Anosov Automorphisms of Nilpotent Lie Algebras

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Outline

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2. Some reductions of the problem
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    - Group action
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Chaotic Dynamical Systems

What is “chaos?”

Roughly speaking, a dynamical system is chaotic if it has sensitivity to initial conditions and a complex orbit structure. There is a nonlinear action-reaction relationship.

What makes “chaos?”

- Stretching
- Compressing
- Wrapping

Examples of “chaotic” maps:

- The baker’s map $f : [0,1]^2 \to [0,1]^2$
- Logistic maps $f(x) = ax(1-x)$ on $\mathbb{R}$
- Geodesic flow for compact manifolds with negative curvature

A family of chaotic maps

Definition

A C^1 diffeomorphism f of a compact Riemannian manifold X is called Anosov if there exist constants λ in $(0,1)$ and $c > 0$, and a df-invariant splitting $TX = E^s \oplus E^u$ of the tangent bundle of X so that for all $n \geq 0$,

- $\|df_x^n v^s\| \leq c\lambda^n \|v^s\|$ for all v^s in $E^s(x)$ and
- $\|df_x^{-n} v^u\| \leq c\lambda^n \|v^u\|$ for all v^u in $E^u(x)$.

It's not geometry

The Anosov property is independent of the metric on M; that is, if a diffeomorphism of a compact manifold M is Anosov with respect to one metric on M, it is Anosov with respect to any metric on M. For this reason, a metric on M is not always specified when discussing Anosov diffeomorphisms on a manifold M.

Also: Expanding maps and partially hyperbolic maps.

The simplest example: the cat map

One of the simplest examples of an Anosov diffeomorphism is Arnold's cat map:

Example

The matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ determines a volume-preserving linear transformation of $\mathbb{R}^2$. The linear transformation preserves the integer lattice $\mathbb{Z}^2$, so the map descends to a diffeomorphism f_A of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The eigenvalues of A are $\frac{1}{2}(3+\sqrt{5}) > 1 > \frac{1}{2}(3-\sqrt{5})$ so the map is Anosov.

(See the illustration.)

Generalizations of the cat map

Example

A matrix A in $GL_n(\mathbb{Z})$ with no eigenvalues of modulus one defines an Anosov map f_A of of the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$.

Example

Let $h : T^2 \to T^2$ be a diffeomorphism of the torus, and let $f : T^2 \to T^2$ be the cat map. Then $g = h \circ f \circ h^{-1}$ is an Anosov map of the torus.

The maps f and g are conjugate. This is our notion of dynamical equivalence.

Even more generally

Example

Let N be a simply connected nilpotent Lie group and let Γ be a torsion-free lattice in N. Let $A : N \to N$ be a hyperbolic automorphism of N that fixes Γ. Then A descends to an Anosov diffeomorphism f_A of the compact homogeneous space N/Γ.

The space N/Γ is called a nilmanifold and f_A is called an Anosov automorphism.

It is also possible that $f_A : N/\Gamma \to N/\Gamma$ finitely covers an Anosov map of a compact quotient of N/Γ (an infra-nilmanifold).

Known examples of Anosov automorphisms

The first example of an Anosov automorphisms of a nontoral nilmanifold was described by Smale in 1967, attributed to Borel. Other examples were presented by Shub (1969), Dani (1978), Dekimpe and Malfait (2000), Dekimpe and Deschamps (2003), Lauret (2003), Mainkar and Dani (2004), and Mainkar and Will (2006).

All known Anosov maps come from automorphisms

Up to dynamical equivalence, all the known examples of Anosov diffeomorphisms arise from automorphisms of nilmanifolds.

To understand Anosov maps in general, one can take a property of Anosov automorphisms and attempt to show that it holds for general Anosov maps.

Are these *all* Anosov automorphisms?

It is conjectured that every Anosov diffeomorphism is topologically conjugate to a quotient of Anosov automorphism of a nilmanifold.

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Theorem (Gromov, 1981)

Any expanding map of a compact manifold is topologically conjugate to an expanding endomorphism of an infra-nilmanifold.

