1}$, so $z \perp x_n$ and $\|T(x_m) - T(x_n)\| = \|z + \lambda x_n\| = \|z\| + |\lambda|\|x_n\| \geq \|\lambda\| > 0$. Thus $\{T(x_n)\}$ can not have a convergent subsequence, contradicting the compactness of T. Now that we know that the above chain is finite, we can pick a minimal n such that $H_n = H_{n+1}$. If $n = 0$ we are done. Otherwise, $n > 0$ and we pick $x \in H_{n-1} - H_n$, and let $y = (\lambda I - T)x \in H_n = H_{n+1}$, we have $y = (\lambda I - T)z$ for some $z \in H_n$. This gives $x - z \neq 0$ and $(\lambda I - T)(x - z) = 0$. So λ is an eigenvalue. ‡//

(2) $\exists c > 0$, *s.t.* $\|(\lambda I - T)x\| \geq c\|x\|$,$\forall x \in H$
Suppose this fails, then $\forall n \in \mathbb{N}$, $\exists x_n \in H$ s.t. $\|(\lambda I - T)x_n\| < \frac{1}{n}\|x_n\| = \frac{1}{n}$. So

$$\lim_{n\to\infty} (\lambda I - T)x_n = 0$$

Choose a convergent subsequence $T(x_{k_n})$ of $T(x_n)$, $\lim_{n\to\infty} T(x_{k_n}) = y$, then $\lim_{n\to\infty} x_{k_n} = \frac{y}{\lambda}$.
Now we have

$$(\lambda I - T)\frac{y}{\lambda} = \lim_{n\to\infty} (\lambda I - T)x_{k_n} = 0 \tag{3}$$

that is, $T(y) = \lambda y$, λ is an eigenvalue of T. //

(3) Now (2) ensures that $\lambda I - T$ is one-to-one and its inverse is bounded (*continuous*), and (1) ensures that its domain is H.
Assume that for some nonzero λ, the eigenspace E_λ is infinite-dimensional, then we can find an orthonormal basis with infinitely-many vectors $\{x_n\}$. This will contradicts the compactness of T as $\{T(x_n)\}$ would not have any convergent subsequences. //

(4) $\{|\lambda| > \delta\} \cap \sigma(T)$ *is a finite set.*
Otherwise we choose one eigenvector x_n for each λ_n, and form $H_n = < x_1, x_2, \cdots, x_n >$. We can choose unit vectors $y_n \in H_n$ s.t. $y_n \perp H_{n-1}$. We check that $T(H_n) = H_n$, and $(\lambda_n I - T)H_n \subseteq H_{n-1}$. Similarly for $m > n$ we have

$$\|T(y_m) - T(y_n)\| = \|\lambda_m y_m - ((\lambda_m I - T)y_m + Ty_n)\| = \|\lambda_m y_m - z\| = \|\lambda_m y_m\| + \|z\| \geq |\lambda_m| > \delta$$

since $z \in H_{m-1}$ and $y_m \perp z$. So $\{T(y_n)\}$ has no convergent subsequences. ‡□
Remark: We will prove later that a bounded normal operator is compact if and only if:

i. $\sigma(T)$ has no limit point except 0 .
ii. If $\lambda \neq 0$, then $\dim \mathscr{N}(\lambda I - T) < \infty$.

4 Spectral Theorem for Bounded Normal Operators

12. **Definition:** A *resolution of the identity* is a self-adjoint projection-valued measure. More precisely, it is a map E on a sigma algebra of Ω satisfying:

(a) $E(\omega)$ is a self-adjoint projection on H.

(b) $E(\emptyset) = 0$, $E(\Omega) = I$, $E(\omega \cap \omega') = E(\omega)E(\omega')$; and if $\omega \cap \omega' = \emptyset$, then $E(\omega \cap \omega') = E(\omega) + E(\omega')$

(c) For every $x,\ y \in H$, $E_{x,y}(\omega) = (E(\omega)x\ ,\ y)$ is a complex measure.

We can then form the Banach algebra $L^\infty(E)$, the equivalent classes of *essentially bounded* E-measurable functions.

Theorem: *Let E be a resolution of the identity, then there is an isometric*-isomorphism Φ of $L^\infty(E)$ onto a closed subalgebra A of $\mathscr{B}(H)$, which is related to E via:*

$$(\Phi(f)x\ ,\ y) = \int_\Omega f\ dE_{x,y} \quad \textit{or simply} \quad \Phi(f) = \int_\Omega f dE \tag{4}$$

Moreover, $\|\Phi(f)x\|^2 = \int_\Omega |f|^2\ dE_{x,x}$. An operator $Q \in \mathscr{B}(H)$ commutes with every $E(\omega)$ if and only if it commutes with A. □

The principle assertion of the spectral theorem is that every bounded normal operator on a Hilbert space induces a canonical resolution of the identity on the Borel subsets of its spectrum $\sigma(T)$, and that T can be reconstructed from E by an integral of the above type.

13. *A theorem of Gelfand-Naimark*
Let A be a commutative Banach algebra and Δ the set of all complex homomorphisms of A. Then Δ is the same as the set of maximal ideals of A. The formula $\hat{x}(h) = h(x)$, $\forall h \in \Delta$ gives the *Gelfand transform* of $x \in A$. The weak A-topology on Δ is called the *Gelfand topology* of Δ , with respect to which Δ is a *compact Hausdorff* space.

Theorem: *Suppose A is a commutative C^*-algebra with maximal ideal space Δ. Then the Gelfand transform gives an isometric*-isomorphism of A onto $C(\Delta)$.*

Corollary: *Suppose that A is the commutative C^*-algebra generated by $I,\ x,$ and x^* , then the formula $(\Phi(f))\hat{} = f \circ \hat{x}$ defines an isometric*-isomorphism of $C(\sigma(x))$ onto A . Moreover, if $f(\lambda) = \lambda$, then $\Phi(f) = x$.*

14. **Theorem:** *Let T be a bounded normal operator on a Hilbert space H . Then there exists a unique resolution of the identity E on the Borel subsets of $\sigma(T)$ with the property that $T = \int_{\sigma(T)} \lambda\ dE(\lambda)$. Furthermore, if S commutes with T, then S commutes with every $E(\omega)$.*
Proof: Let A be the closed subalgebra of $\mathscr{B}(H)$ generated by $I,\ T$ and T^* . The maximal ideal space Δ can be identified with $\sigma(T)$ in such a way that $\hat{T}(\lambda) = \lambda$, for every $\lambda \in \sigma(T)$. Then there exists a resolution of the identity E. (*more details needed here!*)
Conversely if such an E exists, then for any $p \in \mathbb{C}[X,\ Y]$, we have

$$p(T,\ T^*) = \int_{\sigma(T)} p(\lambda\ ,\bar{\lambda})\ dE(\lambda)$$

By the Stone-Weierstrass Theorem, $\mathbb{C}[X,\ Y]$ is dense in $C(\sigma(T))$. Therefore E is *uniquely* determined by T.
If $ST = TS$ then $ST^* = T^*S$, so S commutes with A, hence with every $E(\omega)$ by the Gelfand-Naimark theorem. $\square$