A Simple Case of Siegel-Weil Formula

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0. Introduction

- The *theta correspondence*[1] is a correspondence between automorphic forms on two members of a *dual reductive pair*. Because of the relation between automorphic forms and automorphic representations, it is also a correspondence between automorphic representations. The *Siegel-Weil formula* says that certain natural linear combinations of theta lifts are Siegel-type holomorphic Eisenstein Series.

1. The Heisenberg Group and Stone-von Neumann Theorem

- Let k be a local field, not of characteristic 2, (W, B) a nondegenerate symplectic space over k of dimension $2n$, $W = V \oplus V'$ a *complete polarization* (meaning that $B|_V = B|_{V'} = 0$ and $\dim V = \dim V' = n$). The Heisenberg Lie algebra associated to W is $h(W) = W \oplus k$ as a set on which the Lie bracket is defined as

$$[(w,t),(w',t')] = (0, B(w,w'))$$

If we choose a symplectic basis for W then we have a matrix representation

$$h(W) = \left\{ h \in M_{n+2}(k) \;\middle|\; h = \begin{pmatrix} 0 & x_1 & \dots & x_n & t \\ & 0 & \dots & 0 & y_1 \\ & & \ddots & \vdots & \vdots \\ & & & 0 & y_n \\ & & & & 0 \end{pmatrix} \right\}$$

The Heisenberg group $H = H(W)$ is the Lie group of $h(W)$: we verify that for $(x, y, t) \in h(W)$,

[1] also called *Howe's duality correspondence*

$(x,y,t)^2 = (0,0,x\cdot y)$ and $(x,y,t)^k = 0$, for all $k \geq 3$. Thus we have that

$$e^{(x,y,t)} = I + (x,y,t) + (0,0,\frac{x\cdot y}{2})$$

Then $H \cong k^{2n+1}$ as a set and the multiplication in H is given by

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t' + \frac{x\cdot y' - x'\cdot y}{2})$$

H and $h(W)$ have the same automorphism groups G, any automorphism can be written as a product of a symplectic automorphism, an inner automorphism, a dilation and an inversion.

- Write L =expV'. The center of H is $Z = (0,0,t)$, and M =exp$(V' \oplus k) = L \oplus Z$ is a maximal abelian subgroup. For any character ψ of M, we have that $\psi|_L \equiv 1$. Thus ψ is really a character of Z and such characters are parameterized by $k^\times$. Let

$$\rho_\psi = \mathbf{Ind}_M^H\ \psi = \{|f| \in L^2(H/M) | f(mh) = \psi(m)f(h),\ \forall m \in M,\ h \in H\}$$

We have

Stone-von Neumann Theorem: ([47], [48])

The ρ_ψ's are irreducible unitary representations and all irreducible unitary representations of $H(W)$ arise in this way.

Remark The Stone-von Neumann theorem says that an irreducible representation of $H(W)$ is completely determined by its central character.

Remark There is also the smooth version for the non-archimedean case. But if one takes the smooth version of the theorem in faith, then it follows that every smooth representation of the Heisenberg group actually is (the smooth vectors of some ρ_ψ and hence) unitarizable.

- Note that $H/M \cong V$ so we can realize the representation ρ_ψ on the space $L^2(V)$. This realization is called the *Schrödinger Model.* More precisely:

$$\begin{aligned}
\rho_\psi(v)f(x) &= f(x+v) && \text{for all } x, v \in V \\
\rho_\psi(v')f(x) &= \psi(B(x,v'))f(x) && \text{for all } x \in V,\ v' \in V' \\
\rho_\psi(t)f(x) &= \psi(t)f(x) && \text{for all } x \in V,\ t \in k
\end{aligned}$$

Remark We often address the smooth vectors only and realize ρ_ψ on the Schwartz space $\mathcal{S}(V)$.