Progress on the conjecture

Anosov maps are conjugate to Anosov automorphisms when

- the underlying manifold is a nilmanifold (Manning, 1974)
- when the distributions E^s and E^u are differentiable and an affine connection is preserved (Benoist-Labourie, 1993)
- when $\dim E^s = 1$ (Franks, 1970).

Anosov automorphisms

Some of what is known about Anosov automorphisms:

- Dani and Mainkar gave necessary and sufficient conditions for compact quotients of two-step nilmanifolds defined by graphs to admit an Anosov automorphism (2004).
- Nilmanifolds admitting Anosov automorphisms were classified in dimension ≤ 6 by Ito (1989) and Malfait (2000), and in dimension ≤ 8 by Lauret and Will (2005).

Goal

The ingredients for an Anosov automorphism

We want to find Anosov automorphisms of nilmanifolds. We need:

- a simply connected nilpotent Lie group N
- a torsion-free lattice Γ in N
- a hyperbolic automorphism A of N fixing Γ

The hyperbolic automorphism A has no eigenvalues of modulus one, and the product of the eigenvalues is ± 1.

Rational structures and lattices

Mal'cev and Raghunathan analyzed the existence of lattices in nilpotent Lie groups, showing

Theorem

A simply connected nilpotent Lie group N admits a lattice if and only if its Lie algebra admits a rational structure.

Equivalently, the Lie algebra of N should admit a rational basis; that is, a basis with all rational structure constants.

Rational subspaces

Definition

A subspace E of a Lie algebra $\mathfrak{n}$ is a rational subspace relative to a rational structure defined by basis $\mathcal{B}$ of $\mathfrak{n}$ if E has a basis consisting of vectors that are $\mathbb{Q}$-linear combinations of elements of $\mathcal{B}$.

Example

With respect to the standard orthonormal basis of $\mathbb{R}^2$, one-dimensional rational subspaces are lines with rational slope.

First reduction: integer matrices and structure constants

In order to find an Anosov automorphism of a nilmanifold, we are looking for

- A nilpotent Lie algebra $\mathfrak{n}$ with a basis $\mathcal{B}$ having rational structure constants
- A automorphism f of $\mathfrak{n}$ such that with respect to $\mathcal{B}$, f is represented by a unimodular hyperbolic matrix with integer entries

The Lie algebra $\mathfrak{n}$ is called an Anosov Lie algebra; f is called an Anosov automorphism.

We do not worry about the different rational forms of Anosov Lie algebras.

Second reduction: semisimple matrices

Theorem (Auslander-Scheuneman, 1970)

A nilmanifold admits an Anosov automorphism if and only if it admits a semisimple Anosov automorphism.

We will seek semisimple hyperbolic automorphisms.

Free nilpotent Lie algebras

Definition

The free r-step nilpotent Lie algebra on n generators $\mathfrak{f}_{n,r}$ is defined to be the quotient algebra $\mathfrak{f}_n/\mathfrak{f}_n^{(r+1)}$, where $\mathfrak{f}_n$ is the free nilpotent Lie algebra on n generators.

Can define $\mathfrak{f}_{n,r}$ over any field K. For us, K is a field with characteristic zero.

$\mathfrak{f}_{3,2}$

Example

The free two-step nilpotent Lie algebra $\mathfrak{f}_{3,2}$ is six-dimensional with basis $\{X_1, X_2, X_3, Z_1, Z_2, Z_3\}$ where

$$Z_1 = [X_3, X_2], Z_2 = [X_3, X_1], Z_3 = [X_2, X_1].$$

There are no other bracket relations other than those determined by the ones written above.

We can write $\mathfrak{f}_{3,2} = V_1 \oplus V_2$, where V_1 is the span of the generating set $\mathcal{B}_1 = \{X_1, X_2, X_3\}$ and V_2 is the span of two-fold brackets of elements of $\mathcal{B}_1$ (call this set $\mathcal{B}_2$).

