# Discrete Spectrum of Dual Reductive Pairs

The objective is to compute the explicit correspondence for the discrete part of the spectrum of the dual pair $(Sp(m,\mathbb{R}), O(p,q))$.

The Lie algebra of $G = Sp(m,\mathbb{R})$ is denoted $\mathfrak{g}_0 = \mathfrak{sp}(m,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sp}(m,\mathbb{C})$ its complexification. Then $\mathfrak{g} = \left\{ \left[ \begin{array}{cc} A & B \\ C & -A^t \end{array} \right] \middle| B = B^t, C = C^t, A, B, C \in \mathfrak{gl}(m,\mathbb{C}) \right\}$.
We write $\mathfrak{k} = \left\{ \left[ \begin{array}{cc} A & 0 \\ 0 & -A^t \end{array} \right] \right\}$, $\mathfrak{p}^+ = \left\{ \left[ \begin{array}{cc} 0 & B \\ 0 & 0 \end{array} \right] \middle| B = B^t \right\}$, $\mathfrak{p}^- = \left\{ \left[ \begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right] \middle| C = C^t \right\}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$. The oscillator representation $\omega$ can be realized on $L^2(\mathbb{R}^m)$, and its smooth vectors $\omega^\infty$ on $\mathcal{S}(\mathbb{R}^m)$.

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## 1. The dual pair $(Sp(1,\mathbb{R}), O(p,q))$

We would like to compute action of $O(p,q)$ on $\mathfrak{p}^+$-annihilated vectors in quotients of $\omega^\infty$, dualizing to compute same thing in subrepresentations of $(\omega^\infty)^*$, realized on $\mathcal{S}(\mathbb{R}^n)^*$, the space of tempered distributions.
Now in the case of an anisotropic orthogonal group, $\mathfrak{p}^+$-annihilation requires this distribution to be supported at zero. Fourier transform maps such distribution onto the space of polynomials. And Fourier transform $\mathcal{F}$ interchanges action of $\mathfrak{p}^+$ and $\mathfrak{p}^-$, so we now look at $O(n)$-action on the space of *harmonic* polynomials in the case of $m = 1$. $H$-acts essentially by the degree of the polynomial, and harmonic polynomial of a given degree gives an irreducible $O(n)$ representation. Thus we obtain:

**Proposition:** The decomposition for the dual pair $(Sp(1,\mathbb{R}), O(n))$ is

$$\omega^n = \sum_{m \in \mathbb{Z}^+} D_{-\frac{n}{2}-m} \bigotimes H^n_m$$

Where $H^n_m$ is the space of harmonic polynomials with $n$ variables of degree $m$, and $D_{-\frac{n}{2}-m}$ is the discrete series representation of $\widetilde{Sp}(1,\mathbb{R})$ with highest weight $-\frac{n}{2} - m$.

Let us prove that the highest weights of $H^n_m$ is $(n, 0, \cdots, 0)$. (*to be done...*)

Let us continue with the case $(SL(2,\mathbb{R}), O(p,q))$, two compact pairs are of importance here, namely $(U(1), U(p,q))$ and $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), O(p) \times O(q))$. The proposition gives the decomposition according to the second pair:

$$\omega^{p,q} = \sum_{l \in \mathbb{Z}^+} \sum_{m \in \mathbb{Z}^+} \left( D_{-\frac{p}{2}-l} \check{\otimes} D^*_{\frac{q}{2}+m} \right) \bigotimes \left( H^p_l \check{\otimes} H^q_m \right)$$

With the sign change due to the different sign in the form for the two orthogonal groups. To go back to our original problem, we need only restrict the representation from $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ down to the diagonal copy of $G = SL(2,\mathbb{R})$. The decomposition is computed by Repka:

$$D_{-\frac{p}{2}-l} \otimes D^*_{\frac{q}{2}+m} \simeq \mathrm{Ind}^G_K \, \chi_{\frac{q}{2}+m-\frac{p}{2}-l}$$

Where the latter contains a continuous part and possibly also a discrete part. Depending on the sign and parity of $x = \frac{q}{2} + m - \frac{p}{2} - l$ the discrete part has only holomorphic or anti-holomorphic discrete series of the same parity, and all such with extreme weight bounded by $|x|$ would occur. Thus we easily obtain:

