MATH 3326 EXAM 1
FEBRUARY 10, 2012

Instructions. Show an appropriate amount of work. If you are uncertain about how much detail to include or what you are allowed to assume, please ask me. There are 25 points altogether.

(3 pts) 1. If $a < b < c < d$, prove that $[a, c] \cap [b, d] = [b, c]$.

(3 pts) 2. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^2$.

(a) Is f injective, surjective, both, or neither? No justification necessary.

(b) Determine the direct image $f\big([-1, 4]\big)$. No justification necessary.

(c) Determine the inverse image $f^{-1}\big([-1, 4]\big)$. No justification necessary.

(2 pts) 3. Bernoulli's inequality states that if $x \geq 0$ then $(1+x)^n \geq 1+nx$ for all $n \in \mathbf{N}$. Prove that using mathematical induction.

(3 pts) 4. In this problem, you may assume that $0 \cdot a = 0$ and $a \cdot 0 = 0$ for all $a \in \mathbf{R}$. If you choose part (b), assume all *algebraic* facts about $\mathbf{R}$ and the following:

1. Exactly one of $a < b$, $a = b$, or $a > b$ holds.
2. If $a < b$, then $a + c < b + c$.
3. If $a < b$ and $c > 0$, then $ac < bc$; if $a < b$ and $c < 0$, then $ac > bc$.

Prove either of the following (your choice):

(a) If $x, y \in \mathbf{R}$ and $xy = 0$, then $x = 0$ or $y = 0$.

(b) If $x, y \in \mathbf{R}$ and $xy < 0$, then $x < 0$ or $y < 0$.

(2 pts) 5. Solve the equation $|x - 5| = 2x$.

(2 pts) 6. Give examples of the following. No justification necessary:

(a) A nonempty set $S \subseteq \mathbf{R}$ that is bounded above but not bounded below

(b) A bounded, nonempty set $T \subseteq \mathbf{R}$ that includes its infimum but not its supremum

(3 pts) 7. Answer the following if $S = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$. No justification necessary:

(a) Give two different upper bounds for S.

(b) Give two different lower bounds for S.

(c) Determine $\inf S$ and $\sup S$.

(3 pts) 8. Suppose that S is a bounded, nonempty set of real numbers and that $\sup S = u$. If $T = \{2x : x \in S\}$, prove that $\sup T = 2u$.

(4 pts) 9. True or false. No explanation necessary.

____ (a) The set $\mathbf{Q}$ is countable.

____ (b) The set $\mathbf{R}$ is countable.

____ (c) If $I_1 \supseteq I_2 \supseteq \cdots$ is a nested sequence of closed, bounded intervals $[a_n, b_n]$, then the intersection $\cap_{n=1}^{\infty} I_n$ is either an interval $[a, b]$ or a single point.

____ (c) If $0 < x < y$, then $\dfrac{2xy}{x+y} < \dfrac{x+y}{2}$.