2. Weil Representations[2]

- Let $\rho_\psi^g(v,t) := \rho_\psi(g \cdot v, t)$, where $g \in Sp(W)$ is a symplectic automorphism of $H(W)$. Since $Sp(W)$ acts trivially on Z, the representation ρ_ψ^g also has central character ψ, thus by Schur's lemma we have a *projective representation* M_ψ of $Sp(W)$ so that

$$M_\psi(g)\rho_\psi^g(v,t)M_\psi(g)^{-1} = \rho_\psi(v,t)$$

This lifts uniquely to a linear representation ω_ψ of $Mp(W)$, a central extension of $Sp(W)$ by $\mathbf{Z}/2\mathbf{Z}$, but it is not trivial to see this (see [41]). We call this representation ω_ψ, or the projective representation, the *Weil representation.* On the Schrödinger Model the Weil representation is realized as follows:

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} f(x) = \lambda(\mathrm{det}A)|\mathrm{det}A|^{\frac{1}{2}} f(A^t x) \tag{1}$$

$$\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} f(x) = \psi(\frac{x^t B x}{2}) f(x) \tag{2}$$

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} f(x) = \gamma \hat{f}(x) \tag{3}$$

[2]Also called *Segal-Shale-Weil* representation or *oscillator* representation (see [43][44][49])

Note here although the representation of Heisenberg group depends on its central character ψ, the associated Weil representation only depends on the *square class* of this ψ.

- **Dual reductive pair**([22])

  Two subgroups G and G' of $Sp(W)$ form a *dual reductive pair* if

  1. G and G' are centralizers of each other inside $Sp(W)$.
  2. G and G' both act semisimply on W.

  We only need to look at the irreducible ones, i.e., dual reductive pairs which are not direct sums of smaller pairs. There are essentially three cases([34]):

  Type I

  1. W_1 a symplectic space with $G' = Sp(W_1)$ and V_1 an orthogonal space with $G = O(V_1)$. Then (G, G') is a dual reductive pair in $Sp(W)$ where $W = W_1 \otimes V_1$.
  2. G and G' are both certain unitary groups.

  Type II

  3. V_1 and V_2 are two finite dimensional vector spaces and $V = V_1 \otimes V_2$ and $W = V \oplus V'$ as before. Then $G = GL(V_1)$ and $G' = GL(V_2)$ form a dual reductive pair in $Sp(W)$.

  Let us note that $Mp(W)$ always splits over G and G', unless we are in case 1 and V_1 is odd dimensional (again non-trivial to see, see computation in Appendix). So we may always regard G and G' as subgroups of $Mp(W)$.

- For any irreducible representation π of G, if $\mathrm{Hom}_G(\omega_\psi, \pi) \neq 0$, we say that π occurs in ω_ψ. Howe's conjecture then says that there exists a unique representation π' of G' such that $\mathrm{Hom}_{G\times G'}(\omega_\psi, \pi \otimes \pi') \neq 0$. Moreover, if the π is automorphic, so is the π'.

  Remark: In the *stable range*[3], this correspondence respects unitarity.

- We mention the adelization of the above.

  Let k now be a global field and $\mathbb{A}_k$ its ring of adeles. Fix $\psi = \prod \psi_\nu$ a nontrivial character on $\mathbb{A}_k/k$. For each place ν we have $H_\nu = H(W, k_\nu)$ and $(\rho_{\psi_\nu}, \mathcal{S}_\nu)$ the (smooth) Schrödinger model

[3]It is roughly speaking that split rank of the bigger group is more than the size of the smaller group

of H_ν, that is, $\mathcal{S}_\nu$ is the space of Schwartz functions on V_ν. Let $\mathcal{S} = \mathcal{S}_{\mathbb{A}} = \bigotimes'_\nu \mathcal{S}_\nu$. We have then the Weil representation of $Sp(W_{\mathbb{A}})$ on $\mathcal{S}$. This lifts to a true representation of $Mp(W_{\mathbb{A}})$, a central extension of $Sp(W_{\mathbb{A}})$ by $\mathbf{Z}/2\mathbf{Z}$.

3. Θ-Correspondence

- Here k still is a global field.

