CURVATURE PROPERTIES OF ZERO MEAN CURVATURE SURFACES IN FOUR DIMENSIONAL LORENTZIAN SPACE FORMS

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Abstract

We study the global behaviour of the Gaussian curvature K and normal curvature $K^\perp$ of zero mean curvature spacelike surfaces (*stationary surfaces*) in a 4-dimensional Lorentzian space form $\mathrm{L}^4(c)$. In particular, we show that the only complete station ary surfaces in Minkowski space $\mathbb{E}^4_1$ with $K \geq 0$ are those with $K \equiv 0 \equiv K^\perp$ and we give an explicit description of them. More general results are obtained for stationary surfaces in $\mathrm{L}^4(c)$. We also discuss applications to Willmore surfaces in both Lorentzian and Riemannian 3-spaces. We give new examples of complete stationary surfaces in $\mathbb{E}^4_1$ with finite total curvature.

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1 Introduction

Spacelike surfaces of zero mean curvature in Lorentzian manifolds arise as critical points of the area functional. When the dimension of the ambient space is greater than 3, neither the terms maximal nor minimal is appropriate to describe these surfaces. For example, in the Minkowski space $\mathbb{E}^4_1$ both maximal surfaces in $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$ and minimal surfaces in $\mathbb{E}^3 \subset \mathbb{E}^4_1$ have zero mean curvature. For this reason, we will refer to a spacelike zero mean curvature surface in a 4 dimensional Lorentzian manifold as a *stationary surface*.

Note that there are numerous examples of complete minimal surfaces in $\mathbb{E}^3$ and hence complete stationary surfaces in $\mathbb{E}^4_1$ with non-positive Gaussian curvature. On the other hand, it is well known that maximal surfaces in $\mathbb{E}^3_1$ have non-negative Gaussian curvature and that the only complete immersed maximal surfaces in $\mathbb{E}^3_1$ are planes. It is therefore natural to inquire about the existence of stationary surfaces in $\mathbb{E}^4_1$ with non-negative Gaussian curvature $K \geq 0$. We show

Theorem 1.1 *Let Σ be a parabolic Riemann surface and let $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ be a conformal spacelike immersion with zero mean curvature into a four dimensional Lorentzian space form of curvature c. Suppose the Gaussian curvature of the metric induced by χ satisfies $K - c \geq 0$. Then the curvature $K^\perp$ of the normal bundle of χ satisfies $K^\perp \equiv 0$.*

As a consequence we have the following

Corollary 1.2 *If* $\chi : \Sigma \longrightarrow L^4(0)$ *is a complete stationary immersion with* $K \geq 0$ *then*

$$K \equiv 0 \equiv K^{\perp}. \tag{1}$$

Below we give a complete description of those surfaces in Minkowski space satisfying (1).In regard to upper bounds for the curvature we have the following

Theorem 1.3 *Let* $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ *be stationary and suppose that there exists* $\epsilon > 0$ *such that*

$$0 > c - \epsilon \geq K$$

holds.Then Σ *is not complete.*

It is well known that stationary surfaces in de Sitter space S^4_1 arise naturally from studying the conformal geometry of surfaces in the 3 dimensional sphere.This occurs since S^4_1 can be regarded as the space of oriented 2-spheres in S^3. Given an immersed surface

$$\chi : M^2 \longrightarrow \mathrm{S}^3$$

one can assign to each $p \in M$ the "central sphere" at that point and thus define the *conformal Gauss map*

$$Y : M^2 \longrightarrow \mathrm{S}^4_1.$$

This map contains all information about the extrinsic conformal geometry of χ. In [1] the authors studied the conformal geometry of surfaces in the conformal compactification of $\mathbb{E}^3_1$. Let Q denote this compactification and let

$$\chi : M^2 \longrightarrow Q$$

be a spacelike immersion.The space of oriented 2-spheres in Q can be identified with the anti-deSitter space $\mathbb{H}^4_1$ and one can again define the conformal Gauss map

$$Y : M^2 \longrightarrow \mathbb{H}^4_1.$$

Basicaly, for an immersion of a surface into $\mathbb{E}^3_1$,$Y(p)$ represents the two sphere whose normal and mean curvature agree with those of the surface at p. The area of Y can be used, as in the Riemannian case to define the Willmore functional whose critical points will be called Willmore surfaces.They are characterized by the property that their conformal Gauss map is a stationary map into $\mathbb{H}^4_1$ at points where it is an immersion.We refer the reader to [1] for details.

