

Zero mean curvature surfaces with non-negative curvature in flat Lorentzian 4-spaces

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In this note we study the local geometry of zero mean curvature spacelike surfaces (*stationary surfaces*) with non-negative Gaussian curvature K in a flat, four-dimensional Lorentzian space form L^4 . In particular, at points where the normal curvature K^\perp is non-zero, we are able to estimate the maximum possible geodesic radius in terms of a (local) positive constant. As a consequence of this, we deduce that every complete stationary surface in L^4 with $K \geq 0$ must be flat and satisfy $K^\perp \equiv 0$.

Keywords: mean curvature; stationary surfaces; Lorentz spaces; harmonic measure; geodesic balls

1. Introduction

Spacelike zero mean curvature surfaces (*stationary surfaces*) in Lorentzian 4-manifolds are critical points for the area functional with respect to compactly supported variations of the surface. Such surfaces are neither local maxima nor local minima even when the support of the variation is taken to be sufficiently small. However, we have recently shown (Alías & Palmer 1997) that, under fairly weak conditions on the ambient manifold, stationary surfaces locally minimize area among nearby surfaces having the same boundary and having null mean curvature.

Stationary surfaces in Lorentzian 4-manifolds have well-known important applications in general relativity and conformal geometry. As in the Riemannian case, understanding these surfaces in constantly curved ambient spaces is crucial for understanding the general situation. In Alías & Palmer (1998), the authors studied global properties of the Gaussian curvature K for stationary surfaces in four-dimensional Lorentzian space forms. In the Minkowski space \mathbf{E}_1^4 , minimal surfaces in $\mathbf{E}^3 \subset \mathbf{E}_1^4$ furnish many examples of complete non-positively curved stationary surfaces. On the other hand, it is well known that the only complete spacelike zero mean curvature surfaces in $\mathbf{E}_1^3 \subset \mathbf{E}_1^4$ are the flat spacelike planes. Motivated by this, the authors considered the existence of complete stationary surfaces immersed into four-dimensional Lorentzian space forms with Gaussian curvature bounded below by the curvature of the ambient space and obtained the following (Alías & Palmer 1998, theorem 1.1).

Theorem 1.1. *Let Σ be a parabolic Riemann surface and let $\chi : \Sigma \rightarrow L^4(c)$ be a conformal spacelike immersion with zero mean curvature into a four-dimensional Lorentzian space form of curvature c . Suppose that the Gaussian curvature of the*

metric induced by χ satisfies $K - c \geq 0$. Then the curvature K^\perp of the normal bundle of χ satisfies $K^\perp \equiv 0$.

As a consequence, if $c = 0$ we had the following (Alías & Palmer 1998, corollary 1.2).

Corollary 1.2. *If $\chi : \Sigma \rightarrow \mathbb{L}^4(0)$ is a complete stationary immersion with $K \geq 0$, then*

$$K \equiv 0 \equiv K^\perp.$$

Further, we gave a complete description of the stationary surfaces in the Minkowski 4-space with $K \equiv 0 \equiv K^\perp$.

In this note we wish to prove a local result for the case $c = 0$, which will imply the global result in corollary 1.2.

Theorem 1.3. *Let \mathbb{L}^4 be a flat, four-dimensional Lorentzian space form and let $\chi : \Sigma \rightarrow \mathbb{L}^4$ be a stationary immersed surface with curvature $K \geq 0$ and $K \not\equiv 0$. Suppose for some $p \in \Sigma$ and some $R > 0$, the geodesic ball $B_R(p) \subset\subset \Sigma$. Then for all $0 < r < R$*

$$0 \leq \int_{B_r(p)} K^\perp \arctan\left(\frac{K^\perp}{K}\right) dA \leq \frac{\pi^2}{4} \frac{L(r)}{r \log(R/r)}, \quad (1.1)$$

where $L(r)$ denote the length of $\partial B_r(p)$.

Here the function $\arctan(K^\perp/K)$ makes sense even at points where K vanishes, and the integrand in (1.1) is non-negative everywhere (see §3 for the details).

Remark 1.4. Theorem 1.3 can be paraphrased as follows. Suppose that $\chi : \Sigma \rightarrow \mathbb{L}^4$ is a non-flat stationary surface with $K \geq 0$. Assume that $p \in \Sigma$ is a point with $K^\perp(p) \neq 0$ and let $r > 0$ be such that $B_r(p) \subset\subset \Sigma$. There exists an explicit constant $C_r > 0$ depending only on the geometry of $\chi|_{B_r(p)}$ such that

$$R \leq C_r$$

for every $R > r$ with $B_R(p) \subset\subset \Sigma$. In other words, we are able to estimate the maximum possible geodesic radius in terms of a (local) positive constant C_r .

Similar estimates for stable minimal surfaces in three manifolds of non-negative Ricci curvature were given by Schoen (1983).

