

Rolling Construction for Anisotropic Delaunay Surfaces

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Abstract

Anisotropic Delaunay surfaces are surfaces of revolution with constant anisotropic mean curvature. We show how the generating curves of such surfaces can be obtained as the trace of a point held in a fixed position relative to a curve which is rolled without slipping along a line. This generalizes the classical construction for surfaces of revolution with constant mean curvature due to Delaunay [1]. Our result is given as a corollary of a new geometric description of the rolling curve of a general plane curve. Moreover, we characterize anisotropic Delaunay curves by using their isothermic self-duality.

1 Introduction

When immiscible materials come into contact, the interface which forms between them is often modeled as a surface. According to the *law of least action*, the equilibrium surface will form so as to attempt to minimize its free potential energy subject to whatever constraints and additional forces are imposed by its environment.

In a liquid/air interface with no additional forces present, the free energy assigned to the interface is its surface tension. For homogeneous materials, the surface tension is proportional to the area of the surface interface. Minimization leads to the formation of what are known as minimal surfaces (when no volume constraints are imposed) and constant mean curvature surfaces (when a volume constraint is imposed).

If the temperature is gradually lowered, the liquid in the drop may crystallize. This means that its constituent molecules will be found in an ordered configuration and, as a result, the energy assigned to the surface interface will become anisotropic; it will depend on the direction of the surface at each point. The simplest formulation of an anisotropic surface energy is the following: Let F be a smooth positive function defined on the two dimensional sphere S^2 . If

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ν is a choice of a unit surface normal along the surface Σ , then

$$\mathcal{F} := \int_{\Sigma} F(\nu) d\Sigma \tag{1}$$

defines an anisotropic surface energy called a constant coefficient parametric functional.

The surfaces that we will discuss here are in equilibrium among surfaces with prescribed volume for such parametric functionals which in addition satisfy an ellipticity condition: Denote by DF and D^2F the gradient and Hessian of F on S^2 . Then we require that at each point in S^2 the matrix $D^2F + F1$ is positive definite. The functional appearing in (1) is referred to as a (constant coefficient) parametric elliptic functional. This means that the Euler-Lagrange equation which characterizes equilibria is an elliptic equation. The major consequence of ellipticity is that a maximum principle analogous to that for constant mean curvature surfaces holds. Moreover, for such a parametric elliptic functional \mathcal{F} , there is a unique minimizer (up to translation in \mathbf{R}^3) of \mathcal{F} among closed surfaces bounding prescribed volume which is called the Wulff shape (for \mathcal{F}), and it is a smooth convex surface.

When considering a nonlinear theory, it is essential to have a class of tractable examples at hand. When the energy functional is rotationally invariant (that is, the corresponding Wulff shape is a surface of revolution), an obvious class of examples to consider are the surfaces of revolution. In the classical case of isotropic surface energy, the surfaces of revolution with constant mean curvature were found by C. Delaunay in 1841. The Delaunay surfaces are divided into major types: spheres, unduloids, catenoids, nodoids and cylinders. Delaunay [1] found an ingenious method for describing the generating curves of these surfaces. If a conic section Γ rolls without slipping along a line, then the trace of one of its foci defines a curve Ω which when rotated around the line gives a constant mean curvature surface Σ . In particular, when Γ is a parabola, Σ is a catenoid, when Γ is an ellipse, Σ is an unduloid, and when Γ is a hyperbola, we obtain a nodoid. Among the reasons for the importance of the Delaunay surfaces are the followings:

1. They are the most accessible examples of constant mean curvature surfaces.
2. In many problems involving symmetry, one can conclude, a priori, that the solution is a Delaunay surface.
3. Because of 1, they serve as ideal comparison surfaces in applications of the maximum principle in studying other surfaces of prescribed mean curvature.
4. The ends of any properly embedded surface with nonzero constant mean curvature in Euclidean space are asymptotic to Delaunay surfaces.
5. They are the essential building blocks in the construction of other important examples of constant mean curvature surfaces using “gluing methods”.

It is reasonable to believe that the rotationally invariant equilibrium surfaces should play an analogously important role for other functionals besides surface area. In view of 2 and 3 above, this should particularly be the case for functionals having a maximum principle as do the elliptic parametric functionals. In fact, in [9]-[11] the maximum principle has been applied to conclude that solutions of capillary problems for rotationally invariant elliptic functionals with free boundaries on horizontal planes are surfaces of revolution.

This paper is organized as follows. Section 2 re-derives the basic equations defining anisotropic Delaunay surfaces. We also give a generalization to the case where the Wulff shape need not be rotationally symmetric but is a product of convex curves. Section 3 reviews a general rolling construction of Hsiang and Yu and relates it to the classical concept of isothermic duality. We also give a refinement of the general rolling construction by determining the half-plane which contains the rolling curve (Theorem 3.1). In section 4, we give the definition and some basic properties of the mean curvature profile associated with a surface S of revolution. This is a curve whose curvature $\kappa(s)$ is equal to twice the mean curvature $H(s)$ measured along a meridian of S . Section 5 applies the concept of the mean curvature profile of a surface of revolution S to give a geometric description of the rolling curve of the generating curve of S . The rolling curve Γ is obtained as a type of dual curve of the mean curvature profile (Theorem 5.1). We then proceed to apply the general rolling construction to the anisotropic Delaunay surfaces, beginning with the Wulff shape in section 6 (Theorem 6.1) and continuing with the other types of anisotropic Delaunay surfaces in section 7 (Theorems 7.1, 7.2). In the applications to anisotropic unduloids and nodoids, the isothermic self-duality of their generating curves (Lemmas 7.4, 7.5) is used essentially. And in section 8, we use isothermic duality to characterize the periodic curves which arise as generating curves of anisotropic unduloids and nodoids (Theorem 8.1).

2 Generalized anisotropic Delaunay surfaces

In this section we will give a description of some equilibrium surfaces for functionals whose Wulff shape may not be rotationally symmetric. The Wulff shape is assumed to have the property that all of its intersections with horizontal planes are mutually homothetic. In the case when the Wulff shape is a surface of revolution, the construction reduces to that of the anisotropic Delaunay surfaces which were extensively studied in [8]. The derivation appearing here is more elementary than that in [8].

Let

$$\Omega_W : (u(\sigma), v(\sigma)), \quad -2L_1 \leq \sigma \leq 2L_2 \quad (2)$$

be a closed convex curve in the two-dimensional Euclidean space \mathbf{R}^2 parametrized by arc length σ . Here we mean by “a closed convex curve” a closed C^∞ -curve with positive curvature with respect to the inward pointing normal. We assume

that ω_W is symmetric with respect to the v -axis. We may assume that

$$u \geq 0, \quad v_\sigma \geq 0, \quad -L_1 \leq \sigma \leq L_2, \quad (3)$$

$$u < 0, \quad v_\sigma < 0, \quad \sigma \in [-2L_1, -L_1) \cup (L_2, 2L_2] \quad (4)$$

hold. Also we may assume that u and v' have zeros only at $\sigma = -L_1, L_2$ and u' and v have zeros only at $\sigma = 0, -2L_1, 2L_2$. Let $C : \tau \rightarrow (\alpha(\tau), \beta(\tau))$ be a closed convex curve parameterized by arc length in the plane. We assume that the origin is inside the domain bounded by C . Consider the surface W in \mathbf{R}^3 given by

$$\chi(\sigma, \tau) = (u(\sigma)\alpha(\tau), u(\sigma)\beta(\tau), v(\sigma)).$$

When $(\alpha, \beta) = (\cos \tau, \sin \tau)$, this gives a surface of revolution.

In general, W is a convex surface such that all the curves obtained by intersecting W with horizontal planes (that is, the third component = constant) are homothetic to each other.

For a suitable plane curve $(x(s), z(s))$ parameterized by arc length with $x > 0$, we have the surface Σ defined by

$$X(s, \tau) = (x(s)\alpha(\tau), x(s)\beta(\tau), z(s)). \quad (5)$$

A straightforward calculation shows that

$$\chi_\sigma \times \chi_\tau = u(-v_\sigma \beta_\tau, v_\sigma \alpha_\tau, u_\sigma(\alpha\beta_\tau - \beta\alpha_\tau)). \quad (6)$$

Note that because of the assumption that $C = (\alpha, \beta)$ is convex, and the origin $(0, 0)$ is inside of C , $\alpha\beta_\tau - \beta\alpha_\tau$ has a definite sign, and we can choose the orientation of this curve so that the sign is positive.

The same calculation holds for X and so

$$X_s \times X_\tau = x(-z_s \beta_\tau, z_s \alpha_\tau, x_s(\alpha\beta_\tau - \beta\alpha_\tau)) \quad (7)$$

holds.

Formulas (6) and (7) show that the normals to the two surfaces agree exactly when

$$x_s = u_\sigma, \quad z_s = v_\sigma \quad (8)$$

hold if $z_s \geq 0$ holds, and when

$$x_s = -u_\sigma, \quad z_s = -v_\sigma \quad (9)$$

hold if $z_s < 0$ holds. Recall that the Gauss map of a convex surface is a diffeomorphism onto the 2-sphere. The assignment to a point $X(s, \tau)$ on Σ the point $\chi(\sigma(s), \tau)$ on W where (8) or (9) holds defines a map which we call the *anisotropic Gauss map* of X . This map clearly factors through the 2-sphere. We denote this map by $\underline{\chi}$.

We have

$$\begin{aligned}\underline{\chi}_s &= \sigma_s(u_\sigma\alpha, u_\sigma\beta, v_\sigma) = \pm\sigma_s(x_s\alpha, x_s\beta, z_s) \\ &= \pm\sigma_s X_s ,\end{aligned}$$

$$\begin{aligned}\underline{\chi}_\tau &= (u\alpha_\tau, u\beta_\tau, 0) = (u/x)(x\alpha_\tau, x\beta_\tau, 0) \\ &= (u/x)X_\tau .\end{aligned}$$

Since $\underline{\chi}_s = d\underline{\chi}(X_s)$ and $\underline{\chi}_\tau = d\underline{\chi}(X_\tau)$ we see that X_s, X_τ are eigenvectors of $d\underline{\chi}$ with eigenvalues $\pm\sigma_s, u/x$ respectively. The negative of the trace of $d\underline{\chi}$ is called the *anisotropic mean curvature*, which we will denote by Λ . In the case where W is the 2-sphere, $\underline{\chi}$ is the Gauss map of Σ and $\Lambda = 2H$ holds where H is the usual mean curvature.

The anisotropic mean curvature, of course, has a more important variational definition. If an immersion X is subjected to a normal variation $X_\epsilon = X + \epsilon\psi\nu + \mathcal{O}(\epsilon^2)$, $\psi \in C_0^\infty$, then the first variation of \mathcal{F} is

$$\partial_\epsilon \mathcal{F}(X_\epsilon)_{\epsilon=0} = - \int_\Sigma \psi \Lambda \, d\Sigma .$$

In particular, $\Lambda \equiv \text{constant}$ characterizes surfaces which are in equilibrium for the functional \mathcal{F} subject to a volume constraint (cf. [8]).

From our computations, the anisotropic mean curvature is constant Λ if and only if

$$\pm\sigma_s + u/x = -\Lambda = \text{constant} . \quad (10)$$

However, by (8) and (9), we have, $u_s = u_\sigma\sigma_s = \pm x_s\sigma_s$ and so

$$\frac{du}{dx} = \frac{u_s}{x_s} = \pm\sigma_s . \quad (11)$$

Therefore, (10) is equivalent to

$$du/dx + u/x = -\Lambda . \quad (12)$$

The equation (12) is the same as

$$(xu)_x = -\Lambda x .$$

Integrating this equation gives

$$ux = -\Lambda(x^2/2) + c/2 , \quad (13)$$

where c is a constant. In the case where $\Lambda \neq 0$, we obtain

$$x = (-1/\Lambda)(u \pm \sqrt{u^2 + \Lambda c}) , \quad (14)$$

while if $\Lambda = 0$ we have

$$x = \frac{c}{2u} . \quad (15)$$

Assume $(\Lambda, c) \neq (0, 0)$. Then, we can also use (8) and (9) to give a formula for z . We have, using (11),

$$dz = (dz/ds)ds = \pm v_\sigma s_\sigma d\sigma = v_\sigma x_u d\sigma = x_u dv. \quad (16)$$

The function z can then be defined on a suitable subset of W as

$$z := \int x_u dv \quad (17)$$

with x_u computed from (14) when $\Lambda \neq 0$ and from (15) when $\Lambda = 0$. When inserted in (5), the functions x defined by (14) or (15) and z defined by (17) define an immersion on a suitable subset of W which has constant anisotropic mean curvature Λ .

Let us consider the case where $(\alpha, \beta) = (\cos \tau, \sin \tau)$, that is, W is a smooth, closed convex surface of revolution generated by the curve Ω_W in (2). The surfaces $X(s, \tau)$ of revolution with constant anisotropic mean curvature Λ were studied in detail in [8], [9], [10] and [11]. We refer to them as *anisotropic Delaunay surfaces*. In the special case where W is the 2-sphere, the surfaces $X(s, \tau)$ are just the usual Delaunay surfaces.