Any geometric invariants of Y is a conformal invariant of χ and vice versa. In particular,this is true of the Gaussian curvature K_Y of the metric induced by Y. We have as an application of Theorem 1.1

Theorem 1.4 *Let* $\chi : M \longrightarrow Q$ *be a Willmore immersion.Let* U *denote the set of umbilics points and assume that* $\Sigma = M - U$ *is parabolic as a Riemann surface with respect to the conformal structure induced by* χ*. Then the Gaussian curvature of its conformal Gauss map* Y *satisfies*

$$K_Y \geq -1$$

on Σ *if and only if for each point* p *in* Σ *there exits a neighborhood* V *of* $\chi(p)$ *in* Q *and a conformal map* $\varphi : V \longrightarrow \mathbb{L}^3(c')$ *with* $c' \leq 0$ *such that* $\varphi \circ \chi$ *is a maximal immersion into* $\mathbb{L}^3(c')$*.*

Theorem 1.5 *Let $\chi : M \longrightarrow \mathrm{S}^3$ be a Willmore immersion.Let U denote the set of umbilics points and assume that $\Sigma = M - U$ is parabolic as a Riemann surface with respect to the conformal structure induced by χ. Then the Gaussian curvature of its conformal Gauss map Y satisfies*

$$K_Y \geq 1$$

on Σ if and only if for each point p in Σ there exits a neighborhood V of $\chi(p)$ in S^3 and a conformal map $\varphi : V \longrightarrow \mathbb{R}^3(c')$ with $c' \leq 0$ such that $\varphi \circ \chi$ is a minimal immersion into $\mathbb{R}^3(c')$.

Here $\mathbb{L}^3(c')$, respectively $\mathbb{R}^3(c')$, denotes the standard model of a Lorentzian, respectively Riemannian, space form of constant curvature c'.

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2 Preliminaries

Let Σ be a Riemann surface and let $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ be a conformal, spacelike immersion. Let us denote by $\mathbb{L}^4(c)$ the standard 4-dimensional Lorentzian space form of constant curvature c and let $\mathbb{E}^n_s$ denote the corresponding pseudo-Euclidean space where $\mathbb{L}^4(c)$ is lying. Then, it is well known [8, Theorem 2.4.9] that $\mathrm{L}^4(c)$ is isometric to a quotient of one of the $\mathbb{L}^4(c)$'s, and hence we may locally assume that $\chi : \Sigma \longrightarrow \mathbb{L}^4(c) \subseteq \mathbb{E}^n_s$. If z is a complex coordinate on Σ, then the metric induced by χ has a local expression

$$ds^2 = e^\rho |dz|^2.$$

We can locally choose an orthonormal frame $\{\xi, \eta\}$ for the Lorentzian normal bundle which satisfies

$$\langle \xi, \xi \rangle = -\langle \eta, \eta \rangle = 1, \quad \langle \xi, \eta \rangle = 0.$$

Then the normal bundle valued second fundamental form of χ can be locally expressed as

$$II(v, w) = II^\xi(v, w)\xi - II^\eta(v, w)\eta,$$

for tangent vectors v and w, where $II^\xi = -\langle d\xi, d\chi \rangle$ and $II^\eta = -\langle d\eta, d\chi \rangle$. The trace of the second fundamental form defines the mean curvature vector,which can be written in terms of the chosen coordinate as

$$\vec{H} = \frac{1}{2} trace(II) = 2e^{-\rho} II(\partial_z, \partial_{\bar{z}}).$$