2. Auxiliary results

Let Σ be a Riemannian manifold and let B_r denote the geodesic ball of radius r about a fixed point $p \in \Sigma$. For $0 < r < R$ let $A_{r,R}$ be the geodesic annulus,

$$A_{r,R} := B_R - \overline{B_r}.$$

Let $\omega_{r,R}$ denote the *harmonic measure* of ∂B_R with respect to $A_{r,R}$, that is $\omega = \omega_{r,R}$ is the solution of

$$\begin{aligned} \Delta\omega &= 0 && \text{in } A_{r,R} \\ \omega &\equiv 0 && \text{on } \partial B_r \\ \omega &\equiv 1 && \text{on } \partial B_R. \end{aligned}$$

The *capacity* of the annulus is defined to be

$$\frac{1}{\mu_{r,R}} := \int_{A_{r,R}} |\nabla \omega_{r,R}|^2.$$

It is well known that Σ is parabolic if and only if

$$\lim_{R \rightarrow \infty} \frac{1}{\mu_{r,R}} = 0.$$

Lemma 2.1. *Let Σ be a Riemannian manifold and let u be a C^2 function that satisfies*

$$u\Delta u \geq 0$$

on Σ . Then for $0 < r < R$

$$\int_{B_r} u\Delta u \leq \frac{2}{\mu_{r,R}} \sup_{B_R} u^2.$$

Proof. For any $\zeta \in C_0^\infty(B_R)$

$$\int_{B_R} (\zeta^2 |\nabla u|^2 + \zeta^2 u\Delta u + 2\zeta u \langle \nabla \zeta, \nabla u \rangle) = \oint_{\partial B_R} \zeta^2 u * du = 0.$$

Therefore,

$$\begin{aligned} \int_{B_R} \zeta^2 (|\nabla u|^2 + u\Delta u) &\leq 2 \int_{B_R} |\zeta u \langle \nabla \zeta, \nabla u \rangle| \\ &\leq a^2 \int_{B_R} \zeta^2 |\nabla u|^2 + \frac{1}{a^2} \int_{B_R} u^2 |\nabla \zeta|^2 \end{aligned}$$

for all $a > 0$, and hence taking $a = 1/\sqrt{2}$ we obtain

$$\int_{B_R} \zeta^2 (|\nabla u|^2 + u\Delta u) \leq \frac{1}{2} \int_{B_R} \zeta^2 |\nabla u|^2 + 2 \int_{B_R} u^2 |\nabla \zeta|^2.$$

Therefore,

$$\int_{B_R} \zeta^2 u\Delta u \leq 2 \int_{B_R} u^2 |\nabla \zeta|^2 \leq 2 \sup_{B_R} u^2 \int_{B_R} |\nabla \zeta|^2.$$

Define ζ by

$$\zeta(x) := \begin{cases} 1, & x \in \overline{B_r}, \\ 1 - \omega_{r,R}, & x \in A_{r,R}. \end{cases}$$

Although ζ is not smooth it can be approximated by a smooth function, and so we obtain

$$\int_{B_r} u\Delta u \leq 2 \sup_{B_R} u^2 \int_{A_{r,R}} |\nabla \omega_{r,R}|^2 = \frac{2}{\mu_{r,R}} \sup_{B_R} u^2.$$

■

Let ∂B_t be the geodesic sphere of radius t about a fixed point $p \in \Sigma$ and let $S(t)$ denote the $(n - 1)$ -dimensional volume of ∂B_t . The function $S(t)$ is a continuous piecewise smooth function of t that satisfies

$$\frac{1}{\mu_{r,R}} \leq \frac{1}{\int_r^R dt/S(t)}. \quad (2.1)$$

To see this, define $f : [r, R] \rightarrow \mathbf{R}$ by

$$f(t) := \frac{\int_r^t ds/S(s)}{\int_r^R ds/S(s)}.$$

Then $f(r) = 0$, $f(R) = 1$ and $f(t)$ satisfies

$$S(t)f'(t) = \frac{1}{\int_r^R ds/S(s)}.$$

Let τ denote the geodesic distance from the fixed point p and let $\eta = f(\tau)$ defined on the annulus $A_{r,R}$. Then $\eta \equiv 0$ on ∂B_r , $\eta \equiv 1$ on ∂B_R , and by Dirichlet's principle

$$\frac{1}{\mu_{r,R}} = \int_{A_{r,R}} |\nabla \omega_{r,R}|^2 \leq \int_{A_{r,R}} |\nabla \eta|^2.$$

Therefore, since $|\nabla \eta|^2 = f'(\tau)^2$, it follows that

$$\begin{aligned} \frac{1}{\mu_{r,R}} &\leq \int_{A_{r,R}} |\nabla \eta|^2 = \int_r^R \left(\int_{\partial B_t} f'(t)^2 d\sigma_t \right) dt \\ &= \int_r^R f'(t)^2 S(t) dt = \frac{1}{\int_r^R ds/S(s)} \int_r^R f'(t) dt \\ &= \frac{1}{\int_r^R ds/S(s)}, \end{aligned}$$

which proves (2.1).