**Claim:** If both p and q are even, *all* discrete series of $SL(2,\mathbb{R})$ occur in $\omega^{p,q}$, and the corresponding $K'$-types for $O(p,q)$ of the form $H_l^p \otimes H_m^q$, where $x = \frac{q}{2} + m - \frac{p}{2} - l = k, k+2, \cdots$ Therefore $\theta(D_k)$ is the discrete series of $O(p,q)$ with lowest $K'$-type $H^p_{k+\frac{p-q}{2}} \otimes H_0^q$, whose highest weight is then $(k + \frac{p-q}{2}, 0 \cdots, 0)$
Proof:

- All discrete series of $SL(2,\mathbb{R})$ occur.
- Discrete series of $SL(2,\mathbb{R})$ corresponds to discrete series of $O(p,q)$
- The $K'$-types $H_l^p \otimes H_m^q$ described above lie inside a discrete series of $O(p,q)$

**Remark:** This can't be exactly true, since if both $p$ and $q$ are odd, $G' = O(p,q)$ doest not have discrete series. But whenever $G'$ does have discrete series, the above claim gives the correspondence. The case $q = 2$ has already been verified using distributions.

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## 2. The case $Sp(2,\mathbb{R})$

The first step is to consider $G' = O(p)$. As before, we use the theory of Fourier transform to justify considering only the Fock model in the anisotropic case. Let us give description of the action of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ . For $\mathfrak{k}$:

$$H_1 = \sum_{j=1}^{p} x_j \frac{d}{dx_j} + \frac{p}{2} \,, \qquad H_2 = \sum_{j=1}^{p} y_j \frac{d}{dy_j} + \frac{p}{2}$$

$$E_{e_1 - e_2} = \sum_{j=1}^{p} x_j \frac{d}{dy_j} \,, \qquad E_{e_2 - e_1} = \sum_{j=1}^{p} y_j \frac{d}{dx_j}$$

For $\mathfrak{p}^-$,

$$E_{2e_1} = \frac{i}{2} \sum_{j=1}^{p} \frac{d^2}{dx_j^2} \,, \qquad E_{2e_2} = \frac{i}{2} \sum_{j=1}^{p} \frac{d^2}{dy_j^2} \,, \qquad E_{e_1+e_2} = \sum_{j=1}^{p} \frac{d^2}{dx_j dy_j}$$

To identify a discrete series, we need to look inside the Fock model for lowest ($\mathfrak{p}^-$-killed) $\mathfrak{k}$-types. We label these $\mathfrak{k}$-types by its highest weight vectors, namely simultaneous eigenvectors for $H_1$ and $H_2$ that are killed by $E_{e_1 - e_2}$.

Let us just comment that for $p = 1$, we have a decomposition

$$\omega_2 = V_{(\frac{1}{2}, \frac{1}{2})} \bigoplus V_{(\frac{3}{2}, \frac{1}{2})}$$

As a representation of $\mathfrak{k}$, $V_{(\frac{1}{2},\,\frac{1}{2})} = \bigoplus_{n=0}^{\infty} (\frac{1}{2}+n,\ \frac{1}{2})$, $V_{(\frac{3}{2},\,\frac{1}{2})} = \bigoplus_{n=0}^{\infty} (\frac{3}{2}+n,\ \frac{1}{2})$ .

For the case $p = 2$, $\mathfrak{g}' = \mathfrak{o}(2,\mathbb{C})$ is one-dimensional. The action is

$$t = i\left(x_1\frac{d}{dx_2} - x_2\frac{d}{dx_1} + y_1\frac{d}{dy_2} - y_2\frac{d}{dy_1}\right)$$

Degree-n homogeneous harmonic polynomials in the variables $x_1$ and $x_2$ are examples which are highest weight vectors of lowest $\mathfrak{k}$-type. They are of the form

$$v = (x_1 \pm ix_2)^n$$

and we check that $t\cdot(x_1 \pm ix_2)^n = \mp n(x_1 \pm ix_2)^n$ . The only representation of $G'$ that is missing here is $-\mathbb{1}$, the determinant representation, which is given on $x_1y_2 - x_2y_1$. With all of $\check{G}'$ accounted for, we obtain the decomposition:

$$\omega_2^2 = \big(V_{(1,1)}\otimes\mathbb{1}\big)\bigoplus\big(V_{(2,2)}\otimes\det\big)\bigoplus_{n=1}^{\infty}\big(V_{(n+1,1)}\otimes\pi_n\big)$$

