Equivariant Distributions Supported on a Light Cone

0. Motivation

The pair $(SL(2,\mathbb{R}), O(p,q))$ forms a dual reductive pair as subgroups of $Sp(n,\mathbb{R})$, where $n = p+q$. Realizing the Weil representation of $Sp(n,\mathbb{R})$ on the Schrodinger model (the space being $\mathcal{S}(\mathbb{R}^n)$), the action of $\mathfrak{sl}(2)$ is

$$\begin{aligned} \hbar &= x_1\frac{\partial}{\partial x_1} + \cdots + x_{p+q}\frac{\partial}{\partial x_{p+q}} + \frac{p+q}{2} \\ e &= \frac{i}{2}(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2) \\ f &= \frac{i}{2}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right) \end{aligned}$$

To look in quotients of $\mathcal{S}(\mathbb{R}^n)$ for $\mathfrak{sl}(2)$-modules, it is enough to dualize and look for submodules in $\mathcal{S}(\mathbb{R}^n)^*$. Thus we need to analyze $O(p,q)$-equivariant tempered distributions that are annihilated by e and are $\hbar$-eigendistributions.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions, that is, smooth functions on $\mathbb{R}^n$ all of whose derivatives are of rapid decay. $\mathcal{D} = \mathcal{S}(\mathbb{R}^n)^*$ the dual of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions. Consider $A = \mathcal{S}(\mathbb{R}^n)_0$ the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of Schwartz functions vanishing to infinite order at the origin. We have a short exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow C \longrightarrow 0$$

Dualizing we get

$$0 \longrightarrow C^* \longrightarrow \mathcal{S}(\mathbb{R}^n)^* \longrightarrow A^* \longrightarrow 0$$

where C^* is the space of distributions concentrated at the origin (Dirac delta and its derivatives). We consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & C^* & \longrightarrow & \mathcal{S}(\mathbb{R}^n)^* & \longrightarrow & A^* & \longrightarrow 0 \\ & \uparrow e_1 & & \uparrow e_2 & & \uparrow e_3 & \\ 0 \longrightarrow & C^* & \longrightarrow & \mathcal{S}(\mathbb{R}^n)^* & \longrightarrow & A^* & \longrightarrow 0 \end{array}$$

Where all the e_j's are (induced by) the operator e. Both rows are exact. By the snake lemma we have an exact sequence

$$0 \longrightarrow \ker e_1 \longrightarrow \ker e_2 \longrightarrow \ker e_3 \longrightarrow \operatorname{coker} e_1 \longrightarrow \operatorname{coker} e_2 \longrightarrow \operatorname{coker} e_3 \longrightarrow 0$$

With e_1 being surjective (this can be easily proved by taking Fourier transform), $\operatorname{coker} e_1 \simeq 0$. $\ker e_2$ are tempered distributions killed by e, the space we are mainly interested in. In other words we have a short exact sequence

$$0 \longrightarrow C_e^* \longrightarrow \mathcal{S}(\mathbb{R}^n)_e^* \longrightarrow A_e^* \longrightarrow 0$$

2. **Uniqueness of Invariant Distributions**

If we look inside A^* for $O(p,q)$-invariant distributions, a distribution *supported* on the light cone $\langle x,\ x\rangle_{p,q} = 0$ is uniquely expressible as a locally finite sum of transverse derivatives followed by distributions on the light cone. Since transverse derivative commutes with the $O(p,q)$-action, the distributions that follow also have to be invariant.