Let us assume from now on that $n = 2$ and Σ is an analytic surface endowed with an analytic Riemannian metric. In that case, ∂B_t consists of a finite number of piecewise analytic arcs that are joined together to make up a finite number of connected closed curves. The function $L(t) = S(t)$ denotes now the length of ∂B_t and, at the points where the derivative of L exists, $L'(t)$ is given by

$$L'(t) = \int_{\partial B_t} \kappa_g(t) d\sigma_t - \sum_{j=1}^{N(t)} (\tan \theta_j(t) + \tan \bar{\theta}_j(t)). \quad (2.2)$$

Here $\kappa_g(t)$ is the geodesic curvature of ∂B_t , and $\theta_j(t)$ and $\bar{\theta}_j(t)$ denote the acute angles formed between the cut locus C of p and the minimizing geodesics that join p to the singular points of ∂B_t (for the details see Fiala (1940–41, § 9.4) or Barbosa & do Carmo (1978, pp. 247–249)). Since $0 \leq \theta_j(t), \bar{\theta}_j(t) \leq \pi/2$, it follows from (2.2) that

$$L'(t) \leq \int_{\partial B_t} \kappa_g(t) d\sigma_t. \quad (2.3)$$

On the other hand, at smooth points of ∂B_t , the outward-pointing unit normal is given by $\nabla\tau$, and the geodesic curvature of ∂B_t is then

$$\kappa_g(t) = \langle \nabla_v \nabla\tau, v \rangle,$$

where v is a local unit tangent along ∂B_t . Since $|\nabla\tau|^2 = 1$, we obtain on ∂B_t

$$\Delta\tau = \langle \nabla_v \nabla\tau, v \rangle + \langle \nabla_{\nabla\tau} \nabla\tau, \nabla\tau \rangle = \kappa_g(t).$$

Under the assumption $K \geq 0$, a standard estimate (Li & Schoen 1984) gives $\Delta\tau \leq 1/\tau$, so that

$$\kappa_g(t) \leq \frac{1}{t}$$

on ∂B_t . Using this in (2.3) we obtain

$$\frac{dL}{dt} \leq \int_{\partial B_t} \frac{d\sigma_t}{t} = \frac{L(t)}{t},$$

and therefore

$$\frac{d \log L}{dt} \leq \frac{1}{t}$$

holds. It then follows easily that

$$\frac{L(t)}{t} \leq \frac{L(r)}{r}$$

holds for all $t > r$. Combining this with (2.1) gives

$$\frac{1}{\mu_{r,R}} \leq \frac{L(r)}{r \log(R/r)}. \quad (2.4)$$

3. Proof of theorem 1.3

Let $\chi : \Sigma \rightarrow \mathbb{L}^4$ be a stationary immersed surface for which $K \neq 0$ holds. In Alías & Palmer (1998, proposition 3.1 and lemma 3.2), we showed that Σ is an analytic surface for which the set B of zeros of K has an empty interior, and on $\Sigma - B$ the following holds:

$$\Delta \arctan(K^\perp/K) = 2K^\perp.$$

Further, we showed (Alías & Palmer 1998, lemma 3.3) that under the condition that $K \geq 0$ holds, the function $\arctan(K^\perp/K)$ has a smooth extension u to all of Σ . Note that on $\Sigma - B$

$$u\Delta u = 2K^\perp \arctan(K^\perp/K) \geq 0,$$

and by continuity u satisfies $u\Delta u \geq 0$ on Σ . Thus we can apply lemma 2.1 jointly with (2.4) to obtain

$$0 \leq \int_{B_r} K^\perp \arctan(K^\perp/K) \, dA \leq \frac{1}{\mu_{r,R}} \sup_{B_R} u^2 \leq \frac{\pi^2}{4} \frac{L(r)}{r \log(R/r)}. \quad (3.1)$$

This finishes the proof of theorem 1.3.

To derive the global result in corollary 1.2 from theorem 1.3, observe that every complete stationary surface Σ in \mathbb{L}^4 with $K \geq 0$ must be flat. Otherwise R could

approach infinity in (3.1), yielding $K^\perp \equiv 0$ on Σ . Reasoning now as in the proof of corollary 1.2 in Alías & Palmer (1998), we would have that $K \equiv 0$, in contradiction with the fact that Σ is not flat. Summing up, the surface must be flat, and the Toda system for Σ (or, equivalently, equations (2.11) and (2.12) in Alías & Palmer (1998)) implies that $K^\perp \equiv 0$.

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