The structural equations of the immersion are given by

$$\text{(SE)} \qquad \begin{cases} \chi_{zz} &= \rho_z \chi_z + \frac{1}{2}\phi\xi + \frac{1}{2}\psi\eta \\ \chi_{z\bar{z}} &= \frac{1}{2} e^\rho \vec{H} - \frac{c}{2} e^\rho \chi \\ \xi_z &= -\frac{1}{2} trace(II^\xi)\chi_z - \phi e^{-\rho}\chi_{\bar{z}} + \sigma\eta \\ \eta_z &= -\frac{1}{2} trace(II^\eta)\chi_z + \psi e^{-\rho}\chi_{\bar{z}} + \sigma\xi \end{cases}$$

where $\phi = 2II^\xi(\partial_z, \partial_z)$ and $\psi = 2II^\eta(\partial_z, \partial_z)$. The integrability conditions for this system are the Gauss equation

$$2\rho_{z\bar{z}} = e^{-\rho}(|\phi|^2 - |\psi|^2) - e^\rho\langle \vec{H}, \vec{H}\rangle - ce^\rho, \tag{2}$$

the Codazzi equation,

$$e^\rho \nabla_{\bar{z}}^\perp \vec{H} = (\phi_{\bar{z}} + \psi\bar{\sigma})\xi + (\psi_{\bar{z}} + \phi\bar{\sigma})\eta, \tag{3}$$

and the Ricci equation

$$Im(\sigma_{\bar{z}}) = \frac{e^{-\rho}}{2} Im(\phi\bar{\psi}). \tag{4}$$

The Gaussian curvature K of the metric ds^2 is given by

$$K = -\frac{1}{2}\Delta\rho, \tag{5}$$

where $\Delta = 4e^{-\rho}\partial_z\partial_{\bar{z}}$ is the Laplace operator of ds^2. Then (2) can be written as

$$K - c = -e^{-2\rho}(|\phi|^2 - |\psi|^2) + \langle \vec{H}, \vec{H}\rangle. \tag{6}$$

As for the normal curvature $K^\perp$ of the surface, it follows from the formula

$$K^\perp \star 1 = d\omega,$$

where ω is given locally by $\omega = 2Re(\sigma dz)$, that

$$K^\perp = -4e^{-\rho} Im(\sigma_{\bar{z}}), \tag{7}$$

so that (4) can be written as

$$K^\perp = -2e^{-2\rho} Im(\phi\bar{\psi}). \tag{8}$$

Let us assume from now on that χ is stationary, $\vec{H} \equiv 0$. Then, it follows from (3) that the differential defined locally by

$$q := \langle \chi_{zz}, \chi_{zz}\rangle dz^4 = \frac{1}{4}(\phi^2 - \psi^2)dz^4 \tag{9}$$

is holomorphic. From (6) , (8) and (9) one finds the intrisic norm of q is given by

$$||q||^2 = 16e^{-4\rho}|\langle \chi_{zz}, \chi_{zz}\rangle|^2 = (K - c)^2 + (K^\perp)^2. \tag{10}$$

Since q is holomorphic, if $q \not\equiv 0$ then its zeros are isolated and away from them we can arrange by a local complex coordinate z such that

$$\langle \chi_{zz}, \chi_{zz}\rangle = 1/4,$$

and a local orthonormal frame $\{\xi, \eta\}$ for the normal bundle such that

$$\chi_{zz} = \rho_z\chi_z + \frac{1}{2}\cosh\lambda\xi + \frac{1}{2}\sinh\lambda\eta,$$

where λ is some local complex valued function. Writing now $\{\xi', \eta'\}$ for the new normal frame given by

$$\begin{aligned} \xi' &= \cosh Re\lambda \xi + \sinh Re\lambda \eta \\ \eta' &= \sinh Re\lambda \xi + \cosh Re\lambda \eta \end{aligned}$$

it is easy to see that

$$\chi_{zz} = \rho_z \chi_z + \frac{1}{2} \cos\theta \xi' + \frac{i}{2} \sin\theta \eta',$$

where $\theta = Im\lambda$. Working now with this new frame, equation (3) gives

$$\sigma = i\theta_z,$$

so that using (7) one has

$$K^{\perp} = -\Delta\theta. \tag{11}$$

The integrability conditions for (SE) reduce now to the following system of Toda equations

$$\text{(T)} \qquad \begin{cases} 2\rho_{z\bar{z}} - e^{-\rho}\cos 2\theta + ce^{\rho} & = 0 \\ 2\theta_{z\bar{z}} + e^{-\rho}\sin 2\theta & = 0 \end{cases}$$