Theorem *There is a unique H_k-invariant linear functional on the Schwartz space $\mathcal{S}$.* ([22])

This functional is denoted by Θ. On the Schrödinger model it is realized as

$$\Theta(\Phi) = \sum_{x \in V_k} \Phi(x) \tag{4}$$

Define the Θ-functions on $Mp_{\mathbb{A}}$ by $\Theta(\Phi, g) = \Theta(\rho_\psi(g)\Phi)$. From the H_k-invariance we also have the $Sp(W,k)$-invariance by definition of the Weil representation. Thus, this gives an *automorphic form* on $Mp(W_{\mathbb{A}})$. These are very *special* automorphic forms but when restricted to dual reductive pairs they give rise to surprisingly *general* automorphic forms.

- Let (G, G') be a dual reductive pair in $Sp(W)$. Regard $G(\mathbb{A})$ and $G(\mathbb{A}) \times G'(\mathbb{A})$ as subgroups of $Sp(W_{\mathbb{A}})$. For any cusp form φ on G define

$$\theta(\Phi, \varphi)(g') = \int_{G(k)\backslash G(\mathbb{A})} \Theta(\Phi, gg')\varphi(g)\ dg \tag{5}$$

Convergence is guaranteed by the behavior of φ at the cusps, and invariance is then formal. So we have built an automorphic form on G' by way of the θ-correspondence.

Remark: For our application, assume (V, Q) is a non-degenerate quadratic form, *anisotropic* over k, then reduction theory (see [11]) gives us a *compact* quotient $O(Q)_k\backslash O(Q)_{\mathbb{A}}$! In this case it is fairly easy to see that our theta kernel is of moderate growth, which together with rapid decay of the cusp form give the convergence. But in general it is not trivial.

4. Eisenstein Series

- Let $\phi(g)$ be a $\mathbb{C}$-valued function on $G(\mathbb{A}) = SL(2,\mathbb{A})$, right $\mathbf{K}$-finite, left $P(k)$ and $N(\mathbb{A})$-invariant, and $\phi(g)$ is almost everywhere locally in a principal series I_{χ_ν}, where the product of χ_ν is a *Hecke character*. We can build an (*inhomogeneous*) Eisenstein series out of such a function by "averaging" over $G(k)$:

$$E(g,\phi) = \sum_{\gamma \in P(k)\backslash G(k)} \phi(\gamma g) \tag{6}$$

It is, at least formally, left $G(k)$-invariant. The *constant term* (along N) is defined to be:

$$E_N(g,k) := \int_{N(k)\backslash N(\mathbb{A})} E(ng,\phi)\, dn \tag{7}$$

Use the Bruhat Decomposition $G(k) = P(k) \bigsqcup P(k)wN(k)$, we get

$$\begin{aligned} E_N(g,k) &= \int_{N(k)\backslash N(\mathbb{A})} \sum_{\gamma\in P(k)\backslash G(k)} \phi(\gamma ng)\, dn \\ &= \int_{N(k)\backslash N(\mathbb{A})} \phi(ng) dn + \int_{N(k)\backslash N(\mathbb{A})} \sum_{\gamma \in N(k)} \phi(w\gamma ng)\, dn \\ &= \phi(g) + \int_{N(\mathbb{A})} \phi(wng)\, dn \end{aligned}$$

Call this last integral $T_w\, \phi(g)$.

5. Proof of the Formula

- Now we take $H = G_1 = O(Q)$, $G = G_2 = SL(2) = Sp(1)$, where (V,Q) is an even-dimensional nondegenerate *anisotropic* quadratic space. Let us examine

$$\theta(\Phi, \mathbf{1})(g) = \int_{H(k)\backslash H(\mathbb{A})} \Theta(\Phi, hg)\ dh \tag{8}$$

Our objective is to prove that $\theta(\Phi, \mathbf{1})(g)$ is $E(g, \phi)$ (for certain choice of Φ and ϕ). First compute the constant term of $\theta(\Phi, \mathbf{1})(g)$:

$$\begin{aligned}
\theta_{\Phi,P}(\mathbf{1})(g) &= \int_{N(k)\backslash N(\mathbb{A})} \theta_\Phi(\mathbf{1})(ng)\ dn \\
&= \int_{N(k)\backslash N(\mathbb{A})} \int_{H_k\backslash H_{\mathbb{A}}} \mathbf{1} \sum_{x\in V} \rho(ng, h)\Phi(x) dh dn \\
&= \int_{H_k\backslash H_{\mathbb{A}}} \sum_{x\in V} \int_{N(k)\backslash N(\mathbb{A})} \rho(n_b g, h)\Phi(x) dn_b dh \\
&= \int_{H_k\backslash H_{\mathbb{A}}} \sum_{x\in V} \int_{N(k)\backslash N(\mathbb{A})} \psi(\frac{b}{2}\langle x, x\rangle)\rho(g, h)\Phi(x) dn_b dh \\
&= \int_{H_k\backslash H_{\mathbb{A}}} \sum_{\langle x,x\rangle=0} \rho(g, h)\Phi(x) dh \\
&= \int_{H_k\backslash H_{\mathbb{A}}} (\rho(h)\rho(g)\Phi)(0) dh \\
&= \int_{H_k\backslash H_{\mathbb{A}}} (\rho(g)\Phi)(0\cdot h) dh = \operatorname{mes}(H_k\backslash H_{\mathbb{A}})\ (\rho(g)\Phi)(0)
\end{aligned}$$

Here we used the assumption that Q is *anisotropic*. It can be easily checked that

$$\phi(g) = \rho(g)\Phi(0)$$

satisfies

$$\phi(pg) = \chi(p)\phi(g)$$

where

$$p = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}, \quad \chi(p) = \lambda(a)|a|^m$$

Here $2m$ is the dimension of the quadratic space, and λ as in Appendix. Now since $\phi(g)$ is in the principal series, use the computation in section 4, we need only show that

$$T_w \ \phi(g) = \int_{N(\mathbb{A})} (\rho(wng)\Phi)(0)dn = 0$$

Formally this is a product of the local factors

$$\int_{N(k_\nu)} \rho(wng)\Phi_\nu(0) \ dn \tag{9}$$

If we can prove that at least one of these local factors is zero for *some* choice of Φ, then the global integral would be zero. We will check that the above integral vanish at real primes, if we assume that Q is anisotropic at this real prime and use the Gaussian $\Phi_\nu(x) = \varphi(x) = e^{-Q(x)/2}$.

$$\begin{aligned}
& \int_{N(\mathbb{R})} \rho(wng)\varphi(0) \ dn \\
= \ & \int_{N(\mathbb{R})} \rho(wn)\varphi_g(0) \ dn \\
= \ & \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{2m}} \psi\left(\frac{bQ(y)}{2}\right) \varphi_g(y)\psi(-Q(x,y)) \ dy \right)\Big|_{x=0} \ db \\
= \ & \int_{\mathbb{R}} \int_{\mathbb{R}^{2m}} \psi\left(\frac{bQ(y)}{2}\right) \varphi_g(y) \ dy \ db
\end{aligned}$$

After a change of variable in y, we can assume this anisotropic quadratic form is $Q(x) = \sum_{j=1}^{n} x_j^2 = r^2$. We have computed in the next section that the $SL(2,\mathbb{R})$-module generated by $\varphi(x) = e^{-r^2/2}$ is a holomorphic discrete series representation V_m (there denoted $V_{\frac{n}{2}+k}$, where n is the dimension of quadratic space and take $k = 0$). So we need only prove that the above integral is zero for the functions $r^{2l}e^{-r^2/2}$, $l = 0, 1, 2, \cdots$, since these span a dense subspace of V_m.

Below we prove the case $l = 0$ and the general case follows from induction and integration by parts.

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^{2m}} \psi\left(\frac{br^2}{2}\right) e^{-\frac{r^2}{2}}\, dy\, db \\ = & \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi\left(\frac{by^2}{2}\right) e^{-\frac{y^2}{2}}\, dy \right)^{2m} db \\ = & \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{(\frac{bi-1}{2})y^2}\, dy \right)^{2m} db \\ = & \int_{\mathbb{R}} \left(\frac{2\pi}{bi-1} \right)^{m} db \end{aligned}$$

which is zero as soon as $m > 1$. Thus the constant term of theta lift agrees with the constant term of the Eisenstein series. So their difference, if non-zero, would be a (*holomorphic*) cusp form. This cusp form would generate a unitarizable representation a.e. locally([8]). Since the $\theta(\Phi, \mathbf{1})$ and the $E(g, \phi)$ both generate an irreducible principal series I_m a.e. locally, this cusp form also does. But the principal series I_m is *never* unitarizable if $m \geq 2$ ([14]), so this contradiction proves the desired formula.