Denote by μ_2 the principal curvature of the Wulff shape W along the ‘equator’ with respect to the inward pointing normal. Then, $\mu_2 = v_\sigma/u$ holds. Therefore,

$$\mu_2 = \pm z'/u \quad (18)$$

holds. μ_2 can be regarded as a function of the unit normal $\nu = (\nu_1, \nu_2, \nu_3) = (z' \cos \tau, z' \sin \tau, -x')$. In the present case, μ_2 can be regarded as a function of one variable:

$$\mu_2 = \mu_2(\nu_3) = \mu_2(-x').$$

(13) with (18) implies

$$\pm 2\mu_2^{-1} z' x + \Lambda x^2 = c.$$

The constant of integration c will be called the *flux parameter*.

Let S be an anisotropic Delaunay surface for the Wulff shape W . Let \hat{S} be a reflection of S with respect to a horizontal plane. Then, \hat{S} is also an anisotropic Delaunay surface for the same Wulff shape W . If the anisotropic mean curvature of S is Λ for the ‘outward’ pointing normal, then the anisotropic mean curvature of \hat{S} is $-\Lambda$ for the ‘inward’ pointing normal. Therefore, from now on, we will identify an anisotropic Delaunay surface with its reflection with respect to a horizontal plane, and we may assume that $\Lambda \leq 0$ holds. We remark that, for the CMC case, this normalization for the sign of Λ corresponds to a suitable choice of the orientation of the surface normal.

Now, we may consider only solutions of

$$2\mu_2^{-1} z' x + \Lambda x^2 = c \quad (19)$$

with $\Lambda \leq 0$. Then, the anisotropic Delaunay surfaces fall into six cases as follows:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane*.
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid*.
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to vertical translation and homothety)*.
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *vertical cylinder*.
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid*.
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid*.

The global behavior of these surfaces is strikingly similar to that of the classical Delaunay surfaces. For example, an anisotropic unduloid is a periodic embedded surface while an anisotropic nodoid is a periodic surface with self-intersections. The reader is referred to section 5 of [8] for details. Here we note only the correspondence between these classes and the formulas for x and z given before. The case (I-2) corresponds to (x, z) defined by (15) and (17). The case (II-4) is obtained with (x, z) defined by (14) for $-u_0 \leq u \leq u_0$ with the + sign and (17), where $u_0 := \max u = u(0) = -u(-2L_1) = -u(2L_2)$. The part of the surface defined in $u > 0$ (resp. $u < 0$) corresponds to where the Gaussian curvature is positive (resp. negative). Case (II-3) is obtained by using (14) and (17) for $\sqrt{-\Lambda c} \leq u \leq u_0$. The part of the surface obtained using the + sign (resp. - sign) in (14) corresponds to where the Gaussian curvature is positive (resp. negative) and the two parts can be smoothly joined together. Also, we remark that now

$$x_s = u_\sigma, \quad z_s = v_\sigma \tag{20}$$

holds at $\sigma = \sigma(s)$.

Finally, it should be pointed out that, if we adopt $\mu_2 = v_\sigma/u$ as the definition of μ_2 , then the calculation of the curve (x, z) from W only depended on the curve (u, v) . Therefore the curve (α, β) can be simultaneously varied in both the formulas for X and χ without affecting the conclusion. And these surfaces X have similar classification and properties to those of anisotropic Delaunay surfaces mentioned above. We will call these surfaces generalized anisotropic Delaunay surfaces. Examples of some generalized anisotropic Delaunay surfaces and the corresponding Wulff shapes are shown in figures 1 through 8.

We conclude this section with a remark about the relationship between harmonic maps and anisotropic Delaunay surfaces.

Proposition 2.1 *For any anisotropic Delaunay surface $X : \Sigma \rightarrow \mathbf{R}^3$, the anisotropic Gauss map $\chi : \Sigma \rightarrow W$ is harmonic. In particular, when the surface is an anisotropic catenoid or the Wulff shape, χ is \pm holomorphic.*

Proof. The metric induced by $X(s, \tau) = (x(s) \cos \tau, x(s) \sin \tau, z(s))$ is $dS_X^2 = ds^2 + x^2 d\tau^2$. We can introduce isothermal coordinates (ρ, τ) for this metric by defining $d\rho = ds/x$ so that $dS_X^2 = x^2(d\rho^2 + d\tau^2)$.

The induced metric on W is $dS_W^2 = d\sigma^2 + u^2 d\tau^2$. Then

$$\chi^*(dS_W^2) = \sigma_s^2 ds^2 + u^2 d\tau^2 = (-\Lambda - u/x)^2 x^2 d\rho^2 + u^2 d\tau^2 = (-\Lambda x - u)^2 d\rho^2 + u^2 d\tau^2.$$

We recall that a sufficient condition for the harmonicity of a map from a surface with isothermal coordinates (ρ, τ) is that the pull back of the metric on the target space can be expressed as $E d\rho^2 + 2F d\rho d\tau + G d\tau^2$ with $(E - G) - 2iF$ holomorphic as a function of $\rho + i\tau$.

In our case, we have

$$(E - G) - 2iF = (-\Lambda x - u)^2 - u^2 = \Lambda(2ux + \Lambda x^2) = \Lambda c = \text{constant}$$

by (13), so this function is holomorphic. In the case of the anisotropic catenoid ($\Lambda = 0$) or the Wulff shape ($c = 0$), this function vanishes and we obtain the stronger result that χ is \pm holomorphic. **q.e.d.**

It does not appear that the anisotropic Gauss map is harmonic if the anisotropic Delaunay surface is replaced by a general constant anisotropic mean curvature surface.

3 Rolling construction for general curves

We consider a smooth curve

$$\Omega(s) = (x(s), z(s)), \quad x > 0, \quad 0 \leq s \leq l,$$

represented by the arc length s . The curve Ω will always be regarded as the generating curve of a surface of revolution $S : X(s, t) = (x(s)e^{it}, z(s))$ with respect to the z -axis, where we identify $\mathbf{C} \times \mathbf{R}$ with \mathbf{R}^3 . Our point of departure will be the paper by Hsiang-Yu [5] which investigates a rolling construction for general curves. We also refer the reader to [4].

We will consider a configuration which consists of the following:

- The curve Ω itself, which in the current context will be called *roulette*.
- A second curve Γ which will be called the *rolling curve*.
- A point P called the *pole*, whose location relative to Γ is kept fixed when Γ is moved. We will express this by saying that P is *stationary with respect to Γ* .
- A fixed line L called the *base* which is somewhere tangent to Γ .

According to [5], away from umbilic points of the surface X , there exists a configuration as above, such that Ω is obtained as the trace of the pole as Γ is rolled without slipping along the line L . A classical example is given by taking Ω to be the generating curve of a catenoid S , Γ to be the parabola such that the distance between the vertex and the focus is equal to the distance between the vertex of Ω and the rotation axis of S , and P to be the focus of Γ . Using formulas from [5], it follows that the curve Γ is given in polar coordinates $r = r(\theta)$ around the pole by

$$r = \frac{x}{|z'|}, \quad (21)$$

$$\theta = -\int \left(\frac{z'}{x} - \kappa \right) ds = \int (k_2 - k_1) ds, \quad (22)$$

where

$$\begin{aligned} \kappa &:= x'z'' - x''z', \\ k_1 &= x''z' - x'z'', \quad k_2 = -x^{-1}z'. \end{aligned}$$

κ is the curvature of the plane curve $(x(s), z(s))$. And k_1, k_2 are the principal curvatures of S with respect to the outward (resp. inward) normal if $z' > 0$ (resp. $z' < 0$) holds.

Recall that a surface S is called *isothermic* if, away from umbilic points, its lines of curvature are given as the level sets of a pair of locally defined conjugate harmonic functions. CMC surfaces and surfaces of revolution are examples of isothermic surfaces.

According to a classical theorem of Bour and Christoffel (cf. [2]), to each isothermic surface corresponds an *isothermic dual* \tilde{S} , defined up to homothety and translation, with the property that S and \tilde{S} are anticonformal and they share the same Gauss map. The construction of the isothermic dual involves integration and is therefore only well defined, in general, over simply connected domains. In the case of a surface of revolution, the duality is global, as Proposition 3.1 below will show.

Another case of isothermic duality which the reader is probably familiar with involves a nonspherical surface X with constant mean curvature $H \neq 0$. In this case, the parallel surface $\tilde{X} := HX + \nu$ is the isothermic dual. Although \tilde{X} will have branch points when X has umbilics, it will however be globally defined.

Proposition 3.1 *Set*

$$\tilde{x}(s) := \frac{a}{x(s)}, \quad \tilde{z}(s) := -a \int_0^s \frac{z'}{x^2} ds + b, \quad (23)$$

where a is a nonzero constant and b is any constant. Then, the surface $\tilde{S} : \tilde{X}(s, t) = (\tilde{x}(s)e^{it}, \tilde{z}(s))$ is an isothermic dual of $S : X(s, t) = (x(s)e^{it}, z(s))$.

Proof. We have that

$$\tilde{x}' = -\frac{ax'}{x^2}, \quad \tilde{z}' = -\frac{az'}{x^2} \quad (24)$$

holds. Therefore,

$$(\tilde{x}', \tilde{z}') = -\frac{a}{x^2}(x', z'), \quad (25)$$

$$d\tilde{s} := \sqrt{(d\tilde{x})^2 + (d\tilde{z})^2} = \sqrt{(\tilde{x}')^2 + (\tilde{z}')^2} ds = \frac{|a|}{x^2} ds, \quad (26)$$

$$(\tilde{x}_{\tilde{s}}, \tilde{z}_{\tilde{s}}) = \begin{cases} (-x', -z'), & a > 0, \\ (x', z'), & a < 0. \end{cases} \quad (27)$$

If we compute the induced metric $d\tilde{S}^2$ for the surface \tilde{S} , we see that

$$d\tilde{S}^2 = \tilde{x}^2 d\tau^2 + d\tilde{s}^2 = \frac{a^2}{x^4}(x^2 d\tau^2 + ds^2) = \frac{a^2}{x^4} dS^2,$$

where dS^2 is the metric on S . Thus the metrics on the two surfaces are conformally related (cf. pg. 388 of [2]).

On the other hand, the Gauss map ν of X is given as follows.

$$\nu := \frac{X_t \times X_s}{|X_t \times X_s|} = (\operatorname{sgn} x)(z' \cos t, z' \sin t, -x'),$$

where $(\operatorname{sgn} x)$ is the sign of x . Also, by using (23), (24), and (26), the Gauss map $\tilde{\nu}$ of \tilde{X} is given as follows.

$$\begin{aligned} \tilde{\nu} &:= \frac{\tilde{X}_t \times \tilde{X}_s}{|\tilde{X}_t \times \tilde{X}_s|} = (\operatorname{sgn} \tilde{x}) \cdot \frac{(\tilde{z}' \cos t, \tilde{z}' \sin t, -\tilde{x}')}{\sqrt{(\tilde{x}')^2 + (\tilde{z}')^2}} \\ &= -(\operatorname{sgn} x)(z' \cos t, z' \sin t, -x') = -\nu. \end{aligned}$$

Therefore, if we choose the orientation of the surface \tilde{S} so that its Gauss map coincides with ν , then the map $X(s, t) \mapsto \tilde{X}(s, t)$ is anticonformal. **q.e.d.**

We will see in Section 7 that, when S is a Wulff shape (resp. an anisotropic catenoid), $\tilde{\Omega}$ generates an anisotropic catenoid (resp. a homothety of the Wulff shape). Also we will see that *the isothermic dual of any anisotropic unduloid (resp. nodoid) is a part of the same surface, that is, the anisotropic unduloids (resp. nodoids) are self-dual.* In fact, this self-duality will characterize these classes of curves without reference to the functional (Theorem 8.1).

Set

$$\tilde{k}_1 := \tilde{x}_{\tilde{s}} \tilde{z}_{\tilde{s}\tilde{s}} - \tilde{x}_{\tilde{s}\tilde{s}} \tilde{z}_{\tilde{s}}, \quad \tilde{k}_2 := \frac{\tilde{z}_{\tilde{s}}}{\tilde{x}}, \quad \tilde{H} := (\tilde{k}_1 + \tilde{k}_2)/2. \quad (28)$$

Then, \tilde{k}_1, \tilde{k}_2 are the principal curvatures and \tilde{H} is the mean curvature of the surface $(\tilde{x}(\tilde{s})e^{it}, \tilde{z}(\tilde{s}))$ with respect to the outward (resp. inward) pointing normal if $z' > 0$ (resp. $z' < 0$) holds. We now compute

$$\left(\frac{d^2 \tilde{x}}{d\tilde{s}^2}, \frac{d^2 \tilde{z}}{d\tilde{s}^2} \right) = \frac{-1}{a} x^2 (x'', z'').$$

Hence, we obtain

$$\tilde{k}_1 = -\frac{x^2}{|a|} k_1, \quad \tilde{k}_2 = \frac{x^2}{|a|} k_2, \quad \tilde{H} = x^2 (k_2 - k_1) / |2a|. \quad (29)$$

By (21), (22), (26), (29), we obtain

$$r = \left| \frac{a}{\tilde{x}\tilde{z}_{\tilde{s}}} \right| > 0, \quad (30)$$

$$\theta = \int (\tilde{k}_1 + \tilde{k}_2) d\tilde{s} = 2 \int \tilde{H} d\tilde{s} = \int (k_2 - k_1) ds. \quad (31)$$

Lemma 3.1 *The rolling curve Γ of a smooth curve*

$$\Omega(s) = (x(s), z(s)), \quad x(s) > 0, \quad z'(s) \neq 0$$

is a regular curve away from umbilic points of the rotation surface $X(s, t) = (x(s)e^{it}, z(s))$. In other words, the rolling curve Γ is regular if

$$\tilde{H} \neq 0. \quad (32)$$

Moreover, at any non-umbilic point of X , the curvature κ_Γ of Γ is given as follows.

$$\kappa_\Gamma = \frac{-(z')^3}{x^2|k_2 - k_1|} = \frac{-(z')^3}{2|a\tilde{H}|}. \quad (33)$$

Proof. We represent the rolling curve $\Gamma : r = r(\theta)$ as

$$\xi(s) = r(s) \cos \theta(s) + \xi_0, \quad \eta(s) = r(s) \sin \theta(s) + \eta_0. \quad (34)$$

Here, s is arc length of the curve $\Omega = (x(s), z(s))$, but in general, it is not arc length of Γ . By elementary calculations, we obtain

$$(\xi')^2 + (\eta')^2 = (r')^2 + r^2(\theta')^2 = (z' - \kappa x)^2 (z')^{-4} = x^2(k_1 - k_2)^2 (z')^{-4} = 4a^2 \tilde{H}^2 x^{-2} (z')^{-4}.$$

This implies the first half of the lemma.