It is clear from (5) and (11) that the system (T) is equivalent to

$$K - c = -e^{-2\rho}\cos 2\theta, \tag{12}$$

and

$$K^{\perp} = e^{-2\rho}\sin 2\theta. \tag{13}$$

3 Main Results

Proposition 3.1 *Let $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ be a stationary immersion. Then away from points where $c = K$ the following hold*

$$\Delta \mathrm{arctan}(\frac{K^{\perp}}{K - c}) = 2K^{\perp}, \tag{14}$$

and

$$\Delta \log\left((K - c)^2 + (K^{\perp})^2\right) = 8K. \tag{15}$$

Proof. By (10) we know that $q \neq 0$ when $K \neq c$. Hence (12) and (13) yield

$$\mathrm{arctan}(\frac{K^{\perp}}{K - c}) = -2\theta$$

and the first equation follows from (11). As for the second one, since q is holomorphic one gets from (10) and using (5) that

$$\Delta \log\left((K - c)^2 + (K^{\perp})^2\right) = -4\Delta\rho = 8K,$$

as desired. ■

Lemma 3.2 *Let $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ be a stationary immersion for which $K \not\equiv c$ holds. Then the set*

$$B := \{p \in \Sigma | K(p) = c\}$$

has empty interior.

Proof. First note that points where $(K-c)^2 + (K^\perp)^2 = 0$ are isolated since from (10) these coincide with zeros of the holomorphic differential q . Moreover, the local geometry of χ away from points where $(K-c)^2 + (K^\perp)^2 = 0$ is completely determined from a solution of the system (T). It is well known [2, Remark 3, page 505] that any solution of this system is real analytic. Hence the resulting surface is real analytic and the result follows. ∎

Lemma 3.3 *Let $\chi : \Sigma \longrightarrow \mathrm{L}^4(c)$ be a stationary immersion for which*

$$K - c \geq 0 \qquad \text{and} \qquad K \not\equiv c$$

hold. Then there exists a smooth function

$$\alpha : \Sigma \longrightarrow \mathbb{R}$$

satisfying

$$\alpha = \arctan\left(\frac{K^\perp}{K-c}\right) \tag{16}$$

on $\Sigma - B$.

Proof. Let $p \in B$. If $K^\perp(p) > 0$ then $K^\perp > 0$ holds in a neighborhood N of p. Let arcCot denote the branch of the inverse cotangent function with $\mathrm{arCot}(0) = \frac{+\pi}{2}$. Then the elementary formula $\arctan(x) = \mathrm{arcCot}(\frac{1}{x})$ yields

$$\arctan\left(\frac{K^\perp}{K-c}\right) = \mathrm{arcCot}\left(\frac{K-c}{K^\perp}\right)$$

in $N - \{p\}$ and extends $\arctan(\frac{K^\perp}{K-c})$ smoothly to all of N. Similarly if $K^\perp(p) < 0$ then we proceed as above but using the branch of inverse cotangent satisfying $\mathrm{arcCot}(0) = \frac{-\pi}{2}$.

The final possibility is that $K^\perp(p) = 0$. Note that such points in B are isolated since they coincide with zeros of the holomorphic differential q. Given such a point we may assume by the above that $\arctan(\frac{K^\perp}{K-c})$ has already been extended to a punctured neighborhood $N - \{p\}$. Let D denote a small coordinate disc centered at p and contained in N. Let ϕ denote the solution of

$$\begin{aligned} \Delta\phi &= 2K^\perp, \qquad \text{in } D \\ \phi &= \arctan\left(\frac{K^\perp}{K-c}\right), \qquad \text{on } \partial D \end{aligned}$$

Then the function $\arctan(\frac{K^\perp}{K-c}) - \phi$ is both harmonic and bounded in $D - \{p\}$ and vanishes identically on ∂D. The Phragmèn-Lindelöf Principle [6, page 102] can be used to show that $\arctan(\frac{K^\perp}{K-c}) - \phi$ and hence $\arctan(\frac{K^\perp}{K-c})$ has a smooth extension to all of D. ∎

Lemma 3.4 *Let ψ be a bounded smooth function on a parabolic surface such that*

$$\psi \geq 0 \qquad \text{iff} \qquad \Delta\psi \geq 0.$$

Then $\psi \equiv$ constant.

