Some Homework Problems Due January 20

Here are some of your proofs from the first homework assignment. Study them to see if you consider the argument valid, and if it does not feel right see if you can identify what seems to be wrong or missing. Also, please study my comments on your papers. You will sometimes have to think about them awhile to see what I am saying.

It is important that you "get" proofs as soon as possible, and I believe that can happen. Getting them amounts pretty much to recognizing that a mathematical object is what the definition says it is and nothing more. Proofs are therefore very specific, addressing the exact meanings of terms. Please work hard to develop the discipline of asking yourself *exactly* what a mathematical statement says. Casual reading can be deadly in this and other mathematics courses. You will find that, with any but the simplest statements, there is a period of absorbing followed by a definite click of recognition. The same will happen as you write a proof: After some unsatisfactory starts, you will know when you have hit the nail on the head. We can all improve our arguments, but the important thing right now is to develop habits of being unsatisfied with vagueness and resolving it by focusing on definitions.

In reading a solution here, imagine a skeptic's view: He understands the setup well and knows what all the terms mean but nevertheless doubts the assertion.

<u>Problem 1.1.14 (first part)</u>. Let $f : A \to B$. If E and F are subsets of A, prove that $f(E \cup F) = f(E) \cup f(F)$.

Solution 1: ($\subseteq$) Let $x \in E \cup F$. Then $x \in E$ or $x \in F$, and by applying f one sees that if $f(x) \in f(E \cup F)$ then $f(x) \in f(E)$ or $f(x) \in f(F)$. This argument shows that $f(E \cup F) \subseteq f(E) \cup f(F)$.

($\supseteq$) Suppose that $x \in E$ or $x \in F$. Then $x \in E \cup F$. By applying f, we conclude that $f(x) \in f(E)$ or $f(x) \in f(F)$. This argument shows that $f(E) \cup f(F) \subseteq f(E \cup F)$.

Solution 2: If $x \in E$, then $x \in F$, and hence $f(x) \in f(E)$. This is true for all $x \in E$, and therefore $f(E) \subseteq f(F)$. Because E and F are both subsets of $E \cup F$, this argument shows that $f(E) \cup f(F) \subseteq f(E \cup F)$.

Conversely, if $y \in f(E \cup F)$, then there is an element $x \in E \cup F$ such that $y = f(x)$. Because $x \in E$ or $x \in F$, it follows that $y \in f(E)$ or $y \in f(F)$. We conclude that $f(E \cup F) \subseteq f(E) \cup f(F)$.

Solution 3: ($\subseteq$) Assume that $x \in f(E \cup F)$. Then $x \in f(E) \cup f(F)$. This argument shows that $f(E \cup F) \subseteq f(E) \cup f(F)$.

($\supseteq$) Assume that $x \in f(E) \cup f(F)$. Then $x \in f(E \cup F)$. This argument shows that $f(E) \cup f(F) \subseteq f(E \cup F)$.

Solution 4: ($\subseteq$) The set $f(E \cup F)$ is given by

$$\{f(x) : x \in E \cup F\} = \{f(x) : x \in E \text{ or } x \in F\}.$$

Therefore $f(E \cup F) \subseteq f(E) \cup f(F)$.

($\supseteq$) If $x \in E$ or $x \in F$ then $x \in E \cup F$. Thus, $f(E) \cup f(F)$ is given to be

$$\{f(x) : x \in E \text{ or } x \in F\},$$

which is the set $\{f(x) : x \in E \cup F\} = f(E \cup F)$. This shows that $f(E) \cup f(F) \subseteq f(E \cup F)$.

Solution 5: Let $f(x) \in f(E \cup F)$. Then $x \in E \cup F$, so $x \in E$ or $x \in F$. It follows that $f(x) \in f(E)$ or $f(x) \in f(F)$, so that $f(x) \in f(E) \cup f(F)$. This argument shows that $f(E \cup F) \subseteq f(E) \cup f(F)$. Conversely, let $f(y) \in f(E) \cup f(F)$. Then $f(y) \in f(E)$ or $f(y) \in f(F)$, implying that $y \in E$ or $y \in F$. Therefore $y \in E \cup F$, and so $f(y) \in f(E \cup F)$. This argument shows that $f(E) \cup f(F) \subseteq f(E \cup F)$.