Hall bases

The previous basis $\mathcal{B}$ for $\mathfrak{f}_{3,2}$ is the Hall basis defined by the generating set $\mathcal{B}_1$. Any set of n generators for a free nilpotent Lie algebra $\mathfrak{f}_{n,r} = \oplus_{i=1}^{r} V_i$ determines a Hall basis. If $r \geq 3$, elements spanning V_3 are in the sets

$$\begin{aligned} \mathcal{B}_3' &= \cup_{1 \leq j < i \leq 3} \{[[\mathbf{z}_i, \mathbf{z}_j], \mathbf{z}_i], [[\mathbf{z}_i, \mathbf{z}_j], \mathbf{z}_j]\}, \quad \text{and} \\ \mathcal{B}_3'' &= \cup_{1 \leq j < i < k \leq 3} \{[[\mathbf{z}_i, \mathbf{z}_j], \mathbf{z}_k], [[\mathbf{z}_k, \mathbf{z}_j], \mathbf{z}_i]\}. \end{aligned}$$

Define subspaces $F_1(K)$ and $F_2(K)$ of $\mathfrak{f}_{n,r}(K)$ by

$$F_1(K) = \operatorname{span}_K \mathcal{B}_3', \qquad \text{and} \quad F_2(K) = \operatorname{span}_K \mathcal{B}_3''.$$

The subspace $V_3(K)$ is the direct sum of $F_1(K)$ and $F_2(K)$.

A matrix induces an automorphism

Let A be a matrix in $GL_n(\mathbb{Z})$ and let $\mathfrak{f}_{n,r}(K)$ be the free r-step nilpotent Lie algebra on n generators over the field K.

Fix a set of generators $\mathcal{B}_1$ for $\mathfrak{f}_{n,r}(K)$ and the corresponding decomposition $\mathfrak{f}_{n,r}(K) = \oplus_{i=1}^{r} V_i(K)$.

Together A and $\mathcal{B}_1$ define a linear map $V_1 \to V_1$. This map induces an automorphism $f_A : \mathfrak{f}_{n,r} \longrightarrow \mathfrak{f}_{n,r}$.

$f_A([\mathbf{x}_i, \mathbf{x}_j]) = [f_A\mathbf{x}_i, f_A\mathbf{x}_j] = [A\mathbf{x}_i, A\mathbf{x}_j]$, etc.

Third reduction: mod out by the right kind of ideal

Theorem (Auslander-Scheuneman, 1970)

Let a matrix A *in* $GL_n(\mathbb{Z})$ *and a generating set* $\mathcal{B}_1$ *determine an automorphism* f_A *of* $\mathfrak{f}_{n,r}$*, the free* r*-step nilpotent Lie algebra on* n *generators. If there is an ideal* $\mathfrak{i}$ *of* $\mathfrak{f}_{n,r}$ *so that*

- $\mathfrak{i}$ *is rational*
- f_A *preserves* $\mathfrak{i}$*,*
- $\mathfrak{i}$ *contains the eigenspaces for the eigenvalues of* f_A *of modulus one, and*
- f_A *is unimodular on* $\mathfrak{i}$*,*

then there exists an Anosov automorphism $\overline{f_A} : \mathfrak{n} \to \mathfrak{n}$ *where* $\mathfrak{n} = \mathfrak{f}_{n,r}/\mathfrak{i}$*. Furthermore, every Anosov automorphism arises in this manner.*

Definition of the polynomials $p_2, p_3, \ldots$

Given a matrix A in $GL_n(\mathbb{Z})$ and $r \geq 2$ we associate an r-tuple of polynomials

$$(p_1, p_2, \ldots, p_r).$$

The polynomial p_i is the characteristic polynomial for the restriction of f_A to the ith step of $\mathfrak{f}_{n,r}(K)$.

The monic polynomial p_1 in $\mathbb{Z}[x]$ alone determines $(p_1, \ldots, p_r)$. There is no dependence on the field K or the number of steps, r.