Lemma. *Up to constant multiple there is a unique $O(p,q)$-invariant distribution on the light cone*

$$Q = \{x \in \mathbb{R}^{p+q}\backslash\{0\} | \langle x,\ x\rangle_{p,q} = 0\}$$

Proof. This follows from transitivity of the $O(p,q)$-action on Q and the fact there there exists an *approximate identity* on $\mathbb{R}$. The distribution is a constant multiple of

$$u(\phi) = \int_Q \phi\ d\mu$$

□

Let

$$D = \hbar - \frac{p+q}{2} = x_1\frac{\partial}{\partial x_1} + \cdots + x_{p+q}\frac{\partial}{\partial x_{p+q}}$$

be the transverse derivative on Q . The lemma says that an invariant distribution supported on Q is of the form

$$v = \sum_n c_n\ u \circ \mathrm{Res}_Q^{\mathbb{R}^{p+q}} \circ D^n$$

For $f \in \mathcal{S}(\mathbb{R}^n)$, $s \in \mathbb{C}$, define the distribution u_s by

$$u_s(f) := \int_{x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2>0} (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^s\ f\ d\mu$$

where $d\mu$ is the usual Lebesgue measure. This distribution u_s is well-defined for Re $s > 0$ and is easily seen to be $O(p,q)$-invariant. The following theorem is in [Gelfand, *Genrelized Functions. vol. 1*]

Theorem 1. *The distribution u_s is holomorphic in Re $s > 0$ and has analytic continuation to $s \in \mathbb{C}$ except for two possible sets of poles:*

I. $s = -1,\ -2,\ \cdots,\ -k, \cdots$ *with residues being distributions concentrated on the light cone Q .*
II. $s = -\frac{p+q}{2},\ -\frac{p+q}{2} - 1,\ \cdots,\ -\frac{p+q}{2} - k, \cdots$ *with residues being distributions concentrated at the vertex of Q, the origin.*

Thus u_s has poles at $s = -1-k$. We can write some identities in A^* :

$$\delta_Q^{(k)} = \mathrm{Res}(u_s(f),\ -1-k)$$

The distributions $\delta_Q^{(k)}$ are also $O(p,q)$-invariant. Let $\delta_Q = \delta_Q^{(0)}$, we readily compute

$$\begin{aligned} e \cdot \delta_Q &= e \cdot \mathrm{Res}(u_s(f),\ -1) = \mathrm{Res}(e \cdot u_s,\ -1) = \mathrm{Res}(u_{s+1},\ -1) = 0 & (1) \\ D\ \delta_Q &= D\ \mathrm{Res}(u_s,\ -1) = \mathrm{Res}(D\ u_s,\ -1) = \mathrm{Res}(2su_s,\ -1) = -2\delta_Q & (2) \\ e \cdot \delta_Q^{(k)} &= \mathrm{Res}(e \cdot u_s,\ -1-k) = \mathrm{Res}(u_{s+1},\ -1-k) = \delta_Q^{(k-1)} & (3) \\ D\ \delta_Q^{(k)} &= \mathrm{Res}(D\ u_s,\ -1-k) = \mathrm{Res}(2su_s,\ -1-k) = (-2-2k)\delta_Q^{(k)} & (4) \end{aligned}$$

So up to a constant multiple δ_Q is the unique invariant distribution in A_e^*.

3. Identifying Discrete Series

Theorem 2. The discrete series of $O(2,n)$ occurs when its highest K-type has highest weight $(m, 0, \cdots, 0)$ where $m \leq -\frac{n}{2} - 1$ is an integer and the corresponding discrete series of $SL(2,\mathbb{R})$ is $V_{m+\frac{n}{2}-1}$.

Proof. If n is even then

$$\mathfrak{h} = \left\{ H \middle| \text{on the diagonal } H \text{ has} \begin{bmatrix} 0 & ih_j \\ -ih_j & 0 \end{bmatrix} \right\}$$

is a maximal abelian subalgebra of $\mathfrak{o}(n,2)$. Let H_j be in $\mathfrak{h}$ having $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ on the j-th block and 0 elsewhere. We use E_α to denote a matrix in the root space $\mathfrak{g}_\alpha$ and readily compute that