We compute

$$\begin{aligned} \kappa_\Gamma &:= \frac{\xi'\eta'' - \xi''\eta'}{\{(\xi')^2 + (\eta')^2\}^{3/2}} \\ &= \frac{2(r')^2\theta' + rr'\theta'' - rr''\theta' + r^2(\theta')^3}{\{(r')^2 + r^2(\theta')^2\}^{3/2}} \\ &= \frac{-(\kappa x - z')^2}{x(z')^3} \left(\frac{(\kappa x - z')^2}{(z')^4} \right)^{-3/2} \\ &= \frac{-(z')^3}{x|\kappa x - z'|} = \frac{-(z')^3}{x^2|k_2 - k_1|} = \frac{-(z')^3}{2|a\tilde{H}|}. \end{aligned}$$

q.e.d.

Theorem 3.1 *Let*

$$\Omega(s) = (x(s), z(s)), \quad x(s) > 0, \quad z'(s) \neq 0$$

be a smooth curve with arc length s . Denote by $R(s) = (R_x(s), R_z(s))$ the center of curvature of Ω at s . Set $X(s, t) = (x(s)e^{it}, z(s))$, which is the surface of revolution generated by Ω with z -axis as rotation axis. Denote by \hat{k}_1, \hat{k}_2 the principal curvatures of X with respect to inward pointing normal. Here, \hat{k}_1 is the curvature of Ω . Let Γ be the rolling curve of Ω with z -axis as base. Then,

(I) If $\hat{k}_1(s_0) \neq \hat{k}_2(s_0)$, then Γ is a regular smooth curve near $s = s_0$. The condition $\hat{k}_1(s_0) \neq \hat{k}_2(s_0)$ is equivalent to the following condition: Either $\hat{k}_1(s_0) = 0$ or $R_x(s_0) \neq 0$ holds.

(II) Assume $\hat{k}_1(s_0) \neq \hat{k}_2(s_0)$. Then, $\Gamma \subset \{x > 0\}$ (resp. $\Gamma \subset \{x < 0\}$) for $0 < |s - s_0| < \delta$ for some $\delta > 0$ if and only if $\hat{k}_2(s_0) > \hat{k}_1(s_0)$ (resp. $\hat{k}_2(s_0) < \hat{k}_1(s_0)$) holds.

(III) (i) Assume $\hat{k}_1(s_0) > 0$ (that is, the Gauss curvature of X is positive near $s = s_0$). Then, $R_x(s_0) < 0$ (resp. $R_x(s_0) > 0$) if and only if $\Gamma \subset \{x > 0\}$ (resp. $\Gamma \subset \{x < 0\}$) for $0 < |s - s_0| < \delta$ for some $\delta > 0$.

(ii) Assume $\hat{k}_1(s_0) < 0$ (that is, the Gauss curvature of X is negative near $s = s_0$). Then, $R_x(s_0) > 0$, and $\Gamma \subset \{x > 0\}$ for $0 < |s - s_0| < \delta$ for some $\delta > 0$.

Proof. Assume that the rotation surface $(x(s)e^{it}, z(s))$ has no umbilic points. We represent the rolling curve $\Gamma : r = r(\theta)$ of Ω as

$$\xi(s) = r(s) \cos \theta(s) + \xi_0, \quad \eta(s) = r(s) \sin \theta(s) + \eta_0. \quad (35)$$

Then,

$$r = \frac{x}{|z'|}, \quad \theta' = k_2 - k_1,$$

where

$$k_1 = x''z' - x'z'', \quad k_2 = -z'/x,$$

and, by Lemma 3.1, the curvature κ_Γ of Γ is

$$\kappa_\Gamma := \frac{\xi'\eta'' - \xi''\eta'}{\{(\xi')^2 + (\eta')^2\}^{3/2}} \quad (36)$$

$$= \frac{-(z')^3}{x^2|k_2 - k_1|}. \quad (37)$$

Let $Q := (\xi(s_0), \eta(s_0))$ be the point of contact between Γ and the z -axis. Then, if $\eta'(s_0) > 0$, then, by (36), at $s = s_0$, we have the following:

$$\begin{cases} \kappa_\Gamma > 0 \iff \xi'' < 0 \iff \Gamma \subset \{x < 0\} \text{ near } Q \\ \kappa_\Gamma < 0 \iff \xi'' > 0 \iff \Gamma \subset \{x > 0\} \text{ near } Q \end{cases} \quad (38)$$

If $\eta'(s_0) < 0$, then, by (36), at $s = s_0$, we have the following:

$$\begin{cases} \kappa_\Gamma > 0 \iff \xi'' > 0 \iff \Gamma \subset \{x > 0\} \text{ near } Q \\ \kappa_\Gamma < 0 \iff \xi'' < 0 \iff \Gamma \subset \{x < 0\} \text{ near } Q \end{cases} \quad (39)$$

Also, by (37), at $s = s_0$, we have the following:

$$\kappa_\Gamma > 0 \iff z' < 0, \quad \kappa_\Gamma < 0 \iff z' > 0 \quad (40)$$

On the other hand, if $z' > 0$ (resp $z' < 0$), then k_1, k_2 are principal curvatures of the surface $X(s, t) = (x(s)e^{it}, z(s))$ with respect to the outward (resp. inward) pointing normal. Therefore,

$$z'\theta' = z'(k_2 - k_1) = -|z'|(\hat{k}_2 - \hat{k}_1) = -xr^{-1}(\hat{k}_2 - \hat{k}_1). \quad (41)$$

Note that $\eta'\theta' < 0$ holds. Therefore, by (38) and (40), if $\theta'(s_0) < 0$, then

$$\begin{cases} z' < 0 \iff \Gamma \subset \{x < 0\} \text{ near } Q \\ z' > 0 \iff \Gamma \subset \{x > 0\} \text{ near } Q \end{cases} \quad (42)$$

holds. By (39) and (40), if $\theta'(s_0) > 0$, then

$$\begin{cases} z' < 0 \iff \Gamma \subset \{x > 0\} \text{ near } Q \\ z' > 0 \iff \Gamma \subset \{x < 0\} \text{ near } Q \end{cases} \quad (43)$$

holds. Therefore, If $\theta'z'(s_0) > 0$ (resp. $\theta'z'(s_0) < 0$), then $\Gamma \subset \{x < 0\}$ (resp. $\Gamma \subset \{x > 0\}$) for $0 < |s - s_0| < \delta$ for some $\delta > 0$. Hence, by (41), $\Gamma \subset \{x < 0\}$ (resp. $\Gamma \subset \{x > 0\}$) for $0 < |s - s_0| < \delta$ for some $\delta > 0$ if and only if $\hat{k}_2(s_0) < \hat{k}_1(s_0)$ (resp. $\hat{k}_2(s_0) > \hat{k}_1(s_0)$) holds. We have proved (II).

By elementary calculations, we obtain the following:

$$R = (R_x, R_z) = (xk_1^{-1}(k_1 - k_2), z - k_1^{-1}x'), \quad (44)$$

which proves (I).

Lastly we prove (III). It is sufficient to prove it for the case where $z'(s_0) > 0$ holds. In this case,

$$k_1 = -\hat{k}_1, \quad k_2 = -\hat{k}_2$$

holds. Now, (III) follows from (II) and (44). **q.e.d.**

The rolling construction is still applicable in the case where S has umbilic points. Figure 17 depicts the rolling curve for the surface S generated by the curve $u^4 + v^4 = 1$. The cusps of the rolling curve correspond to the umbilics of S . We investigate this situation below.

Proposition 3.2 *Let Ω and Γ be the same as in Theorem 3.1. Let s_0 be an isolated zero of $\eta(s) := k_1 - k_2$. Assume that $z'(s_0) \neq 0$, i.e. r is bounded as $s \rightarrow s_0$ by (21). Then, the one sided limits*

$$\lambda_\pm := \lim_{s \rightarrow s_0 \pm 0} \frac{\Gamma'(s)}{\|\Gamma'(s)\|}$$

both exist. In addition, if $\eta(s)$ does not change sign at s_0 , then $\lambda_- = \lambda_+$ holds. If $\eta(s)$ changes sign at s_0 , then $\lambda_- = -\lambda_+$ holds and consequently at $\Gamma(s_0)$ the unit vectors, the limits of the unit tangent vectors to the two arcs exist and point in opposite directions.

Proof. A straightforward calculation using (21) and (22) shows that, for $s \approx s_0$, $s \neq s_0$, with $\sigma := \text{sign of } z'$,

$$r' = \sigma(k_1 - k_2) \frac{xx'}{(z')^2}.$$

Therefore

$$r' + i\theta'r = \sigma(k_1 - k_2) \left(\frac{xx'}{(z')^2} - i \frac{x}{z'} \right),$$

and so

$$\frac{\Gamma'}{\|\Gamma'\|} = \rho\sigma \left(\frac{xx'}{(z')^2} - i \frac{x}{z'} \right) e^{i\theta} \left\| \frac{xx'}{(z')^2} - i \frac{x}{z'} \right\|^{-1}, \quad (45)$$

where $\rho = 1$ if $k_1 - k_2$ is positive, and $\rho = -1$ if $k_1 - k_2$ is negative. Here $\theta(s)$ has a well-defined limit $\lim_{s \rightarrow s_0} \theta(s)$. Therefore, the result follows. **q.e.d.**

Remark 3.1 It follows from the previous proposition that the part of the rolling curve near $s = s_0$ consists of two C^1 arcs which lie on opposite (resp. the same) sides of the tangent line to Γ if $k_1 - k_2$ changes sign (resp. does not change sign) at s_0 .

4 Mean curvature profile

Let $S : X(s, t) = (x(s)e^{it}, z(s))$ be a surface of revolution generated by a smooth curve $\Omega(s) = (x(s), z(s))$ parameterized by arc length. We assume that either $x(s) > 0$ ($\forall s$) or $x(s) < 0$ ($\forall s$) holds. We set

$$\hat{k}_1 := x'z'' - x''z', \quad \hat{k}_2 := \frac{z'}{x}, \quad H_S := (\hat{k}_1 + \hat{k}_2)/2. \quad (46)$$

$H_S(s)$ is the mean curvature of S measured along the meridian $\Omega(s)$ with respect to the inward (resp. outward) pointing normal, if $xz' > 0$ (resp. $xz' < 0$).

By the *mean curvature profile* of S , we mean the plane curve $C_S(s) = (f(s), g(s))$ defined by the properties:

- (A) s is also the arc length parameter of C_S .
- (B) The curvature $\kappa_{C_S}(s)$ of C_S is given by $\kappa_{C_S}(s) = 2H_S(s)$.

Note that the mean curvature profile is only determined up to rigid motion. The mean curvature profile was used extensively by Kenmotsu to study surfaces of revolution with prescribed mean curvature ([6], [7]).

It is elementary that, up to rotation, the mean curvature profile (f, g) is given by the following formulas:

$$f(s) := \int_0^s \cos \theta(s_1) ds_1 - c_1, \quad g(s) := \int_0^s \sin \theta(s_1) ds_1 - c_2, \quad (47)$$

where

$$\theta(s) := 2 \int_0^s H_S(s) ds, \quad (48)$$

and c_1 and c_2 are constants. In [6], Kenmotsu showed that one can determine constants c_1, c_2 so that the following equalities hold.

$$x = (\operatorname{sgn} x) \sqrt{f^2 + g^2}, \quad (49)$$

$$z' = (\operatorname{sgn} x) \frac{fg' - f'g}{\sqrt{f^2 + g^2}}. \quad (50)$$

The equality (49) can be expressed as follows:

(C) *The distance between a point of Ω and the z -axis is equal to the distance between the corresponding point of C_S and the origin.*

(A), (B), (C) determine the curve $C_S(s) = (f(s), g(s))$ up to rotation around the origin.