Eigenvalues of f_A

Since f_A is an automorphism, if $\alpha_1, \ldots, \alpha_n$ are eigenvalues for eigenvectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ spanning V_1, respectively, then for all distinct i, j, the vector $[\mathbf{z}_i, \mathbf{z}_j]$ in V_2 is an eigenvector with eigenvalue $\alpha_i \alpha_j$. (May need to move to $\mathfrak{f}_{n,r}(K)$ if some eignevalue is not real)

More generally, eigenvalues for $f_A|_{V_i}$ are Hall words of length i in $\alpha_1, \ldots, \alpha_n$.

Then $\mathbb{Q}(p_1) > \mathbb{Q}(p_i) > \mathbb{Q}(q)$, where q is any irreducible factor of p_i. This provides restrictions on the Galois group (hence degree) of q.

p_2 and p_3

For example, if

$$p_1(x) = \prod_{i=1}^{n}(x - \alpha_i),$$

then

$$p_2(x) = \prod_{1 \leq i < j \leq n}(x - \alpha_i\alpha_j).$$

The polynomial p_3 is the product of polynomials q_1 and q_2 having roots of form $\alpha_i\alpha_j^2$ and $\alpha_i\alpha_j\alpha_k$ respectively.

The polynomial p_1 is in $\mathbb{Z}[x]$ with constant term ± 1, and if it is to define an Anosov automorphism, no roots of modulus one. Call such a polynomial an Anosov polynomial.

Conversely, the companion matrix $A \in GL_n(\mathbb{Z})$ to an Anosov polynomial defines f_A in $\mathrm{Aut}(\mathfrak{f}_{n,r})$.

Anosov polynomials

Roots of all polynomials p_i are algebraic units.

Not much is known about the inverse Galois problem for polynomials in $\mathbb{Z}[x]$ with constant term ± 1.

Questions:

- If you can solve the inverse Galois problem for G, can you solve it with an algebraic unit?
- What is an example of p in $\mathbb{Z}[x]$ with constant term ± 1 and with Galois group A_n? A_5?

The free 2-step nilpotent Lie algebra on three generators

Suppose that the matrix A in $GL_3(\mathbb{Z})$ has characteristic polynomial

$$p_1(x) = x^3 + ax^2 + bx - 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3),$$

where none of the roots have modulus one. Note that $\alpha_1\alpha_2\alpha_3 = 1$. The characteristic polynomial p_2 of $[f_A|_{V_2}]_{\mathcal{B}_2}$ is reciprocal to p_1 :

$$\begin{aligned} p_2(x) &= (x - \alpha_1\alpha_2)(x - \alpha_2\alpha_3)(x - \alpha_3\alpha_1) \\ &= (x - \alpha_3^{-1})(x - \alpha_1^{-1})(x - \alpha_2^{-1}) \\ &= x^3 - bx^2 - ax - 1 \end{aligned}$$

The Anosov automorphism f_A can represented by the matrix A on the top step V_1 of $\mathfrak{f}_{3,2}$ and by the matrix A^{-1} on the center V_2 of $\mathfrak{f}_{3,2}$.

Want to find rational invariant subspaces

We are looking for an ideal $\mathfrak{i}$ of $\mathfrak{f}_{n,r} = \oplus_{i=1}^{r} V_i$ satisfying the Auslander-Scheuneman conditions for some (semisimple) f_A.

Minimal rational invariant subspaces of the steps V_i generate the ideal $\mathfrak{i}$ satisfying the Auslander-Scheuneman conditions.

Through the theory of rational canonical forms, the decomposition of $V_i < \mathfrak{f}_{n,r}$ into such subspaces corresponds to factorization of the polynomial p_i into irreducible polynomials.

Goal: Factor the polynomials $p_2, \ldots, p_r$ based on the Galois group of p_1.

When p_1 is quartic

Lemma

Let (p_1, p_2) be the pair of polynomials associated to an irreducible Anosov polynomial p_1 of degree four. Let G denote the Galois group of the splitting field for p_1. Then

1. *$G \cong S_4$ or $G \cong A_4$ if and only if p_2 is irreducible.*
2. *$G \cong C_4$ or $G \cong D_8$ if and only if p_2 has an irreducible quartic factor.*
3. *$G \cong V_4$ if and only if p_2 has no irreducible factors of degree three or more.*

Furthermore, roots of p_2 come in reciprocal pairs β and $\pm\beta^{-1}$.