Proof. For a suitable constant C

$$w := C - \psi^2 \geq 0$$

holds and

$$\Delta w = -2\psi\Delta\psi - 2|\nabla\psi|^2.$$

By the assumptions on ψ ,$\psi\Delta\psi \geq 0$ holds and so $\Delta w \leq 0$. It follows that w and hence ψ is constant. ■

Proof of Theorem 1.1. Let us first assume that $K \equiv c$. Then the set

$$A := \{p \in \Sigma | K^{\perp}(p) = 0\}$$

coincides with the zeros of the holomorphic differential q. That means that if $K^{\perp} \not\equiv 0$ points in A are isolated. In that case, one would get from (12) that θ is locally constant on $\Sigma - A$, but using (11) it would be $K^{\perp} = 0$ on $\Sigma - A$, which is a contradiction.

When $K \not\equiv c$, then on $\Sigma - B$ we have

$$K^{\perp} \geq 0 \Leftrightarrow \frac{K^{\perp}}{K-c} \geq 0 \Leftrightarrow \text{arctan}\frac{K^{\perp}}{K-c} \geq 0 \Leftrightarrow \Delta\text{arctan}\frac{K^{\perp}}{K-c} \geq 0,$$

using Proposition 3.1. By Lemma 3.2, $\Sigma - B$ is dense in Σ. Hence the extension α of the function $\text{arctan}\frac{K^{\perp}}{K-c}$ obtained from Lemma 3.3 satisfies

$$\alpha \geq 0 \Leftrightarrow \Delta\alpha \geq 0$$

on Σ, and we find by Lemma 3.4 that $\arctan(\frac{K^{\perp}}{K-c}) \equiv const$. Then applying Proposition 3.1 we obtain $K^{\perp} \equiv 0$ on Σ, completing the proof. ■

It is clear that if χ is a stationary immersion into $\mathbb{L}^4(c) \subseteq \mathbb{E}^n_s$ such that there exists a constant vector $a \in \mathbb{E}^n_s$ with $\langle \chi, a \rangle \equiv const.$ then the normal curvature $K^{\perp}$ vanishes identically. When $\langle a, a \rangle$ is positive, these immersions are maximal in $\mathbb{L}^3(c) \subset \mathbb{L}^4(c)$ and hence their Gaussian curvature satisfies $K - c \geq 0$. Similarly, if $\langle a, a \rangle$ is negative, these immersions are minimal in the three dimensional Riemannian space form of curvature c contained in $\mathbb{L}^4(c)$ and their curvature satisfies $K - c \leq 0$. Therefore the most interesting case is when $\langle a, a \rangle = 0$.

By the action of an appropriate isometry of $\mathbb{L}^4(c)$,we may assume that $a = (1, \vec{0}, 1) \in \mathbb{E}^n_s$ and $\langle \chi, a \rangle = \chi_1 - \chi_n = 0$. Then the projection of χ onto the coordinate space $(\chi_2, ..., \chi_n)$ gives a local isometry between (Σ, ds^2) and a two dimensional Riemannian space form of curvature c, which implies that $K \equiv c$.

Conversely, let $\chi : \Sigma \longrightarrow \mathbb{L}^4(c) \subseteq \mathbb{E}^n_s$ be a stationary immersion with $K^{\perp} \equiv 0$. Then locally we can choose an orthonormal frame $\{\xi, \eta\}$ for the normal bundle which is parallel with respect to the normal connection. Then $\sigma \equiv 0$ in (SE) so that we have

$$\begin{aligned} \xi_z &= -\phi e^{-\rho} \chi_{\bar{z}}, \\ \eta_z &= \psi e^{-\rho} \chi_{\bar{z}}. \end{aligned} \tag{17}$$

Moreover, using (3) with $\sigma \equiv 0$ implies $\phi_{\bar{z}} = 0 = \psi_{\bar{z}}$, i.e. ϕ and ψ are both holomorphic. Since points where $K = c$ coincide with zeros of the differential q, they are isolated unless $K \equiv c$. Let us first assume that $K \equiv c$. Then we obtain from(6) that $\phi \equiv \pm\psi$ and we find from (17) that $a := \xi \pm \eta$ is a constant null vector in $\mathbb{E}^n_s$ and $\langle \chi, a\rangle \equiv$ const.