6. Further Calculations: Real Case

- The θ-correspondence for small real orthogonal groups (we follow [25] very closely)
- $H = O(1) = \{\pm 1\}$. This group has two representations, namely the trivial representation and sign representation. So the Weil representation splits into two pieces consisting of even/odd functions. For the (class of) character $\tau = \exp \pi i x$ we have two pieces of lowest weight representations, with lowest weight $\frac{1}{2}$ and $\frac{3}{2}$ respectively. For the other class we have two pieces of highest weight representations. Note here the dimension of orthogonal space is odd so we do need to deal with $\widetilde{G}$.
- $H = O(n)$. This is the theory of spherical harmonics. The action of $SL(2, \mathbb{R})$ on $S(\mathbb{R}^n)$ is the n-fold tensor product of its action ω on $S(\mathbb{R})$. The θ-correspondence for H^n_m (the harmonic polynomials of n variable with degree m and $O(n)$ acting naturally), is a lowest weight module

of lowest weight $\frac{n}{2}+m$. This is most directly proved by computing Jacquet modules in the space of tempered distributions. Below is a sketch of proof, using a slightly different approach:

- Notice that we have established the result $\mathbf{P}[x] = V_{\frac{1}{2}} \oplus V_{\frac{3}{2}}$, where V_λ is the lowest weight module of $\mathfrak{sl}(2,\mathbb{R})$ with lowest weight λ. Using the technique in ([42]) we see that

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{k=0}^{\infty} V_{\lambda+\mu+2k} \qquad \text{if } \lambda+\mu \neq 0,-1,-2,\cdots$$

As an immediate corollary we have

$$\mathbf{P}[x_1,\cdots,x_n] \cong (\mathbf{P}[x])^{\otimes n} = (V_{\frac{1}{2}} \oplus V_{\frac{3}{2}})^{\otimes n} = \bigoplus_{k=0}^{\infty} \beta(k,n) V_{\frac{n}{2}+k}$$

Where $\beta(k,n)$ is an integer indicating the multiplicity of $V_{\frac{n}{2}+k}$. Let H_k^n be the space of f-annihilated $\hbar$-eigenvectors with eigenvalue $\frac{n}{2}+k$, then H_k^n consists of harmonic polynomial of degree k. Since $O(n)$ and $\mathfrak{sl}(2,\mathbb{R})$ have a commuting action, the $V_{\frac{n}{2}+2k}$ - isotypic component is also invariant under $O(n)$, so H_k^n is an $O(n)$-module:

$$\mathbf{P}[x_1,\cdots,x_n] = \bigoplus_{k=0}^{\infty} H_k^n \otimes V_{\frac{n}{2}+k}$$

We see that this is in fact a decomposition into $O(n)\times\mathfrak{sl}(2,\mathbb{R})$-modules. Change basis to

$$\hbar = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad n^+ = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix}, \quad n^- = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & -\frac{1}{2} \end{pmatrix}$$

It is elementary to check that n^- acts via $e^{-\frac{r^2}{2}}\Delta e^{\frac{r^2}{2}}$ on $\mathcal{S}(\mathbb{R}^n)$, so we have an isomorphism of $O(n)\times\mathfrak{sl}(2,\mathbb{R})$-modules:

$$\mathbf{P}[x_1,\cdots,x_n] = \bigoplus_{k=0}^{\infty} H_k^n \otimes V_{\frac{n}{2}+k} \longrightarrow \bigoplus_{k=0}^{\infty} \tilde{H}_k^n \otimes V_{\frac{n}{2}+k} = e^{-\frac{r^2}{2}}\cdot\mathbf{P} \subseteq \mathcal{S}(\mathbb{R}^n)$$

Where $\tilde{H}_k^n = e^{-\frac{r^2}{2}}\cdot H_k^n$ is the space of Hermite functions of degree k. The above map has a dense image in $\mathcal{S}(\mathbb{R}^n)$, so $\mathcal{S}(\mathbb{R}^n) = \overline{\bigoplus_{k=0}^{\infty} \tilde{H}_k^n \otimes V_{\frac{n}{2}+k}}$. This says exactly that corresponding to the representation $\tilde{H}^n_k$ of $O(n)$ is a holomorphic discrete series representation $V_{\frac{n}{2}+k}$ of $SL(2,\mathbb{R})$, when n is even.