We can characterize the mean curvature profile in another way as follows:

Proposition 4.1 *Let $C_S(s) = (f(s), g(s))$ be a plane curve defined by the above properties (A) and (C). Then, (B) is automatically satisfied (up to sign). The curve $C_S(s) = (f(s), g(s))$ is determined up to rotation around the origin.*

Proof. Express the curve $C_S(s) = (f(s), g(s))$ by the polar coordinate (ρ, t) , that is

$$f = \rho \cos t, \quad g = \rho \sin t.$$

We regard ρ, t as functions of s . Then,

$$\begin{aligned} \kappa_{C_S}(s) &:= \frac{f'g'' - f''g'}{\{(f')^2 + (g')^2\}^{3/2}} \\ &= \frac{2(\rho')^2 t' + \rho \rho' t'' - \rho \rho'' t' + \rho^2 (t')^3}{\{(\rho')^2 + \rho^2 (t')^2\}^{3/2}} \end{aligned} \quad (51)$$

holds. The condition (A) implies

$$dx^2 + dz^2 = d\rho^2 + \rho^2 dt^2. \quad (52)$$

The condition (C) implies

$$|x| = \rho. \quad (53)$$

Inserting (53) into (52), we obtain

$$dz^2 = x^2 dt^2.$$

Therefore,

$$t' = \pm \frac{z'}{x}, \quad t'' = \pm \frac{z''x - z'x'}{x^2} \quad (54)$$

holds. Inserting (53) and (54) into (51), we obtain

$$\kappa_{C_S}(s) = \pm \left\{ (x'z'' - x''z') + \frac{z'}{x} \right\} = \pm 2H_S(s).$$

q.e.d.

When (f, g) satisfies (A), (B), and (C), we call the curve (f, g) the *mean curvature profile associated with S*.

Lemma 4.1 *If $z'(0) = \pm 1$ holds, then*

$$c_1 = 0, \quad c_2 = (\operatorname{sgn} z'(0))x(0)$$

holds.

Proof. At $s = 0$, it holds that

$$\begin{aligned} f(0) &= -c_1, \quad g(0) = -c_2, \quad f'(0) = 1, \quad g'(0) = 0, \\ x(0) &= (\operatorname{sgn} x(0))\sqrt{c_1^2 + c_2^2}, \quad z'(0) = (\operatorname{sgn} x(0))\frac{c_2}{\sqrt{c_1^2 + c_2^2}}. \end{aligned}$$

These equalities imply the result. **q.e.d.**

5 Rolling curves as dual curves

Let $\gamma = \gamma(s) = (f(s), g(s))$ be a smooth curve with arc length s in the plane with unit normal N . Let $p := \langle \gamma, N \rangle$ be the support function. Away from points where $N'(s) = 0$ (that is, the curvature κ_γ vanishes at s), we can consider p to be (locally) a function on a subset of the unit circle S^1 . We assume that p has at most isolated zeros. The curve γ can be expressed using its *tangential representation*,

$$\gamma = pN + Dp,$$

where Dp is the gradient of p on S^1 . If $N = (\cos \theta, \sin \theta)$, then

$$\gamma = (p \cos \theta - \dot{p} \sin \theta, \quad p \sin \theta + \dot{p} \cos \theta),$$

where $\dot{p} = dp/d\theta$.

We define the *dual curve* by $\gamma^* := p^{-1}N$. Since

$$(\gamma^*)' = -p'p^{-2}N + p^{-1}N', \quad \langle N, N' \rangle = 0,$$

γ^* is a regular smooth curve around s where $p(s) \neq 0$ and $(p'(s), \kappa_\gamma(s)) \neq (0, 0)$ hold.

When γ is the profile curve for the unit sphere of a rotationally invariant norm, then γ^* is the profile curve of the unit sphere of the dual norm. Note that the curve γ^* depends on the choice of origin in the plane.

γ^* is characterized by the conditions

$$\langle d\gamma, \gamma^* \rangle = 0, \quad (55)$$

$$\langle \gamma, \gamma^* \rangle \equiv 1. \quad (56)$$

Note that $\gamma^{**} = \gamma$ holds. This can be seen by differentiating (56) to get $0 = d1 = d\langle \gamma, \gamma^* \rangle = \langle d\gamma, \gamma^* \rangle + \langle \gamma, d\gamma^* \rangle$. Thus (55) and (56) hold when γ and γ^* are interchanged.

Example 5.1 Let γ be the circle of radius ρ and center $(\epsilon, 0)$, where $\epsilon \geq 0$. Then,

$$\begin{aligned} \gamma &= (\epsilon + \rho \cos \theta, \rho \sin \theta), \\ N &= (\cos \theta, \sin \theta), \quad p = \rho + \epsilon \cos \theta, \\ \gamma^* &= (1/p)N = \frac{(1/\rho)}{1 + (\epsilon/\rho) \cos \theta} (\cos \theta, \sin \theta). \end{aligned}$$

This means that γ^* is a curve with polar equation $r = (1/\rho)/(1 + (\epsilon/\rho) \cos \theta)$. These equations describe all the various conic sections if one of the foci is at the origin, the directrix is the line $x = 1/\epsilon$, and ϵ/ρ is the eccentricity.

In the present context it is natural to treat all conic sections in a unified way. In order to do this, we consider the projective plane

$$P^2(\mathbf{R}) := \{X = (x_1, x_2, x_3) \in \mathbf{R}^3; X \neq (0, 0, 0)\} / \sim,$$

where \sim is the equivalence relation which is defined in the following way: $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$ if and only if $(y_1, y_2, y_3) = c(x_1, x_2, x_3)$ holds for some $c \in \mathbf{R}$. Let (ξ, η) be the orthogonal coordinates in the plane where the curve γ lies. We identify each point (ξ, η) in this plane with $(\xi, \eta, 1)$ in $P^2(\mathbf{R})$, and regard γ as a curve in $P^2(\mathbf{R})$.

Lemma 5.1 *Assume that $\kappa_\gamma(s) \neq 0$ for all s in the domain of γ . Assume also that $p = \langle \gamma, N \rangle$ has at most isolated zeros. Then, the dual curve γ^* can be regarded as a connected regular smooth curve in the projective plane $P^2(\mathbf{R})$.*

Proof. Set $\gamma^*(s) = (\varphi(s), \psi(s))$. Then

$$p = fg' - f'g, \quad \varphi = \frac{g'}{fg' - f'g}, \quad \psi = -\frac{f'}{fg' - f'g}$$

holds. Therefore, at each point s (including a point s where $p(s) = 0$ holds),

$$\gamma^*(s) = (g', -f', fg' - f'g)$$

in $P^2(\mathbf{R})$. Hence

$$(\gamma^*)'(s) = (g'', -f'', fg'' - f''g) = \kappa_\gamma(f', g', ff' + gg') \neq (\gamma^*)(s)$$

in $P^2(\mathbf{R})$. Therefore, $\gamma^*(s)$ is well-defined for all s , and it is a connected regular smooth curve in the projective plane $P^2(\mathbf{R})$. **q.e.d.**

For any smooth curve

$$\Omega(s) = (x(s), z(s)), \quad x > 0, \quad z' > 0, \quad l_1 \leq s \leq l_2 \quad (57)$$

($l_1 \leq 0 < l_2$) represented by the arc length s , and for a real constant $a \neq 0$, we recall from Section 3 that the isothermic dual is given by

$$\tilde{x}(s) := \frac{a}{x(s)}, \quad \tilde{z}(s) := -a \int_0^s \frac{z'}{x^2} ds. \quad (58)$$

Now represent the coordinate functions of the rolling curve $\Gamma : r = r(\theta)$ of Ω as

$$\xi(s) = r(s) \cos \theta(s) + \xi_0, \quad \eta(s) = r(s) \sin \theta(s) + \eta_0, \quad (59)$$

where the point (ξ_0, η_0) is the pole of Γ . Then, by elementary calculation, we obtain that the dual curve Γ^* of Γ is

$$\Gamma^* = (1/r)(\cos \theta, \sin \theta) + (1/r)_\theta(-\sin \theta, \cos \theta).$$

We compute, using (21) and (22) and the fact that s is arc length of (x, z) , and obtain

$$r_\theta = -r \frac{x'}{z'}.$$

Therefore, from (27), we obtain

$$(1/r)_\theta = \tilde{x}_z / r.$$

Therefore

$$\Gamma^* = (1/r)[(\cos \theta, \sin \theta) + \tilde{x}_z(-\sin \theta, \cos \theta)].$$

Write $\tilde{x}_s = \cos \vartheta$, $\tilde{z}_s = \sin \vartheta$, then

$$\Gamma^* = (1/(r\tilde{z}_s))(\sin(\vartheta - \theta), \cos(\vartheta - \theta)).$$

This can be expressed, because of (27), (30), (57) and (58), as

$$\Gamma^* = (-1/|a|)\tilde{x} \cdot (\sin(\vartheta - \theta), \cos(\vartheta - \theta)). \quad (60)$$

We consider the mean curvature profile $C_{\tilde{s}}(\tilde{s}) = (f(\tilde{s}), g(\tilde{s}))$ associated with the surface $\tilde{S} : (\tilde{x}(\tilde{s})e^{it}, \tilde{z}(\tilde{s}))$. We obtain from (49) and (50),

$$\frac{\tilde{z}_s}{\tilde{x}} = \frac{fg_s - f_s g}{f^2 + g^2} = -\frac{d}{d\tilde{s}}(\arctan(f/g)). \quad (61)$$

By using (61), a useful formula is obtained by inverting the equations in (49) and (50):

$$g - if = (\operatorname{sgn} \tilde{x}) \tilde{x} \exp\left(i \int_0^{\tilde{z}} \frac{d\tilde{z}}{\tilde{x}}\right). \quad (62)$$

We obtain, from (46) and (61),

$$\tilde{k}_2 = -\frac{d}{d\tilde{s}}(\arctan(f/g)).$$

Also, it follows from (46) and the definition of ϑ above that $\vartheta_{\tilde{s}} = \tilde{k}_1$ holds, and therefore

$$\theta = \int_0^{\tilde{s}} 2\tilde{H} d\tilde{s} = \int_0^{\tilde{s}} (\tilde{k}_1 + \tilde{k}_2) d\tilde{s} = \vartheta - \arctan(f/g) + c_3,$$

where c_3 is a constant. This gives

$$\vartheta - \theta = \arctan(f/g) - c_3.$$

Using this in (60), gives

$$\begin{aligned} \Gamma^* &= (\operatorname{sgn} a)(-1/|a|)\sqrt{f^2 + g^2} \cdot [(\sin(\arctan(f/g) - c_3), \cos(\arctan(f/g) - c_3))] \\ &= (\operatorname{sgn} a)(-1/|a|)\sqrt{f^2 + g^2} \left[\begin{pmatrix} \cos c_3 & -\sin c_3 \\ \sin c_3 & \cos c_3 \end{pmatrix} \begin{pmatrix} \sin(\arctan(f/g)) \\ \cos(\arctan(f/g)) \end{pmatrix} \right]^t \\ &= (\operatorname{sgn} a)(-1/|a|) \left[\begin{pmatrix} \cos c_4 & -\sin c_4 \\ \sin c_4 & \cos c_4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right]^t, \end{aligned} \quad (63)$$

where $c_4 = c_3$ or $c_3 + \pi$. Therefore, the curve Γ^* is a homothety of a rotation of the curve $C_{\tilde{S}}$.

The above discussion and Remark 3.1, and Lemma 5.1 prove the following:

Theorem 5.1 *Let S be a smooth surface of revolution generated by the curve*

$$\Omega(s) = (x(s), z(s)), \quad x > 0, \quad l_1 \leq s \leq l_2$$

($l_1 \leq 0 < l_2$) represented by the arc length s and with the z -axis as the rotation axis. Assume that the surface S restricted to Ω has at most isolated umbilics. Assume also that z' has at most isolated zeros. Also we assume that $z' \neq 0$ at any umbilic of S . And regard Γ as a curve in the projective plane if z' has isolated zeros. Then the rolling curve Γ of Ω is a piecewise C^1 curve which is smooth away from the umbilics of S . Define $\tilde{\Omega} = (\tilde{x}, \tilde{z})$ by (58). And let $C_{\tilde{S}}$ be the mean curvature profile associated with the surface \tilde{S} of revolution generated by $\tilde{\Omega}$. Then, the rolling curve Γ of Ω satisfies, up to homothety,

$$\Gamma = C_{\tilde{S}}^*,$$

i.e. Γ is (up to homothety) dual to $C_{\tilde{S}} = (f, g)$ with respect to the origin of the coordinate plane. Moreover, the pole of the rolling construction is the origin in this plane. Furthermore, Γ is a closed curve if

$$\frac{x(l_1)}{z'(l_1)} = \frac{x(l_2)}{z'(l_2)}, \quad \int_{l_1}^{l_2} (k_2 - k_1) ds = 2n\pi$$

holds for some integer n .

Remark 5.1 Each zero of \tilde{H} corresponds to an umbilic point of S , and vice versa.

Remark 5.2 In [7] Kenmotsu studied periodic surfaces of revolution with a prescribed mean curvature function. He classified the periodic surfaces of revolution into two classes. One class, which appears in Theorem 2 of [7], has a mean curvature function for a one parameter family of periodic surfaces of revolution. In the class in Theorem 3 of the same paper, each mean curvature function has only an isolated periodic surface of revolution. It can be shown that the surfaces appearing in Theorem 2 are exactly those periodic surfaces whose rolling curves are closed while those appearing in Theorem 3 are exactly those periodic surfaces whose rolling curves are not closed.

6 Rolling curve of the Wulff shape

We assume that the Wulff shape W is a smooth convex surface which is rotationally symmetric with respect to the vertical axis (z -axis). We consider the rolling curve $\Gamma : r = r(\theta)$ of the generating curve Ω_W of W , and we will obtain a sufficient condition so that Γ is smooth.