If $K \not\equiv c$ then from the holomorphicity of ϕ and ψ and (8), we deduce that $\phi = r\psi$ for some real constant r or $\psi \equiv 0$. Hence from (17) we find that either $a := \xi + r\eta$ or η is a constant vector with $\langle \chi, a\rangle =$ const. Moreover, in this case (6) can be written as

$$K - c = e^{-2\rho}\langle a, a\rangle|\psi|^2 \qquad \text{or} \qquad K - c = e^{-2\rho}\langle a, a\rangle|\phi|^2$$

which implies that $\langle a, a\rangle \neq 0$ and $K - c \geq 0$ if and only if $\langle a, a\rangle$ is positive.

Proof of Corollary 1.2. It follows from the curvature assumption that the universal cover $\tilde{\Sigma}$ of Σ is conformal to the complex plane **C**. Hence Σ is parabolic and it follows from Theorem 1.1 that $K^\perp \equiv 0$. By [8, Theorem 2.4.9] any four dimensional, flat Lorentzian manifold is a quotient of $\mathbb{E}^4_1$ and we can lift χ to a complete stationary immersion $\tilde{\chi}$ from **C** into $\mathbb{E}^4_1$ with $K \geq 0$ and $K^\perp \equiv 0$. By the above remarks there exists a constant vector $a \in \mathbb{E}^4_1$ such that $\langle a, a\rangle \geq 0$ and $\langle \tilde{\chi}, a\rangle \equiv 0$. If a is spacelike then $\tilde{\chi}$ is a complete spacelike immersion into $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$ and it is then well known that $\tilde{\chi}$ must be an immersion of a plane. If a is null then we can deduce from the above remarks that $K \equiv 0$ even without the completeness assumption. ■

Proof of Theorem 1.3. Suppose to the contrary that the induced metric ds^2 is complete. Then by the assumptions, the curvature is bounded above by a negative constant $-a^2$ and so the universal cover $\tilde{\Sigma}$ of Σ is conformally equivalent to the unit disc D. This can be seen by using the Faber-Krahn inequality. If $\tilde{\Sigma}$ were conformal to the plane, we consider the geodesic disc of radius r in the plane equipped with the lift of the metric ds^2. The first Dirichlet eigenvalue of the Laplacian on this disc is bounded below by the eigenvalue for the disc of the same area in the simply connected surface with constant curvature $-a^2$. This egenvalue has a positive lower bound $\lambda > 0$ independent of r and hence there exists a positive solution of $\Delta u + \lambda u = 0$ in the plane. This contradicts the parabolicity of the plane.

On the other hand, on Σ we write

$$d\sigma^2 = ||q||^{1/2} ds^2 = ((K - c)^2 + (K^\perp)^2)^{1/4} ds^2 \geq \epsilon^{1/2} ds^2,$$

with

$$\Delta \log ||q|| = 4K,$$

from (15). Therefore $d\sigma^2$ is a complete flat metric on Σ and lifting it to D we get a complete flat metric in D. This contradicts the Fialla-Blanc Theorem. ■