For the case $p = 4$ the action of $\mathfrak{h}\subset\mathfrak{g}'$ is

$$H_1' = i\left(x_1\frac{d}{dx_2} - x_2\frac{d}{dx_1} + y_1\frac{d}{dy_2} - y_2\frac{d}{dy_1}\right), \qquad H_2' = i\left(x_3\frac{d}{dx_4} - x_4\frac{d}{dx_3} + y_3\frac{d}{dy_4} - y_4\frac{d}{dy_3}\right)$$

The positive root space acts as follows

$$\begin{aligned}
E'_{e_1-e_2} &= -(x_3+ix_4)\left(\frac{d}{dx_1} - i\frac{d}{dx_2}\right) + (x_1-ix_2)\left(\frac{d}{dx_3} + i\frac{d}{dx_4}\right)\\
&\quad -(y_3+iy_4)\left(\frac{d}{dy_1} - i\frac{d}{dy_2}\right) + (y_1-iy_2)\left(\frac{d}{dy_3} + i\frac{d}{dy_4}\right)\\
E'_{e_1+e_2} &= -(x_3-ix_4)\left(\frac{d}{dx_1} - i\frac{d}{dx_2}\right) + (x_1-ix_2)\left(\frac{d}{dx_3} - i\frac{d}{dx_4}\right)\\
&\quad -(y_3-iy_4)\left(\frac{d}{dy_1} - i\frac{d}{dy_2}\right) + (y_1-iy_2)\left(\frac{d}{dy_3} - i\frac{d}{dy_4}\right)
\end{aligned}$$

We consider the polynomials

$$\begin{aligned}
v_{m,n} &= (x_1-ix_2)^{m-n}[(x_1-ix_2)(y_3-iy_4) - (y_1-iy_2)(x_3-ix_4)]^n\\
v_{m,-n} &= (x_1-ix_2)^{m-n}[(x_1-ix_2)(y_3+iy_4) - (y_1-iy_2)(x_3+ix_4)]^n
\end{aligned}$$

and check that they are killed by $\mathfrak{p}^-$ and $E_{e_1-e_2}$, also not so easily check that they are also killed by $E'_{e_1-e_2}$ and $E'_{e_1+e_2}$. After computing the weights of $\mathfrak{h}$ and $\mathfrak{h}'$ respectively, we reach the conclusion that $V_{m+2,n+2}$ corresponds to $(m,\pm n)$ for the pair $(\mathfrak{g},\mathfrak{g}')$. We label the representation for $O(4)$ by $\pi_{m,n}$ . Since all possible $\mathfrak{k}$-types are of the form $(m+2,n+2)$ with $m\ge n\ge 0$ (again using tensor product $\omega_2^4 = \omega_2^2\otimes\omega_2^2$), all possible representations for $G$ are accounted for, we obtain a decomposition

$$\omega_2^4 = \bigoplus_{m\ge n\ge 0} V_{m+2,n+2}\otimes\pi_{m,n}$$

**Remark:** It can now be guessed that for general $p \geq 4$ the decomposition looks like

$$\omega_2^p = \bigoplus_{m \geq n \geq 0} V_{m+\frac{p}{2}, n+\frac{p}{2}} \otimes \pi_{m,n}$$

and obviously something needs to be addressed about $\pi_{0,0}$ here.

If we consider the case $q = 0$ and $p$ an even integer in general, the same vector $v_{m,n}$ will be annihilated by all the $E'_{e_j \pm e_{j+1}}$ as well, so we have found a $\mathfrak{o}(p)$-type $(m, n)$ corresponding to $V_{m+\frac{p}{2}, n+\frac{p}{2}}$ . Note here the only ambiguity for the $O(p)$-represnetation is its central character, and only one of the central character can occur. So we are done in this case. (*How about odd p's?*)

Next the case $p = q = 2$. Since all the holomorphic discrete series of $G$ are accounted for in the case $p = 4,\ q = 0$, they will NOT occur in this situation!