As usual, we adopt the notations that appeared in (2), (3) and (4) for the generating curve Ω_W of W . We denote by “ ’ ” the derivative with respect to σ . We denote by κ the curvature of Ω_W with respect to the inward pointing normal, that is

$$\kappa := -u''v' + u'v''.$$

We will prove

Theorem 6.1 *If either (i) $\kappa' < 0$ in $-L_1 < \sigma < 0$ and $\kappa' > 0$ in $0 < \sigma < L_2$, or (ii) $\kappa' > 0$ in $-L_1 < \sigma < 0$ and $\kappa' < 0$ in $0 < \sigma < L_2$, then the rolling curve Γ of the interior*

$$(u(\sigma), v(\sigma)), \quad -L_1 < \sigma < L_2$$

of Ω_W is a smooth arc. Moreover, in the former case, $r' > 0$ in $-L_1 < \sigma < 0$ and $r' < 0$ in $0 < \sigma < L_2$. In the latter case, $r' < 0$ in $-L_1 < \sigma < 0$ and $r' > 0$ in $0 < \sigma < L_2$.

In order to prove Theorem 6.1, we prepare a lemma. We write

$$\mu_1 = \kappa = -u''v' + u'v'', \quad \mu_2 = u^{-1}v_\sigma$$

as usual.

Lemma 6.1 *If $\kappa' < 0$ in $-L_1 < \sigma < 0$ and $\kappa' > 0$ in $0 < \sigma < L_2$, then*

$$\mu_1 < \mu_2, \quad -L_1 < \sigma < L_2.$$

If $\kappa' > 0$ in $-L_1 < \sigma < 0$ and $\kappa' < 0$ in $0 < \sigma < L_2$, then

$$\mu_1 > \mu_2, \quad -L_1 < \sigma < L_2.$$

Therefore, in both cases, the Wulff shape W has umbilic points only at $(0, 0, \min v)$, $(0, 0, \max v)$.

Proof. It is clear that

$$\mu_1 = \mu_2 \quad \text{at } \sigma = -L_1, L_2. \quad (64)$$

We compute

$$\mu'_2 = \frac{v''u - v'u'}{u^2} = \frac{\kappa u'u - v'u'}{u^2} = \frac{u'}{u}(\mu_1 - \mu_2). \quad (65)$$

First assume that $\kappa' < 0$ holds in $-L_1 < \sigma < 0$ and $\kappa' > 0$ holds in $0 < \sigma < L_2$. Set

$$f(\sigma) := u^2(\sigma) + v^2(\sigma).$$

We claim that $f(\sigma)$ is a strictly decreasing function of σ in $-L_1 \leq \sigma \leq 0$ and a strictly increasing function in $0 \leq \sigma \leq L_2$. In fact, this is proved by a similar way to the proof of Lemma 3.2 in [11]. This implies that

$$\mu_1 < \mu_2 \quad \text{at } \sigma = 0.$$

Assume

$$\mu_1(\sigma_0) - \mu_2(\sigma_0) \geq 0 \quad (66)$$

holds for some point $\sigma_0 \in (0, L_2)$. Then, because of (64), there exists some $\sigma_1 \in (0, L_2)$ such that

$$\mu_1(\sigma_1) - \mu_2(\sigma_1) \geq 0, \quad \mu'_1(\sigma_1) - \mu'_2(\sigma_1) \leq 0 \quad (67)$$

holds. On the other hand, (65) with the first inequality of (67) implies that

$$\mu'_2(\sigma_1) \leq 0 \quad (68)$$

holds. This with the second inequality of (67) implies that

$$\mu'_1(\sigma_1) \leq \mu'_2(\sigma_1) \leq 0$$

holds. This contradicts the assumption

$$\mu'_1 = \kappa' > 0, \quad \text{in } 0 < \sigma < L_2.$$

Therefore, $\mu_1 < \mu_2$ in $0 \leq \sigma < L_2$. Similarly we can prove that $\mu_1 < \mu_2$ in $-L_1 \leq \sigma < 0$. The proof for the latter case is similar. **q.e.d.**

Proof of Theorem 6.1. Lemma 6.1 combined with Lemma 3.1 implies the first half of Theorem 6.1.

We will prove the second half. We will prove the result only for the first case. The proof of the second case is similar. We know from (21),

$$r = \frac{u}{v'}. \quad (69)$$

Hence, we have

$$r' = \frac{-u'}{(v')^2}(\kappa u - v'). \quad (70)$$

Set

$$g := \kappa u - v'.$$

Then,

$$\begin{aligned} g(-L_1) &= g(L_2) = 0, \\ g' &= \kappa' u + \kappa u' - v'' = \kappa' u. \end{aligned}$$

Hence, in the first case, $g < 0$ in $-L_1 < \sigma < L_2$. This with (70) implies the result. **q.e.d.**

Figure 17 shows the rolling curve for the Wulff shape having profile curve $u^4 + v^4 = 1$. Although the Wulff shape W is smooth, the rolling curve has cusps at points which correspond to umbilics of W . The rolling construction continues to work in this case as long as the curve is rolled as described in section 3.

7 Applications to anisotropic Delaunay surfaces

In this section we will apply the results of the previous sections when Ω is an anisotropic Delaunay curve.

We assume that the Wulff shape W is rotationally symmetric with respect to the vertical axis. This means that, from (19) in §2, there exists a positive function μ_2 of one variable such that the surface $S : (x(s)e^{it}, z(s))$ is an anisotropic Delaunay surface if and only if S satisfies

$$2\mu_2^{-1}(-x')z'x + \Lambda x^2 = c \quad (71)$$

for some constants $\Lambda \leq 0$ and $c \in \mathbf{R}$. Equation (71) determines the surface uniquely up to translation along z direction.

As usual, we adopt the notations that appeared in (2), (3) and (4) for the generating curve Ω_W of W .

First we apply the ‘dual curve method’ constructed in §3 - §5 to anisotropic Delaunay curves Ω . Denote by S the surface of revolution obtained from the curve Ω by rotating it around z -axis. Denote by \tilde{S} the surface of revolution obtained from the curve $(\tilde{x}(s), \tilde{z}(s))$ by rotating it around z -axis, where \tilde{x} and \tilde{z} are defined as in (23). Then, it is clear that, if S is a horizontal plane, then \tilde{S} is also a horizontal plane, and, if S is a vertical cylinder, then \tilde{S} is also a vertical cylinder.

Lemma 7.1 *If S is the Wulff shape, then \tilde{S} is an anisotropic catenoid with flux parameter $c = -2|a|$.*

Proof. In this case $\Lambda = -2$ holds. Now it is easy to see that $2\mu_2^{-1}\tilde{z}_{\tilde{s}}\tilde{x} = -2|a|$ holds. **q.e.d.**

Lemma 7.2 *Assume that S is an anisotropic catenoid with flux parameter c . Then \tilde{S} is homothetic to the Wulff shape W . In fact, \tilde{S} is $2|a/c|$ times W .*

Proof. It is easy to see that $2\mu_2^{-1}\tilde{z}_{\tilde{s}}\tilde{x} + (c/|a|)\tilde{x}^2 = 0$ holds. **q.e.d.**

Before we study the isothermic duals of an anisotropic unduloid and nodoid, we give a lemma.

Lemma 7.3 $\tilde{z} = \int \tilde{x}_u dv$ holds.

Proof. We compute, by using (16) and (23),

$$\begin{aligned}\tilde{z} &= \int d\tilde{z} = \int \frac{d\tilde{z}}{ds} \frac{ds}{dz} dz = -a \int \frac{z'}{x^2} (z')^{-1} x_u dv \\ &= - \int \frac{a}{x^2} x_u dv = \int \tilde{x}_u dv.\end{aligned}$$

q.e.d.

Lemma 7.4 *Assume that S is an anisotropic unduloid \mathcal{U} , that is, S satisfies (71) for some constants $\Lambda < 0$ and $c > 0$. Set*

$$\tilde{x}(s) := \frac{c}{|\Lambda|x(s)}, \quad (72)$$

$$\tilde{z}(s) := -\frac{c}{|\Lambda|} \int_0^s \frac{z'}{x^2} ds. \quad (73)$$

Then, \tilde{S} is an anisotropic unduloid $\tilde{\mathcal{U}}$ which is a translation of \mathcal{U} along z direction, that is, an anisotropic unduloid is isothermic self-dual. Moreover, a positively (resp. negatively) curved part of \mathcal{U} corresponds to a negatively (resp. positively) curved part of $\tilde{\mathcal{U}}$.

Proof. By using (27) and (71), we obtain

$$-2\mu_2^{-1}(\tilde{x}_{\tilde{s}})\tilde{x}\tilde{z}_{\tilde{s}} + \Lambda\tilde{x}^2 = c. \quad (74)$$

Changing the parameter so that $\tau := -\tilde{s}$, we obtain

$$2\mu_2^{-1}(-\tilde{x}_{\tau})\tilde{x}\tilde{z}_{\tau} + \Lambda\tilde{x}^2 = c. \quad (75)$$

The equation (75) means that $(\tilde{x}(\tau), \tilde{z}(\tau))$ generates the same anisotropic unduloid as $(x(s), z(s))$. Therefore, $(\tilde{x}(\tilde{s}), \tilde{z}(\tilde{s}))$ generates the same anisotropic unduloid as $(x(s), z(s))$.

If S is a positively (resp. negatively) curved part of the anisotropic unduloid \mathcal{U} , then,

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda} \quad \left(\text{resp. } x = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda} \right) \quad (76)$$

holds. We obtain, by using (72),

$$\tilde{x} = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda} \quad \left(\text{resp. } \tilde{x} = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda} \right) \quad (77)$$

holds. This with Lemma 7.3 implies that \tilde{S} is a negatively (resp. positively) curved part of \mathcal{U} . **q.e.d.**

Lemma 7.5 *Assume that S is an anisotropic nodoid \mathcal{N} , that is, S satisfies (71) for some constants $\Lambda < 0$ and $c < 0$. Set*

$$\tilde{x}(s) := -\frac{c}{\Lambda x(s)}, \quad (78)$$

$$\tilde{z}(s) := \frac{c}{\Lambda} \int_0^s \frac{z'}{x^2} ds. \quad (79)$$

Then, \tilde{S} is an anisotropic nodoid $\tilde{\mathcal{N}}$ which is a translation of \mathcal{N} along z direction, that is, an anisotropic nodoid is isothermic self-dual. Moreover, a positively (resp. negatively) curved part of \mathcal{N} corresponds to a negatively (resp. positively) curved part of $\tilde{\mathcal{N}}$.

Proof. By using (27) and (71), we obtain

$$2\mu_2^{-1}(-\tilde{x}_{\tilde{s}})\tilde{x}\tilde{z}_{\tilde{s}} + \Lambda\tilde{x}^2 = c. \quad (80)$$

The equation (80) means that $(\tilde{x}(\tilde{s}), \tilde{z}(\tilde{s}))$ generates the same anisotropic nodoid as $(x(s), z(s))$.

Note that, if $(x(s), z(s))$ generates a positively (resp. negatively) curved part of the anisotropic nodoid, then

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad u > 0 \quad (\text{resp. } u < 0)$$

holds. We obtain, by using (78),

$$\tilde{x} = -\frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad u > 0 \quad (\text{resp. } u < 0).$$

This with Lemma 7.3 implies that (\tilde{x}, \tilde{z}) generates a negatively (resp. positively) curved part of $\tilde{\mathcal{N}}$. **q.e.d.**

Applying Theorem 5.1 to an anisotropic catenoid, using Lemmas 7.2 and 4.1, we obtain the following:

Theorem 7.1 *Let $\Omega_W(\sigma)$ ($-L_1 \leq \sigma \leq L_2$) be the generating curve of W as usual. We assume that the mean curvature $H_W(\sigma)$ of W regarded as a function of σ has only isolated zeroes. Let $C_W(\sigma)$ denote the mean curvature profile associated with W . Let Ω be the generating curve of an anisotropic catenoid with flux parameter c normalized by $c = 2$ and denote its rolling curve with z -axis as base by Γ . Then the curve Γ is obtained as the dual curve of C_W , that is*

$$\Gamma = (C_W)^* .$$

Here, the pole of the rolling construction is the origin in the coordinate plane where C_W is expressed as $C_W(\sigma) = (f(\sigma), g(\sigma))$ which satisfies (49) and (50) for $(x(s), z(s)) = (u(\sigma), v(\sigma))$. In addition, Γ is a piecewise C^1 curve which is smooth away from those points which correspond to the zeroes of $H_W(\sigma)$ under the anisotropic Gauss map.

Figures 9 through 12 show an anisotropic catenary, an anisotropic parabola (the rolling curve) and the mean curvature profile for the functional whose Wulff shape is generated by a curve

$$|u|^p + |v|^p = 1 \tag{81}$$

with $p = 4$.