G action on $\mathfrak{f}_{n,r}$

Let the semisimple matrix A in $GL_n(\mathbb{Z})$ define the automorphism f_A of $\mathfrak{f}_{n,r}$, and let $(p_1, p_2, \ldots, p_r)$ be the associated tuple of polynomials.

Let G denote the Galois group of p_1. Let K denote the splitting field of p_1 over $\mathbb{Q}$. There is a natural $\mathbb{Q}$-linear G action on $\mathfrak{f}_{n,r}(K)$.

Can visualize this action using complexes. (Examples: $\mathfrak{f}_{4,2}, \mathfrak{f}_{4,3}, \mathfrak{f}_{5,2}$)

The Lie bracket commutes with the action, and f_A is G-equivariant. The action permutes eigenvectors for f_A in $\mathfrak{f}_{n,r}(K)$.

How to find rational invariant subspaces

Theorem (T.P., 2008)

The decomposition of $\mathfrak{f}_{n,r}$ *into irreducible subspaces for the* G *action is a decomposition into rational invariant subspaces* E *for* f_A *such that the characteristic polynomial of* $f|_E$ *is a power of an irreducible for each* E.

Basic Strategy

Recall that the goal is to find minimal rational f_A-invariant subspaces of $\mathfrak{f}_{n,r}$. Any ideal satisfying the Auslander-Scheueneman conditions is generated by such subspaces.

1. Fix a group $G < S_n$ and consider degree n polynomials with Galois group G

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3. Decompose each rational invariant subspaces into minimal rational invariant subspaces by seeing if the characteristic polynomial factors. (Number theory: need to understand sets of algebraic numbers of full rank)
4. Existence theorems: Need to find Anosov polynomials with given Galois groups, and need to understand Galois groups for algebraic numbers with modulus one

Theorem (T.P., 2008)

Suppose that $\mathfrak{n}$ is a two-step Anosov Lie algebra of type (n_1, n_2) with associated polynomials (p_1, p_2). Let G denote the Galois group of p_1.

1. *If $n_1 = 3, 4$ or 5, then $\mathfrak{n}$ is one of the Anosov Lie algebras listed in the table.*
2. *If p_1 is irreducible and the action of G on the roots of p_1 is doubly transitive, then $\mathfrak{n}$ is isomorphic to the free nilpotent Lie algebra $\mathfrak{f}_{n,2}$.*

Theorem (T.P., 2008)

Let f be a semisimple Anosov automorphism of an r-step Anosov Lie algebra. Let $(p_1, \ldots, p_r)$ be the r-tuple of polynomials associated to the automorphism $\overline{f}$ of the free nilpotent Lie algebra $\mathfrak{f}_{n,r}$ induced by f. Suppose that p_1 is irreducible.

1. *If the polynomial p_1 is of prime degree with cyclic Galois group, then $\mathfrak{n}$ is one of the Lie algebras of "cyclic type." Conversely, if n is prime, and $\mathfrak{i}$ is an ideal of $\mathfrak{f}_{n,r}$ of cyclic type containing the ideal $\mathfrak{j}_{n,r}$, then the Lie algebra $\mathfrak{n} = \mathfrak{f}_{n,r}/\mathfrak{i}$ is Anosov.*

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2. *If the Galois group of p_1 is symmetric, then*
    - *If $r = 2$, then $\mathfrak{n}$ is isomorphic to $\mathfrak{f}_{n,2}$,*
    - *If $r = 3$, then $\mathfrak{n}$ is isomorphic to one of the following five Lie algebras: $\mathfrak{f}_{n,3}$, $\mathfrak{f}_{n,3}/F_1, \mathfrak{f}_{n,3}/F_2$, $\mathfrak{f}_{n,3}/(F_1 \oplus F_{2a})$, and $\mathfrak{f}_{n,3}/F_{2a}$*

The generic 2-step Anosov automorphism is of a free nilpotent Lie algebra

Matrices in $GL_n(\mathbb{Z})$ having characteristic polynomial with symmetric Galois group are dense in the sense of thick/thin sets (see Serre); hence, the second part of the previous theorem describes Anosov automorphisms of two- and three-step Lie algebras that are generic in this sense.