- $H = O(p,q)$. We are looking at $\omega^{p,q} = (\otimes^p \omega) \otimes (\otimes^q \omega)^*$. The Lie algebra representation of $\mathfrak{sl}(2,\mathbb{R})$ on the Schrödinger model is:

$$\begin{aligned}
\omega^{p,q}(\hbar) &= \sum_{j=1}^{p} x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j} + \frac{p+q}{2} \\
\omega^{p,q}(e^+) &= \frac{i}{2}(\sum_{j=1}^{p} x_j^2 + \sum_{j=1}^{q} y_j^2) = \frac{i}{2}(r_p^2 - r_q^2) \\
\omega^{p,q}(e^-) &= \frac{i}{2}(\sum_{j=1}^{p} \frac{\partial}{\partial^2 x_j} + \sum_{j=1}^{q} \frac{\partial}{\partial^2 y_j}) = \frac{i}{2}(\triangle_p - \triangle_q)
\end{aligned}$$

Note here $O(p,q)$-invariant distributions supported on the light cone $\{r_p^2 - r_q^2 = 0\}$ amounts to identifying the image of $\mathbf{1}$ under θ-correspondence. For more details, see [10] and [13].

  - $H = O(1,1)$ The subgroup of index two $SO(1,1)$ is abelian and isomorphic to $\mathbb{R}^\times$. So all representations of $SO(1,1)$ are characters $|a|^s$ and $|a|^s \text{sgn}\, a$. And unless $s = 0$, every representation of $SO(1,1)$ is of class 2 and hence induces to an irreducible 2-dimensional representation of $O(1,1)$; Corresponding to the family $|a|^s$ is a continuous family of representations of $SL(2,\mathbb{R})$, so the parameters already suggest that they are the unitarizable principal series. Since the $G \times H$-module $\chi_s \otimes I_s$ is irreducible, each I_s, if it appears, is realized twice. Continuity of parameter also suggest that they appear as *quotients* of the Schrödinger model. Suppose that $C = I_s$ is a quotient of $B = \mathcal{S}(\mathbb{R}^2)$ by A,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

  Dualizing we get

$$0 \longrightarrow C^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$$

  Noticing $C^* = I_{-s}$, the e^+-annihilated vectors are distributions supported in

$$\{(x,y) | x^2 - y^2 = 0\}$$

After a change of basis this is $\{(x,y)|xy=0\}$. Such distributions are given by(see [19] [18] for "regularization")

$$u_{s,x,l} = \int_{\mathbb{R}^\times} |x|^s \frac{\partial^l}{\partial\, y^l} f(x,0)\; d^\times x \;\text{ and }\; u_{-s,y,l} = \int_{\mathbb{R}^\times} |y|^{-s} \frac{\partial^l}{\partial\, x^l} f(0,y)\; d^\times y$$

The condition of e^+-annihilation is more than the support condition here: only $u_{s,x,0}$ and $u_{-s,y,0}$ give lowest weight vectors. An element in $\mathfrak{so}(1,1)$ acts by $\ddot{s} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, it is easily checked that

$$\ddot{s}\; u_{s,x,l} = s \cdot u_{s,x,l} \;\text{ and }\; \ddot{s}\; u_{-s,y,l} = s \cdot u_{-s,y,l}$$

For $\mathfrak{sl}(2)$, we have

$$\begin{aligned} \hbar &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1 \\ e &= ixy \\ f &= i\frac{\partial^2}{\partial x \partial y} \end{aligned}$$