Applying Theorem 5.1 to an anisotropic unduloid and nodoid, using Lemmas 7.4, 7.5 and 4.1, we obtain the following:

Theorem 7.2 *Let $\Omega : (x(s), z(s))$, ($-l \leq s \leq l$), be one period (from a bulge to the next bulge) of the generating curve of an anisotropic unduloid \mathcal{U} (resp. anisotropic nodoid \mathcal{N}). Denote by $H_\Omega(s)$ the restriction of the mean curvature of \mathcal{U} (resp. \mathcal{N}) to $\Omega(s)$. We assume that $H_\Omega(s)$ vanishes only at isolated points. Define a curve $C_\Omega(s) = (f(s), g(s))$ using (47) with $c_1 = 0$ and $c_2 = -B$, where B is the radius of the bulge. Then, C_Ω is the mean curvature profile associated with \mathcal{U} (resp. \mathcal{N}). And the dual curve C_Ω^* of C_Ω , with respect to the origin, is a homothety of the rolling curve of one period (from a neck to the next neck) of the generating curve of \mathcal{U} (resp. \mathcal{N}) with z -axis as base. Here, the pole of the rolling construction is the origin in the coordinate plane where C_Ω is expressed as $C_\Omega(s) = (f(s), g(s))$. If we normalize \mathcal{U} (resp. \mathcal{N}) so that $|c/\Delta| = 1$ holds, then C_Ω^* is the rolling curve of one period (from a neck to the next neck) of the generating curve of \mathcal{U} (resp. \mathcal{N}) (up to rigid motion).*

Figures 13 and 14 show the rolling curves for anisotropic unduloids having Wulff shape of the form (81). Figure 15 shows the rolling curve of an anisotropic nodoid having Wulff shape of the form (81). Moreover, Figure 18 demonstrates the rolling construction for an anisotropic unduloid.

In Theorem 7.2 we assumed that the mean curvature of the anisotropic Delaunay curve has at most isolated zeros. Here we give sufficient conditions for the Wulff shape so that this assumption is satisfied.

Lemma 7.6 *Assume that the curvature κ of the generating curve Ω_W of W satisfies (i) in Theorem 6.1. Then, the mean curvature H of an anisotropic unduloid with respect to the outward pointing normal is everywhere negative.*

Proof. In Lemma 3.1, we obtained that the curvature κ_Γ of the rolling curve Γ of a general curve $(x(s), z(s))$ is given by

$$\kappa_\Gamma = \frac{-(z')^3}{x|\kappa x - z'|} = \frac{-(z')^3}{2|a\tilde{H}|}. \quad (82)$$

Now let $(x(s), z(s))$ be an anisotropic undulary. Then,

$$x = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad \Lambda > 0, \quad c > 0.$$

Set

$$k_1 := -x'z'' + x''z', \quad k_2 := -z'/x.$$

We compute:

$$k_1 = -\mu_1\sigma_s, \quad x\sigma_s = \pm\sqrt{u^2 + \Lambda c}, \quad z' = u\mu_2, \quad k_2 = -\mu_2\frac{u}{x},$$

where

$$\mu_1 = \kappa = u_\sigma v_{\sigma\sigma} - u_{\sigma\sigma} v_\sigma, \quad \mu_2 = v_\sigma/u.$$

Hence, we obtain

$$H := (k_1 + k_2)/2 = -\frac{1}{2x}(\pm\mu_1\sqrt{u^2 + \Lambda c} + \mu_2u). \quad (83)$$

Now, by the assumption on κ and Lemma 6.1,

$$0 < \mu_1 \leq \mu_2, \quad -L_1 < \sigma < L_2 \quad (84)$$

holds. (83) combined with (84) implies $H < 0$. **q.e.d.**

Lemma 7.7 *The mean curvature H of an anisotropic nodoid with respect to the outward pointing normal is everywhere negative, if*

$$\Lambda c > \max\left\{\frac{\mu_2^2 - \mu_1^2}{\mu_1^2}u^2\right\}$$

holds. Especially, if the curvature κ of the generating curve Ω_W of W satisfies (ii) in Theorem 6.1, then $H < 0$.

Proof. Let $(x(s), z(s))$ be an anisotropic nodary. Then,

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad \Lambda < 0, \quad c < 0.$$

As in the proof of Lemma 7.6, we obtain

$$H = -\frac{1}{2x}(\mu_1\sqrt{u^2 + \Lambda c} + \mu_2 u). \quad (85)$$

Therefore, if $u \geq 0$, then $H < 0$.

Suppose $u < 0$. Then, $\mu_1\sqrt{u^2 + \Lambda c} + \mu_2 u > 0$ holds if and only if

$$\Lambda c > \frac{\mu_2^2 - \mu_1^2}{\mu_1^2} u^2 \quad (86)$$

holds. This implies the first half of the lemma.

Now, assume (ii) in Theorem 6.1. Then by Lemma 6.1,

$$\mu_1 \geq \mu_2 > 0, \quad -L_1 < \sigma < L_2 \quad (87)$$

holds. Hence, (86) is satisfied. **q.e.d.**

We will now briefly discuss how these theorems apply to the classical Delaunay surfaces. Let $(x(s)e^{it}, z(s))$ be a (CMC) catenoid. We make the normalization $c = 2$. We take $a = 1$. Then, (\tilde{x}, \tilde{z}) is a half circle with radius 1 and center in the z -axis, and $\tilde{H} \equiv -1$. We may assume $\tilde{z}_s(0) = 1$. Then, by Lemma 4.1, $c_1 = 0$, $c_2 = -\tilde{x}(0) = -1$. Hence, (f, g) is a circle with radius $1/2$ and center at $(0, 1/2)$. Therefore, the rolling curve Γ (the dual curve of (f, g)) is a parabola.

We will apply Theorem 7.2 to the classical unduloid. Let $(x(s)e^{it}, z(s))$ be one period (from a bulge to the next bulge) of an unduloid with mean curvature $H = \Lambda/2 < 0$. We take $a = |c/\Lambda| = |c/(2H)|$. We normalize $|c/\Lambda| = |c/(2H)| = 1$. Then, (\tilde{x}, \tilde{z}) gives the same undulary as $(x(s), z(s))$ (up to translation). We may assume $z'(0) = 1$. Then, $B = (1 + \sqrt{1 - 2|Hc|})/|2H|$. Since the mean curvature of (x, z) is constant H ,

$$f(s) = \int_0^s \cos(2Hs) ds = \frac{1}{2H} \sin(2Hs),$$

$$g(s) = \int_0^s \sin(2Hs) ds + B = -\frac{1}{2H} \cos(2Hs) + \frac{\sqrt{1 - 2|Hc|}}{2|H|}.$$

Therefore, (f, g) is a circle with radius $1/|2H|$ and center at $(0, \sqrt{1 - 2|Hc|}/|2H|)$. The origin is inside of the circle (f, g) . Therefore, the rolling curve Γ (the dual curve of (f, g)) is an ellipse.

Finally, we apply Theorem 7.2 to the nodoid. Let $(x(s)e^{it}, z(s))$ be one period (from a bulge to the next bulge) of a nodoid with mean curvature $H = \Lambda/2 < 0$. We take $a = -|c/\Lambda| = -|c/(2H)|$. We normalize $|c/\Lambda| = |c/(2H)| = 1$. Then, (\tilde{x}, \tilde{z}) gives the same nodary as $(x(s), z(s))$ (up to translation). We

may assume $z'(0) = 1$. Then, $B = (1 + \sqrt{1 + 2|Hc|})/|2H|$. Since the mean curvature of (\tilde{x}, \tilde{z}) is constant H ,

$$f(s) = \int_0^s \cos(2Hs) ds = \frac{1}{2H} \sin(2Hs),$$

$$g(s) = \int_0^s \sin(2Hs) ds + B = -\frac{1}{2H} \cos(2Hs) + \frac{\sqrt{1 + 2|Hc|}}{2|H|}.$$

Therefore, (f, g) is a circle with radius $1/|2H|$ and center at $(0, \sqrt{1 + 2|Hc|}/|2H|)$. The origin is outside of the circle (f, g) . Therefore, the rolling curve Γ (the dual curve of (f, g)) is a hyperbola.

In concluding this section, we give expressions of the rolling curves Γ of the anisotropic undulary and nodary in terms of the Wulff shape, which are useful to draw pictures of Γ . It is sufficient to give formulas only for the ‘half’ period which corresponds to a part of the ‘upper half’ $(u(\sigma), v(\sigma))$, $0 \leq \sigma \leq L_2$ of Ω_W .

Proposition 7.1 *Let S be a ‘half’ period of an anisotropic unduloid \mathcal{U} with anisotropic mean curvature Λ and flux parameter c . Parameterize the corresponding half period of the generating curve of the Wulff shape as $u = u(v)$, $0 \leq v \leq \bar{v} := \max v$, $u \geq 0$ and let $v_1 > 0$ be defined by $u(v_1) = \sqrt{-\Lambda c}$. Then, the x coordinate of the generating curve C of S can be represented as*

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad v = 0 \quad \text{to} \quad v = v_1, \quad (88)$$

$$x = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad v = v_1 \quad \text{to} \quad v = 0. \quad (89)$$

Then, the rolling curve $\Gamma : (r, \theta)$ is given as follows:

$$r = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda} \cdot \sqrt{1 + u_v^2}, \quad v = 0 \quad \text{to} \quad v = v_1, \quad (90)$$

$$r = \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda} \cdot \sqrt{1 + u_v^2}, \quad v = v_1 \quad \text{to} \quad v = 0, \quad (91)$$

$$\theta = \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) - \int_0^v \frac{dv}{\sqrt{u^2 + \Lambda c}}, \quad v = 0 \quad \text{to} \quad v = v_1, \quad (92)$$

$$\theta = \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) + \int_{v_1}^v \frac{dv}{\sqrt{u^2 + \Lambda c}} - \int_0^{v_1} \frac{dv}{\sqrt{u^2 + \Lambda c}}, \quad v = v_1 \quad \text{to} \quad v = 0. \quad (93)$$

In particular, if W is symmetric with respect to the horizontal plane, the rolling curve of one period of \mathcal{U} is a closed curve if and only if

$$\int_0^{v_1} \frac{dv}{\sqrt{u^2 + \Lambda c}} = \frac{\pi}{2} \quad (94)$$

holds.

Proof. We have, from (21),

$$r = x/z' = x/v_\sigma.$$

On the interval in question, we have $v_\sigma^2 = (1 + u_v^2)^{-1}$, from which we obtain (90), (91).

Recall (20) and (22). We compute

$$\begin{aligned} \theta &= - \int \left(\frac{z'}{x} - \kappa \right) ds = - \int \left(\frac{v_\sigma}{x} - (u_\sigma v_{\sigma\sigma} - u_{\sigma\sigma} v_\sigma) \frac{d\sigma}{ds} \right) ds \\ &= - \int \frac{dv}{x\sigma_s} - \int \frac{u_{\sigma\sigma}}{v_\sigma} d\sigma = - \int \frac{x_u}{x} dv + \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) + \text{constant} \\ &= \mp \int \frac{dv}{\sqrt{u^2 + \Lambda c}} + \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) + \text{constant}. \end{aligned}$$

q.e.d.

In a similar way, and using Lemma 5.1, we have the following.

Proposition 7.2 *Let S be a ‘half’ period of an anisotropic nodoid \mathcal{N} with anisotropic mean curvature Λ and flux parameter c . Parameterize the upper half of the generating curve of the Wulff shape as*

$$v = v(u) \geq 0, \quad -\bar{u} \leq u \leq \bar{u} := \max u.$$

On this interval the x coordinate of the generating curve C of S can be represented as, by setting $\bar{v} := \max v$,

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad 0 \leq u \leq \bar{u}, \quad v = 0 \quad \text{to} \quad v = \bar{v}, \quad (95)$$

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, \quad -\bar{u} \leq u < 0, \quad v = \bar{v} \quad \text{to} \quad v = 0. \quad (96)$$

Then, the rolling curve $\Gamma : (r, \theta)$ is given as follows:

$$r = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda} \cdot \sqrt{1 + u_v^2}, \quad 0 < u \leq \bar{u}, \quad v = 0 \quad \text{to} \quad v = \bar{v}, \quad (97)$$

$$r = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda} \cdot \sqrt{1 + u_v^2}, \quad -\bar{u} \leq u < 0, \quad v = \bar{v} \quad \text{to} \quad v = 0, \quad (98)$$

$$\theta = \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) - \int_0^v \frac{dv}{\sqrt{u^2 + \Lambda c}}, \quad v = 0 \quad \text{to} \quad v = \bar{v}, \quad (99)$$

$$\theta = 2\pi - \arcsin\left(\frac{|u_v|}{\sqrt{1 + u_v^2}}\right) - \int_0^v \frac{dv}{\sqrt{u^2 + \Lambda c}}, \quad v = \bar{v} \quad \text{to} \quad v = 0, \quad (100)$$

where $\arcsin(|u_v|/\sqrt{1 + u_v^2})$ takes values in the closed interval $[0, \pi/2]$. In particular, the total variation of θ for one period of \mathcal{N} is 2π , and therefore its rolling curve is a closed curve (in the projective plane).

8 Characterization of anisotropic Delaunay curves

What surfaces of revolution arise as anisotropic Delaunay surfaces? Here, using isothermic duality, we characterize the surfaces which can arise as anisotropic unduloids and nodoids without making explicit reference to the functional.

Let

$$\Omega(s) = (x(s), z(s)), \quad x(s) > 0$$

be a smooth curve with arc length s which is periodic with period L in the following sense: x and z satisfy

$$x(s + L) = x(s), \quad \forall s \in \mathbf{R},$$

$$z(s + L) - z(s) = z(L) - z(0), \quad \forall s \in \mathbf{R}.$$

We assume that one period of Ω contains a unique local maximum and a unique local minimum of x , which will be called a bulge and a neck of Ω , respectively. We also assume that Ω satisfies either the following condition (I) or (II):

(I) There is only one inflection point between a bulge and the next neck, and there is no zero of z' .