Problem 1.1.19(b) (first part). If $f : A \to B$ is surjective and $H \subseteq B$, prove that $f(f^{-1}(H)) = H$.

Solution 1: ($\supseteq$) Let $y \in H$. Then $y \in B$, because $H \subseteq B$ and the function is surjective. Let $G = f^{-1}(H)$. Applying f shows that $f(G) = f(f^{-1}(H))$, so that $y \in f(G) = f(f^{-1}(H))$.

($\subseteq$) Let $y \in f(f^{-1}(H))$. Unwinding the definitions, this means that y is an element of $\{f(x) : x \in A \text{ and } f(x) \in H\}$. Because f is surjective, $y = f(x)$, and so y is an element of $\{f(x) : x \in A \text{ and } y \in H\}$. Therefore $y \in H$.

Solution 2: The set $f^{-1}(H)$ consists of all the inputs to f that yield an output in H. Because $f(f^{-1}(H))$ is the direct image of that set under f, all of its elements must be in H. Conversely, because f is surjective, for each $y \in H$ there exists $x \in f^{-1}(H)$ such that $f(x) = y$. It is therefore clear the set $f(f^{-1}(H))$ will contain every element of H.

Solution 3: Let $b \in H$. Because f is surjective, for all $b \in H$ there exists $x \in A$ such that $f(x) = b$. It follows that $x = f^{-1}(b)$ and $f(x) = f(f^{-1}(b))$. Because $f(x) = b$ for all $b \in H$, one concludes that $b = f(f^{-1}(b))$ and that $H = f(f^{-1}(b))$.

Solution 4: ($\subseteq$) Consider an element $y \in f(f^{-1}(H))$. Then $y \in \{f(x) : x \in f^{-1}(H)\}$, where $f^{-1}(H)$ is the set of all $x \in A$ such that $f(x) \in H$. That is given to exist since f is surjective.

($\supseteq$) Consider an element $y \in H$. Then there is some $f(K)$ that maps the set $K \subseteq A$ in to H such that y is also in $f(K)$. By definition, K is the set $f^{-1}(H)$. Therefore if $y \in H$ then $y \in f(K) = f(f^{-1}(H))$.

Solution 5: The set $f^{-1}(H)$ is defined as $\{x \in A : f(x) \in H\}$. If this set is mapped by f then its image is H. Therefore $f(f^{-1}(H)) = H$.

Problem 1.1.22(b). If $f : A \to B$ and $g : B \to C$ and $g \circ f$ is surjective, prove that g is surjective.

Solution 1: Suppose that $g \circ f$ is surjective. Then for all $c \in C$ there exists $a \in A$ such that $g(f(a)) = c$. The point $b = f(a)$ is in B, and $g(b) = c$. Therefore g is surjective.

SOLUTION 2: Choose an element $x \in C$. Because g is surjective, there is an element $z \in B$ such that $g(z) = x$. Because f is surjective, there is an element $y \in A$ such that $f(y) = z$. Then $(g \circ f)(y) = x$. Hence y is the element we are looking for.

SOLUTION 3: The assertion "If $g \circ f$ is surjective then g is surjective" is the same as saying "If g is not surjective then $g \circ f$ is not surjective." Assume, then, that g is not surjective. Then there exists $c \in C$ such that none of the values $g(b)$ for $b \in B$ equals c. If $a \in A$, then since $f(a) \in B$ it follows that $g(f(a)) \neq c$. Thus $g \circ f$ never attains the value c and so is not surjective.

SOLUTION 4: Because $g \circ f$ is surjective, for all $z \in C$ there exist $y \in B$ and $x \in A$ such that $f(x) = y$ and $g(y) = z$. Therefore z is in the image of g.

SOLUTION 5: Because $g \circ f$ is surjective, its range is all of C. Because $g \circ f$ is a composite, its range must be a subset of the range of g. Conversely, the range of g is a superset of the range of $g \circ f$. It follows that the range of g must be all of C.