It is easily checked that

$$\begin{aligned} \hbar\; u_{s,x,l} &= (s+1)u_{s,x,l} \\ e\; u_{s,x,l} &= 0 \\ f\; u_{s,x,l} &= u_{s-1,x,l+1} \cdot \operatorname{sgn}\, x\;; \\ \hbar\; u_{-s,y,l} &= (-s+1)u_{-s,y,l} \\ e\; u_{-s,y,l} &= 0 \\ f\; u_{-s,y,l} &= u_{-s-1,y,l+1} \cdot \operatorname{sgn}\, y\;. \end{aligned}$$

The Casimir operator is $\Omega = \frac{1}{2}\hbar^2 + ef + fe = \frac{1}{2}\hbar^2 - \hbar - 2ef$, so from the above equations we have

$$
\begin{aligned}
\Omega\ u_{s,x,l} &= (\frac{1}{2}\hbar^2 - \hbar)u_{s,x,l} = (\frac{1}{2}(s+1)^2 - (s+1))u_{s,x,l} = \frac{1}{2}(1+s)(1-s)u_{s,x,l} \\
\Omega\ u_{-s,y,l} &= (\frac{1}{2}\hbar^2 + \hbar)u_{-s,y,l} = (\frac{1}{2}(-s+1)^2 - (-s+1))u_{-s,y,l} = \frac{1}{2}(1-s)(1+s)u_{-s,y,l}
\end{aligned}
$$

These indeed are the eigenvalues of Casimir operator we expected from the *normalized* principal series $I_{\pm s}$. As a byproduct the continuous family of unitary representations of $O(1,1)$ corresponds to the unitary principle series of $SL(2,\mathbb{R})$.

– $H = O(2,1)$ This group and G have the same Lie algebra. Again the dimension is odd and we need to deal with $\widetilde{G}$. The resulting correspondence is calculated in [16]: discrete series of H with parameter $k-1$ corresponds to discrete series of $\widetilde{G}$ with parameter $\frac{k}{2}$; principal series I_s corresponds to principal series $\tilde{I}_{\frac{s}{2}}$ of $\widetilde{G}$. This is related to *part of* the Shimura correspondence.

Remark The case for $(GL(n), GL(m))$ over $\mathbb{R}$ is less exciting, since the correspondence is essentially the identity map *twisted* by a character ([1]).

7. Some ideas for further work

- Continuing along this line, we hope to reconsider Oda's work ([36]) on dual reductive pairs $O(p,2) \times Sp_n(\mathbb{R})$. We note that at archimedean places the orthogonal groups of signature $(p,2)$ are the only orthogonal groups with holomorphic discrete series representations. Orthogonal groups of signature $(p,4)$ have *quaternion* discrete series (in the sense of Gross and Wallach [20]), and $O(p,q)$ with pq even have discrete series. Theta correspondences with the archimedean representations on the orthogonal groups lying in some discrete series are perhaps the most interesting, because we have more reasons to expect some arithmetic content in the corresponding automorphic forms on symplectic groups.

Appendix. Compatibility Check

- This computation amounts to check explicitly that the 2-cocycle from the Weil representation *splits* when restricted to certain pairs $O(Q) \times SL_2(\mathbb{R})$. We use Lemma 4.1.2 in [5]:

  Lemma: *$SL(2)$ is the universal group generated by the elements $t(y)$, $n(z)$, ω where $y \in k^*, z \in k$ with the generating relations:*

$$\begin{aligned} t(a)t(b) &= t(ab), & n(a)n(b) &= n(a+b), & t(a)n(z)t(a^{-1}) &= n(a^2z), \\ wt(a)w &= t(-a^{-1}), & wn(b)w &= t(b^{-1})n(-b)wn(-b^{-1}) \end{aligned}$$

Proof. This is the Bruhat decomposition for $SL(2)$. See [5]. □

Remark Here we used different notation from Bump's: $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now let V be a vector space over k of *even* dimension $2n$ and Q a nondegenerate quadratic form on V. Define an inner product on V by

$$\langle u, v \rangle = \frac{1}{4}(Q(u+v) - Q(u-v))$$

so that $Q(v) = \langle v, v \rangle$. The action of $SL(2)$ on $\mathcal{S}(V)$ is as follows:

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(v) &= \lambda(a)|a|^n f(av); \\ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(v) &= \psi(\frac{b}{2}Q(v))f(v); \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(v) &= \gamma\tilde{f}(v) := \gamma \int_V f(u)\psi(-\langle v, u \rangle)\, du \end{aligned}$$