(II) There is no inflection point, and there is only one zero of z' between a bulge and the next neck.

We may assume that $s = 0$ corresponds to a bulge, $s = -l_1, l_2$ ($l_1, l_2 > 0$) correspond to the next necks, and $z(0) = 0$. We denote by $-s_I, s_J$ ($-l_1 < -s_I < 0 < s_J < l_2$) the unique point in (I) or (II) above.

Define a curve $\tilde{\Omega}(s) := (\tilde{x}(s), \tilde{z}(s))$,

$$\tilde{x}(s) := \frac{a}{x(s)}, \quad \tilde{z}(s) := -a \int_0^s \frac{z'}{x^2} ds, \quad (101)$$

as before. Here, a is a positive constant if Ω satisfies (I), and a is a negative constant if Ω satisfies (II).

Theorem 8.1 *If*

$$\Omega(\mathbf{R}) = \tilde{\Omega}(\mathbf{R}) \quad (102)$$

holds up to vertical translation and reflection with respect to the vertical axis (z -axis), then Ω is an anisotropic undulary or nodary for some anisotropic surface energy $\mathcal{F} = \int F(\nu_3) d\Sigma$ with rotationally symmetric energy integrand.

Conversely, if Ω is an anisotropic undulary or nodary for an anisotropic surface energy $\mathcal{F} = \int F(\nu_3) d\Sigma$ with rotationally symmetric energy integrand, then Ω is periodic, either (I) or (II) holds, and $\Omega(\mathbf{R}) = \tilde{\Omega}(\mathbf{R})$ holds for some constant $a \neq 0$ up to vertical translation.

Proof. We need to prove only the first half. We know, from (27), it holds that

$$(\tilde{x}_{\tilde{s}}, \tilde{z}_{\tilde{s}}) = \begin{cases} (-x', -z'), & a > 0, \\ (x', z'), & a < 0. \end{cases} \quad (103)$$

First we assume that (I) is satisfied. We may assume that $z'(s) > 0$ holds for all s . The range $z'([-l_1, 0])$ is a closed interval $[\beta_1, 1]$, and $z'([0, l_2])$ is a closed interval $[\beta_2, 1]$ for some $\beta_1, \beta_2 \in (0, 1)$. Each point satisfying $x' > 0$ and $z' = \beta_1$ (resp. $x' < 0$ and $z' = \beta_2$) corresponds to the inflection point $s = -s_I$ (resp. $s = s_J$). We may assume that $s = 0$ corresponds to $\tilde{s} = 0$. Because of (103) with $a > 0$, the range $\tilde{z}_{\tilde{s}}([-l_1, 0])$ is a closed interval $[-1, -\beta_1]$, and $\tilde{z}_{\tilde{s}}([0, l_2])$ is a closed interval $[-1, -\beta_2]$.

Then, at $\tilde{s} = l_2 - s_J$,

$$x(s_J) = \tilde{x}(l_2 - s_J), \quad (x'(s_J), z'(s_J)) = (-\tilde{x}_{\tilde{s}}(l_2 - s_J), -\tilde{z}_{\tilde{s}}(l_2 - s_J))$$

holds. Therefore, $a = x^2(s_J)$ holds. Similarly, we obtain $a = x^2(-s_I)$. Hence,

$$a = x^2(s_J) = x^2(-s_I)$$

must hold. Now set

$$\tilde{\mu}_2 := \frac{2z'}{x + \tilde{x}}.$$

Then, $\tilde{\mu}_2$ can be regarded as a function of $\tilde{\nu}_3 := -\tilde{x}_{\tilde{s}}$. For any negative number Λ , set

$$\mu_2(\nu_3) := (-\Lambda)^{-1} \tilde{\mu}_2(-\tilde{\nu}_3), \quad c := -a\Lambda.$$

Then,

$$2\mu_2^{-1}z'x + \Lambda x^2 = c, \quad c > 0$$

holds. Set

$$u := \mu_2^{-1}z', \quad v := \int_0^z \frac{z'}{x'} u' ds.$$

Then, it is easy to see that $z = \int x_u dv$ holds. Denote arc length of (u, v) by σ . Then, by elementary computation, we obtain

$$(d\sigma/ds)^2 = (u')^2/(x')^2.$$

Therefore, we obtain

$$dv/d\sigma = (dv/ds)(ds/d\sigma) = z'.$$

Hence,

$$2\mu_2^{-1}v_\sigma u - 2u^2 = 0 \tag{104}$$

holds. (104) is the equation for the generating curve of the Wulff shape. Now, u and v can be regarded as functions of ν_3 for $x'(s_J) \leq \nu_3 \leq x'(-s_I)$. We compute

$$u'(s)v''(s) - u''(s)v'(s) = (u'(s))^2(x')^{-2}(x'z'' - x''z').$$

Since the arc $(x(s), z(s))$ ($-s_I \leq s \leq s_J$) is convex with respect to the inward normal, $u'(s)v''(s) - u''(s)v'(s) > 0$. Therefore, (u, v) is also convex. (u, v) can

be extended to a convex closed curve, which we will denote by (u, v) also. Then, (x, z) is an anisotropic undulary with constant anisotropic mean curvature Λ for a Wulff shape generated by (u, v) .

If (II) is satisfied, then, in a similar way, we can prove that (x, z) is an anisotropic nodary with constant anisotropic mean curvature Λ for a Wulff shape generated by a convex closed curve (u, v) . **q.e.d.**

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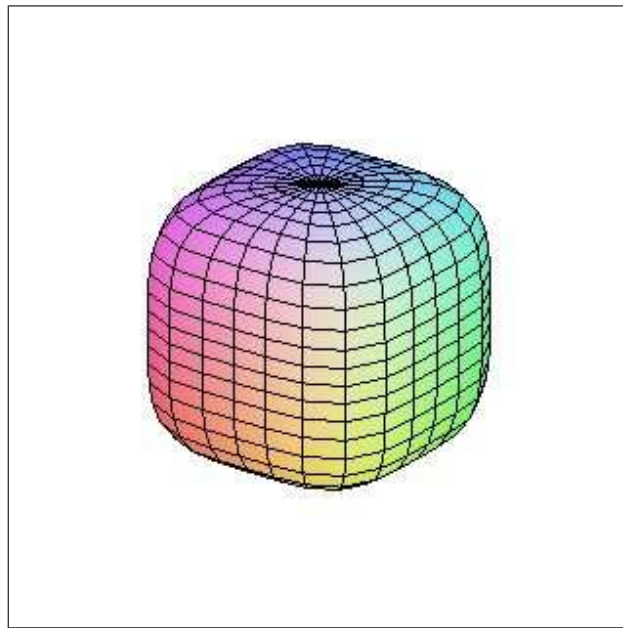


Figure 1: This Wulff shape is the superellipsoid $u^4 + v^4 + w^4 = 1$.

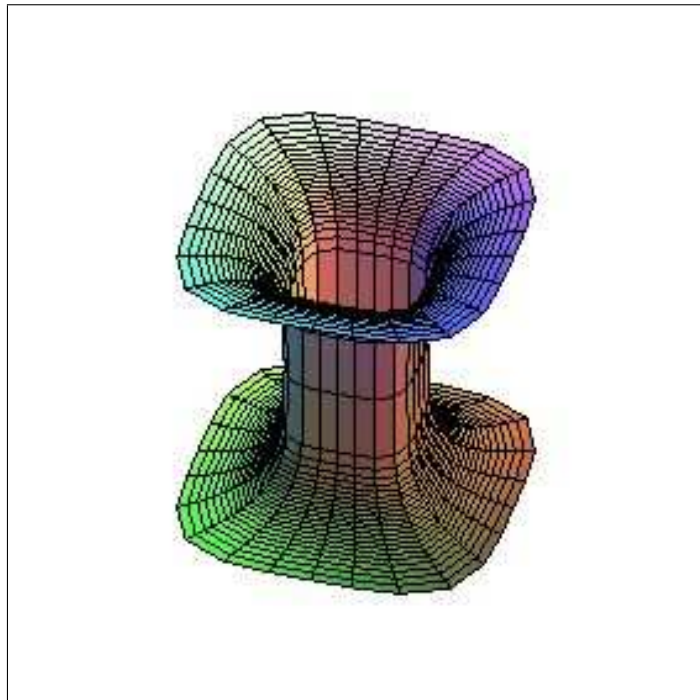


Figure 2: A generalized anisotropic catenoid for the Wulff shape shown above, $c = 1$.

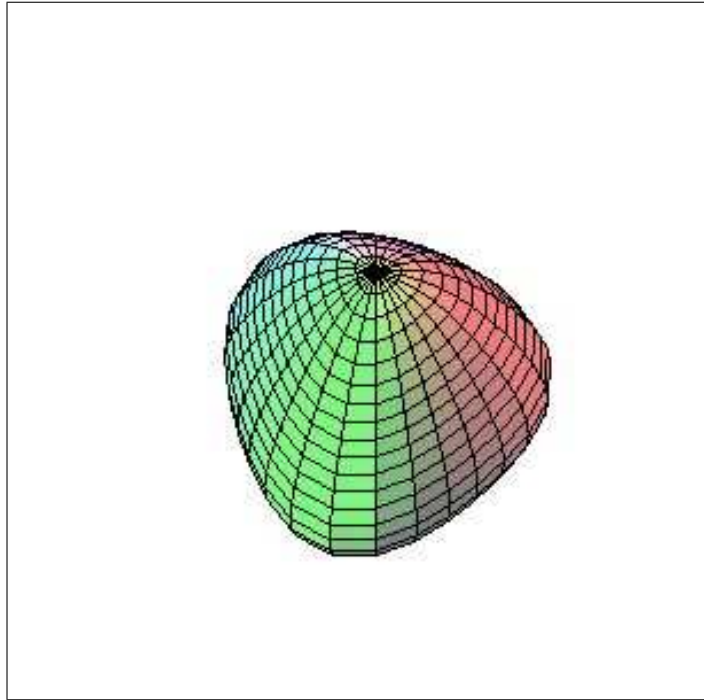


Figure 3: This Wulff shape is generated from the superformula [3]. $r = \{ |\cos(\frac{m\theta}{4})|^{n_2} + |\sin(\frac{m\theta}{4})|^{n_3} \}^{-n_1}$ with $(m, n_1, n_2, n_3) = (3, 3.2, 3, 3)$.

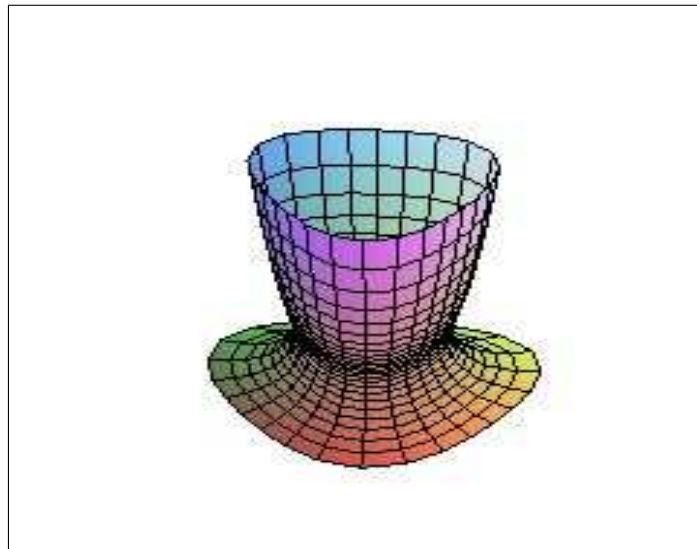


Figure 4: A generalized anisotropic catenoid for the Wulff shape shown above.

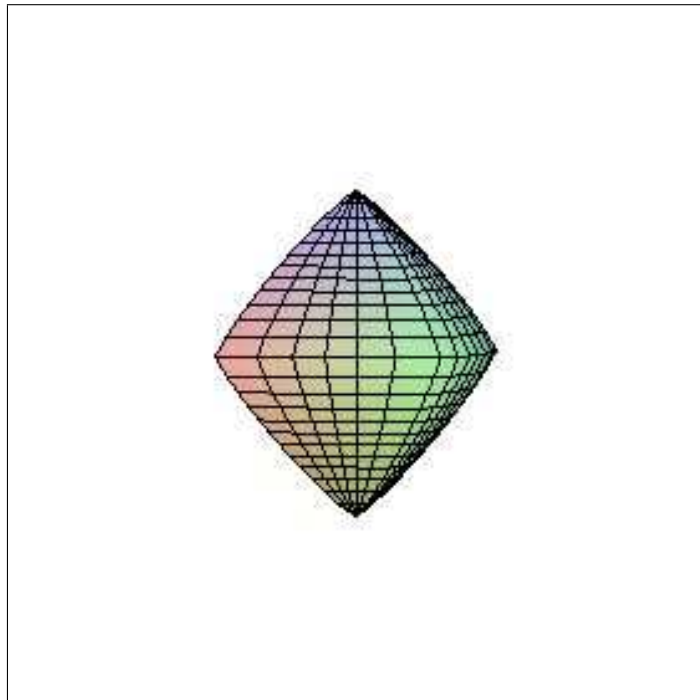


Figure 5: This Wulff shape is the subellipsoid $|u|^{8/7} + |v|^{8/7} + |w|^{8/7} = 1$.