$$\begin{aligned} H_j &= i\left(-x_{2j}\frac{\partial}{\partial x_{2j-1}} + x_{2j-1}\frac{\partial}{\partial x_{2j}}\right) \\ E_{e_j-e_k} &= (-x_{2j-1} + ix_{2j})\left(\frac{\partial}{\partial x_{2k-1}} + i\frac{\partial}{\partial x_{2k}}\right) + (-x_{2k-1} - ix_{2k})\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right) \\ E_{e_j+e_k} &= (-x_{2j-1} + ix_{2j})\left(\frac{\partial}{\partial x_{2k-1}} - i\frac{\partial}{\partial x_{2k}}\right) + (-x_{2k-1} + ix_{2k})\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right) \end{aligned}$$

Again the action of $\mathfrak{sl}(2)$ is

$$\begin{aligned} \hbar &= x_1\frac{\partial}{\partial x_1} + \cdots + x_{n+2}\frac{\partial}{\partial x_{n+2}} + \frac{n+2}{2} \\ e &= \frac{i}{2}(x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2) \\ f &= \frac{i}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_{n+2}^2}\right) \end{aligned}$$

We look for H_j and $\hbar$-weight vectors in A^* that are annihilated by e and by $E_{e_j \pm e_k}$. Annihilation by e implies that this distribution is supported on the light cone $Q(\bar{x}) = 0$ where

$$Q(\bar{x}) = x_1^2 + x_2^2 - x_3^2 - \cdots - x_{n+2}^2$$

Let $v_{m,\pm} = (x_1 \pm ix_2)^m \cdot \delta_Q$, (1) and (2) give immediately

$$e\ v_{m,\pm} = 0$$

$$\hbar\ v_{m,\pm} = (\frac{n+2}{2} + m - 2)v_{m,\pm} = (\frac{n}{2} + m - 1)v_{m,\pm}$$

This vector generates a holomorphic discrete series of $SL(2,\mathbb{R})$ if and only n is even and

$$\frac{n}{2} + m - 1 \leq -2$$

We check that

$$H_1\ v_{m,\pm} = \pm m\ v_{m,\pm}\ , \quad H_j\ v_{m,\pm} = 0\ \ \forall j > 1$$

$$E_{e_j \pm e_k}\ v_{m,\pm} = 0 \quad \forall j,k \quad \text{s.t.} \quad 1 < j < k \leq \frac{n+2}{2}$$

For odd n, besides the operators above, we also need to consider

$$E_{e_j} = x_{n+2}\left(-\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\right) - (x_{2j-1} - ix_{2j})\frac{\partial}{\partial x_{n+2}}$$

And we can also check that $E_{e_j}\ v_{m,\pm} = 0$. □

Remark. This result confirms previous computations using the tensor product method.

4. **Uniqueness Proof**
We can see that the proof for Theorem 2 generalizes to the case of $O(p,q)$ when $p \geq 2$ as well as the case of a principle series representation. The only missing item being the uniqueness of the constructed distributions of which we will sketch a proof.
Let u be a tempered distribution in A_e^* satisfying

$$H_1\ u = s\ u\ , \quad H_j\ u = 0 \quad \forall j > 1 \tag{5}$$

$$E_{e_j \pm e_k}\ u = 0 \quad \forall j,k \quad \text{s.t.} \quad 1 < j < k \leq \frac{p+q}{2} \tag{6}$$

Regard u as a functional on $A_1 = C_c^\infty(\mathbb{R}^n)_0$, the space of smooth function on $\mathbb{R}^n$ with compact-support away from zero. We can define another distribution v via

$$(v,\ \phi) = (u,\ (x_1 + ix_2)^{-s}\phi)$$

Clearly for $\phi \in A_1$, $(x_1 + ix_2)^{-s}\phi$ is also in A_1. We can check the v is an invariant distribution in A_1^* annihilated by e . By the uniqueness lemma in section 2, we have up to a constant multiple $v = \delta_Q$. Therefore $u = (x_1 + ix_2)^s\delta_Q$ is the unique such distribution satisfying (5) and (6).