Let's check that these do give a real representation of $SL(2, k)$, i.e. we have to check that the equations in the above Lemma are satisfied. The first three equations are trivial. For the fourth one:

$$\begin{aligned}
\text{LHS} &= \gamma \int_V \int_V \lambda(a)|a|^n f(u)\psi(-\langle aw, u\rangle)\psi(-\langle v, w\rangle)\ du\ dw \\
&= \gamma \int_V \int_V \lambda(a)|a|^n f(u)\psi(-\langle aw, u\rangle)\psi(-\langle v, w\rangle)\ dw\ du \\
&= \gamma \int_V \lambda(a)|a|^n f(u) \int_V \psi(-\langle w, au+v\rangle)\ dw\ du \\
&= \gamma\lambda(a)|a|^n f(-a^{-1}v) = \gamma\lambda(a^2)|a|^{2n} \begin{pmatrix} -a^{-1} & 0 \\ 0 & -a \end{pmatrix} f(v) = \gamma\lambda(a^2)|a|^{2n}\text{RHS}
\end{aligned}$$

The last equation:

$$\begin{aligned}
\text{LHS} &= \gamma \int_V \int_V f(u)\psi(-\langle w, u\rangle)\psi(\frac{b}{2}\langle w, w\rangle)\psi(-\langle v, w\rangle)\ du\ dw \\
&= \gamma \int_V \int_V f(u)\psi(\frac{b}{2}\langle w, w\rangle - \langle w, u\rangle - \langle v, w\rangle)\ du\ dw \\
&= \gamma \int_V \int_V \psi(\frac{b}{2}\langle w - \frac{1}{b}(u+v), w - \frac{1}{b}(u+v)\rangle)\ dw f(u)\psi(-\frac{1}{2b}\langle (u+v), (u+v)\rangle)\ du \\
&= \gamma \int_V \psi(\frac{b}{2}\langle w, w\rangle)\ dw \int_V f(u)\psi(-\frac{1}{2b}\langle (u+v), (u+v)\rangle)\ du
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \gamma t(b^{-1})\psi(\frac{-b}{2}\langle v, v\rangle) \int_V f(u)\psi(\frac{-1}{2b}\langle u, u\rangle)\psi(-\langle v, u\rangle)\ du \\
&= \gamma\lambda(b^{-1})|b|^n\psi(\frac{-b}{2}\langle -b^{-1}v, -b^{-1}v\rangle) \int_V f(u)\psi(\frac{-1}{2b}\langle u, u\rangle)\psi(-\langle b^{-1}v, u\rangle)\ du \\
&= \gamma\lambda(b^{-1})|b|^{-n} \int_V f(u)\psi(-\frac{1}{2b}\langle v, v\rangle - b^{-1}\langle v, u\rangle - \frac{1}{2b}\langle u, u\rangle)\ du \\
&= \gamma\lambda(b^{-1})|b|^{-n} \int_V f(u)\psi(-\frac{1}{2b}\langle u+v, u+v\rangle)\ du
\end{aligned}$$

Now it is clear that we should choose

$$\gamma = \left(\int_V \psi(\frac{1}{2}\langle w, w\rangle)\ dw\right)^{-1}$$

the $\lambda(a)$ is a character on $k^\times/(k^\times)^2$, its value being

$$\lambda(a) = \frac{\int_V \psi(\frac{1}{2}\langle w, w\rangle)\ dw}{|a|^n \int_V \psi(\frac{a}{2}\langle w, w\rangle)\ dw}$$

This factor arises naturally and depends only on the square class of a.

- Now if we want to examine Schrödinger model of the Weil representation of $Sp(n)$, we will have that in (1) – (3):

$$\begin{aligned} \gamma &= \frac{1}{\int_{k^n} \psi(Q(x))\ dx} \\ \lambda(\det B) &= \frac{\int_{k^n} \psi(Q(x))\ dx}{|\det B|^{\frac{1}{2}} \int_{k^n} \psi(\langle x, Bx\rangle)\ dx} \end{aligned}$$

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