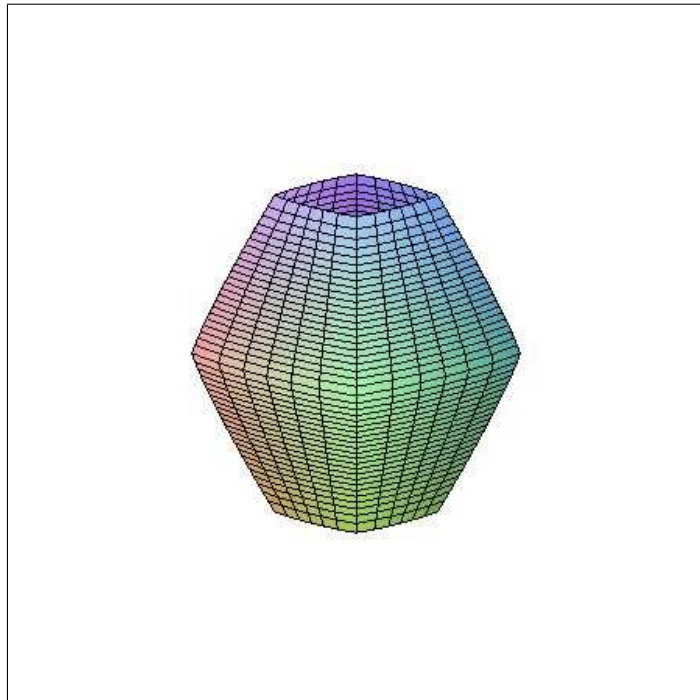


Figure 6: The convex part of a generalized anisotropic unduloid for the Wulff shape shown above. $\Lambda = -0.5$, $c = 1$.

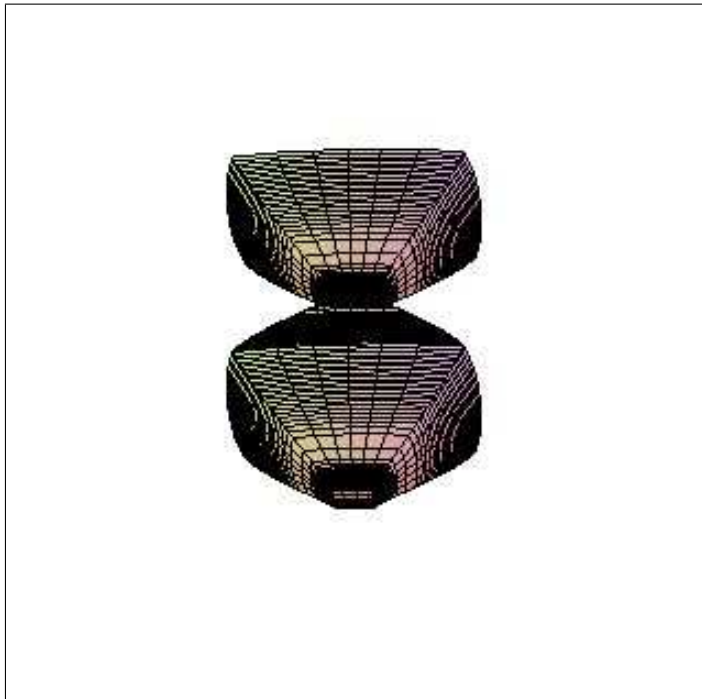


Figure 7: More of the same surface.

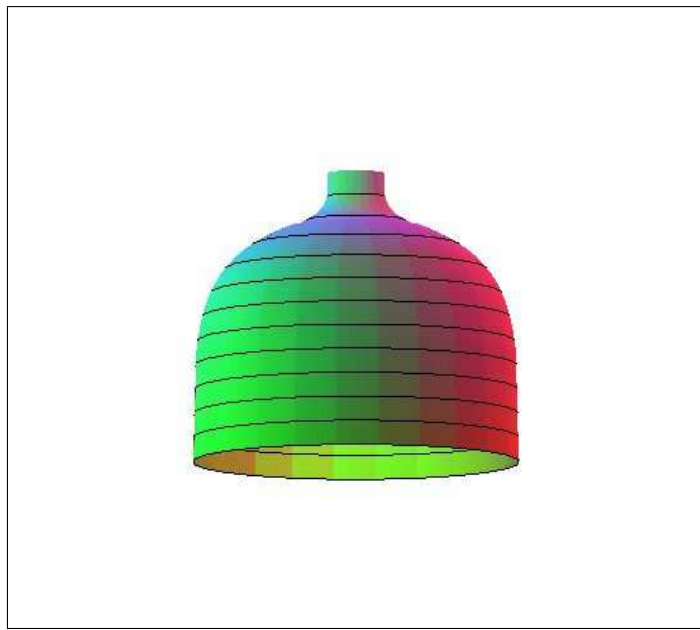


Figure 8: A half period of an anisotropic unduloid with Wulff shape $u^4 + v^4 = 1$

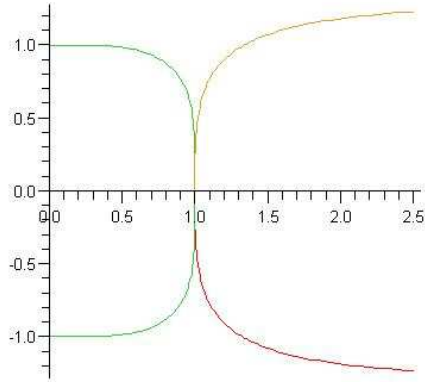


Figure 9: The profile curves of a Wulff shape defined by $u^4 + v^4 = 1$ (left) and an anisotropic catenoid. These surfaces are isothermic dual to each other.

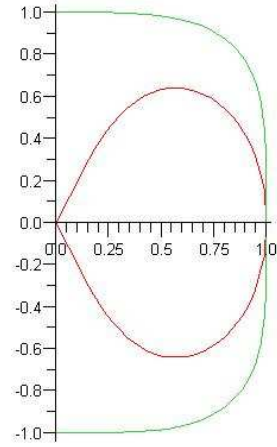


Figure 10: The profile curve of a Wulff shape W defined by $u^4 + v^4 = 1$ (outer curve) and the mean curvature profile associated with W .

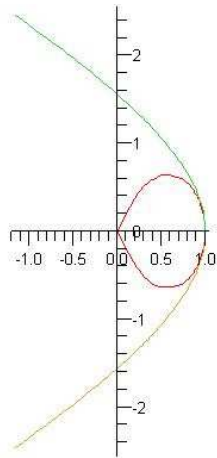


Figure 11: The same mean curvature profile as one in Figure 10 and its dual curve. The dual curve is an anisotropic parabola: the rolling curve of an anisotropic catenoid in Figure 9 with the origin as the pole.

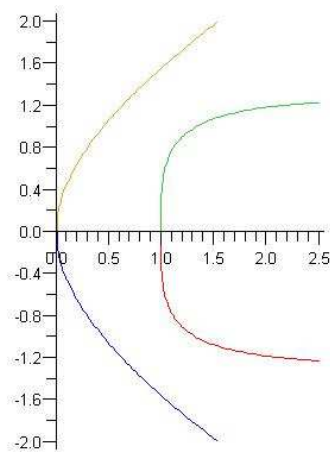


Figure 12: anisotropic catenary (right) and anisotropic parabola. The base of the rolling construction is the vertical axis.

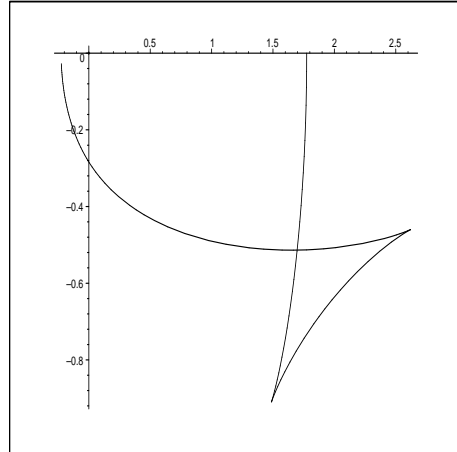


Figure 13: Rolling curve of a half period of an anisotropic undulary $p=4$, $\Lambda = -1$, $c=0.4$. The pole is the origin.

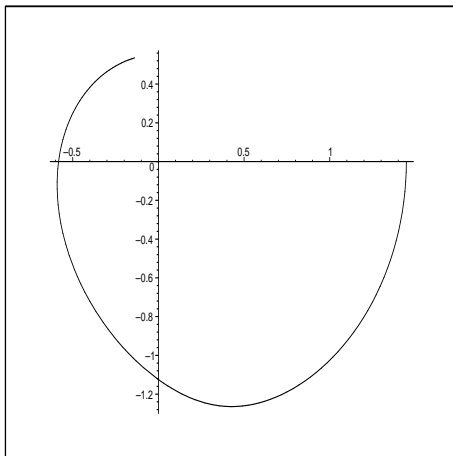


Figure 14: Rolling curve of a half period of an anisotropic undulary $p=4$, $\Lambda = -1$, $c=0.8$. The pole is the origin.

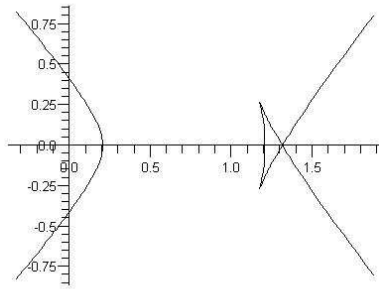


Figure 15: Rolling curve of an anisotropic nodary $p=4$, $\Lambda = -2$, $c=-0.5$. The pole is the origin.

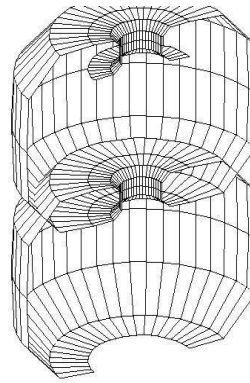


Figure 16: A part of the corresponding anisotropic nodoid

Figure 17: This series of pictures demonstrates the rolling construction for the Wulff shape whose generating curve is given by $u^4 + v^4 = 1$.

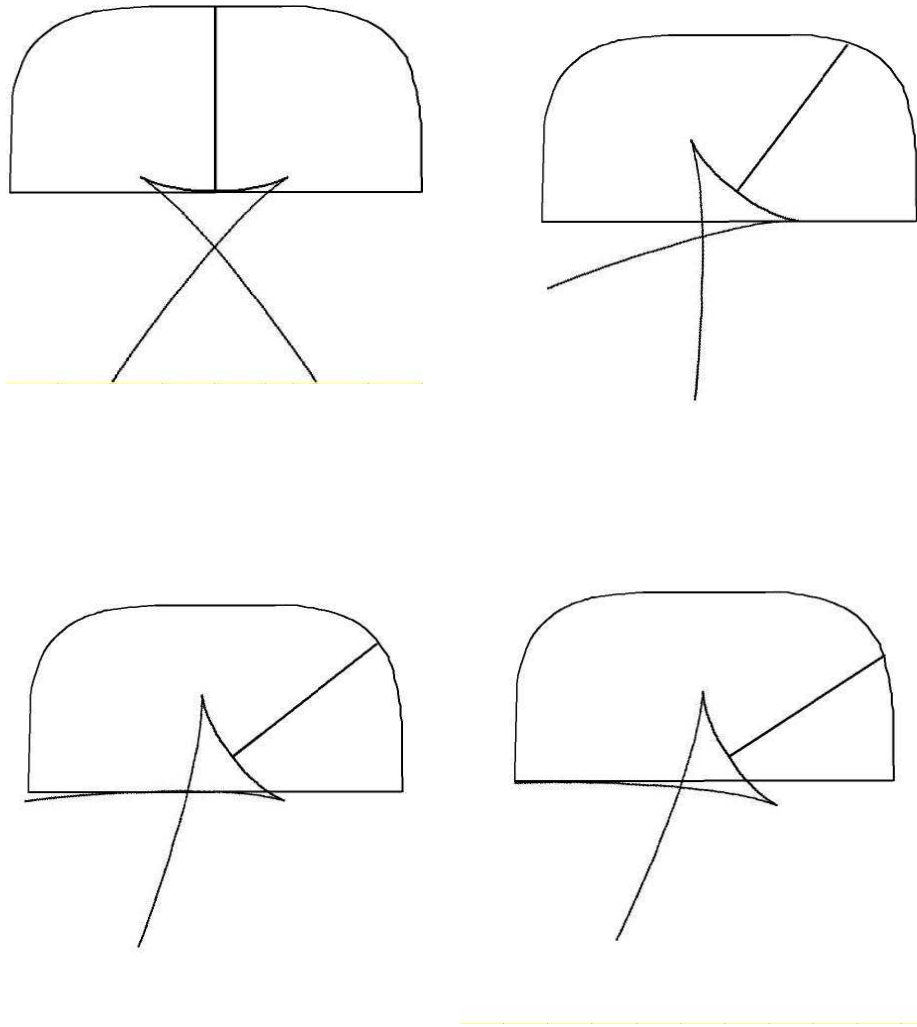


Figure 18: This series of pictures demonstrates the rolling construction for an anisotropic unduloid. The generating curve of the Wulff shape is given by $u^4 + v^4 = 1$, $c = 0.5$ and $\Lambda = -1$.