Dimensional restrictions

Theorem (T.P., 2008)

Suppose that $\mathfrak{n}$ *is an Anosov Lie algebra of type* $(n_1, \ldots, n_r)$*. If* $n_1 = 3$*, then* n_i *is a multiple of* 3 *for all* $i = 2, \ldots, r$*, and if* $n_1 = 4$*, then* n_i *is even for all* $i = 2, \ldots, r$*. If* n_1 *is prime and the polynomial* p_1 *is irreducible, then* n_1 *divides* n_i *for all* $i = 2, \ldots, n-1$*.*

Theorem (T.P., 2008)

Let f be an Anosov automorphism of a two-step Lie algebra $\mathfrak{n}$, and let (p_1, p_2) be the associated pair of polynomials. Suppose that p_1 is a product of quadratics. Then $\mathfrak{n}$ is one of the Anosov Lie algebras of "quadratic type".

Number theoretic results

Suppose that p_1 is an Anosov polynomial in $\mathbb{Z}[x]$ with roots $\alpha_1, \ldots, \alpha_n$. We will want to know when the equation

$$\alpha_1^{d_1} \alpha_2^{d_2} \cdots \alpha_n^{d_n} = 1 \tag{1}$$

has nonnegative integer solutions $d_1, \ldots, d_n$. Note that if p_1 has constant term $(-1)^n$, then $\alpha_1 \cdots \alpha_n = 1$, and $d_1 = \cdots = d_n = d$ is a solution for any integer d.

Definition

Let $\Lambda = \{\alpha_1, \ldots, \alpha_n\}$ be the set of roots of a polynomial p in $\mathbb{Z}[x]$ with constant term $(-1)^n$ and degree $n \geq 2$. The set Λ is said to be of full rank if the only integral solutions to Equation (1) are of form $d_1 = d_2 = \cdots = d_n$.

Number theoretic results

Theorem (T.P., 2008, and Bell-Hare, 2005)

Suppose that $\alpha_1, \ldots, \alpha_n$ are roots of a degree n irreducible monic polynomial p in $\mathbb{Z}[x]$ with constant term $(-1)^n$, and suppose that none of $\alpha_1, \ldots, \alpha_n$ are roots of unity. Let G denote the Galois group for p.
The set $\{\alpha_1, \ldots, \alpha_n\}$ is of full rank in the following situations.

1. *When the permutation representation of G on $\mathbb{Q}^n$ is the sum of the principal representation and a representation that is irreducible over $\mathbb{Q}$.*
2. *When the action of G on the set of roots of p is doubly transitive.*
3. *When p is Anosov, and precisely one of its roots has modulus greater than one (Bell-Hare).*

Proof of the dimension theorem

- The $n = 3$ and $n = 4$ cases: see how G/N can act
- If n is prime, then the set of roots of p_1 has full rank
- Suppose that p_i has a factor of degree k
- The constant term $s = \pm 1$ of the factor is $(\alpha_1 \alpha_2 \cdots \alpha_n)^d$
- On the other hand, s is the product of k Hall words in $\alpha_1, \ldots, \alpha_n$, each of length i
- Count the α_i's in s both ways: $nd = ki$
- Since n is prime and $i < n$, i divides d and the degree $k = n(d/i)$.

Proof of the quadratic theorem

- A quadratic Anosov polynomial has real roots
- Say q and r are Anosov quadratics with roots α, α^{-1} and β, β^{-1} respectively
- Case One: $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Use Dirichlet fundamental unit.
- Case Two: $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. Then $\mathbb{Q}(qr)$ is a biquadratic extension of $\mathbb{Q}$.