Proof of Theorem 1.4. Using Theorem 1.1, we find that the conformal Gauss map is a stationary immersion of Σ into $\mathbb{H}^4_1 \subset \mathbb{E}^5_2$ with $K_Y \geq -1$ and $K^\perp_Y \equiv 0$. By the remarks preceeding the proof of Corollary 1.2 ,on each connected component of Σ there is a constant vector $a \in \mathbb{E}^5_2$ such that $\langle a, a\rangle \geq 0$ and $\langle Y, a\rangle \equiv 0$. We recall from [4] that the group of conformal transformations of Q is isomorphic to the projectivized orthogonal group $\mathbb{P}O(3, 2)$. Through the action of an appropriate element of this group, we can arrange that either $a = (1, 0, 0, 0, 1)$ or $a = (1, 0, 0, 0, 0)$ depending on whether a is null or spacelike. Let c' be either 0 or -1 in the respective cases and let $\varphi_{c'} : \mathbb{L}^3(c') \longrightarrow Q$ be a conformal embedding (see [1]). Again using a suitable conformal transformation if necessary, we can find for each $p \in \Sigma$ a neighborhood V of $\chi(p)$ in Q such that

$V \subset \varphi_{c'}(\mathbb{L}^3(c'))$. We thus replace χ locally by a conformal map into $\mathbb{L}^3(c')$ which we continue to denote by χ.Now Y is explicitely given by

$$Y = h\left(\frac{\langle\chi,\chi\rangle - 1}{2}, \chi, \frac{\langle\chi,\chi\rangle + 1}{2}\right) + (\langle\chi,\nu\rangle, \nu, \langle\chi,\nu\rangle) \qquad \text{if } c' = 0, \tag{18}$$

or

$$Y = h(1,\chi) + (0,\nu) \qquad \text{if } c' = -1, \tag{19}$$

where h and ν are respectively the mean curvature and normal of the immersion χ into $\mathbb{L}^3(c')$. Hence $\langle Y, a\rangle \equiv 0$ implies $h \equiv 0$ and χ is maximal in $\mathbb{L}^3(c')$.

Conversely, if χ is maximal in $\mathbb{E}^3_1$, then (18) with $h = 0$ defines a local isometric embedding of the hyperbolic plane into $\mathbb{H}^4_1$. On the other hand, if χ is maximal in $\mathbb{H}^3_1$ then (19) with $h = 0$ defines a maximal immersion of Σ into $\mathbb{H}^3_1$ and it follows from the Gauss equation that the induced metric has curvature greater or equal to -1. ∎

We omit the proof of Theorem 1.5 since it is similar to the proof of Theorem 1.4. The above proof uses an idea from a theorem of Thomsen [7].

4 Stationary surfaces in Minkowski space

Every zero mean curvature immersion in $\mathbb{E}^4_1$ may be represented by a non-compact Riemann surface Σ and four holomorphic 1-forms on Σ, α_k, $1 \le k \le 4$, without real periods such that, if we locally write $\alpha_k = \varphi_k dz$, they satisfy

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \varphi_4^2 = 0, \tag{20}$$

and

$$|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 - |\varphi_4|^2 > 0. \tag{21}$$

The corresponding immersion $\chi : \Sigma \longrightarrow \mathbb{E}^4_1$ is given by

$$\chi(z) = \mathrm{Re}\int_{z_0}^{z} (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \tag{22}$$

and the metric induced on Σ is written as $ds^2 = e^\rho |dz|^2$ with

$$e^\rho = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 - |\varphi_4|^2). \tag{23}$$

We refer the reader to [3] for details.

It follows from (22) that solutions to (20) and (21) satisfying $\alpha_1 = \alpha_4$ correspond to stationary immersions contained in a degenerate hyperplane $\chi_1 - \chi_4 =$ constant. In that case, we deduce from (20) that $\alpha_2 = \pm i\alpha_3$, where $|\alpha_2| > 0$ by (21). Hence, we can locally arrange by a complex coordinate w such that

$$\frac{dw}{dz} = \varphi_2 \neq 0,$$

so that the holomorphic 1-forms are written as

$$\begin{aligned}\alpha_1 &= \alpha_4 = \varphi dw,\\ \alpha_2 &= dw,\\ \alpha_3 &= -idw,\end{aligned}$$

for some holomorphic function φ. Let us recall that stationary immersions in $\mathbb{E}^4_1$ with $K \equiv 0$ and $K^\perp \equiv 0$ are precisely those contained in a degenerate hyperplane (see remarks preceeding the proof of Corollary 1.2). Thus, we can give the following description of those immersions.

Example 4.1 Let Σ be an open domain of the complex plane $\mathbf{C}$ and let $\varphi : \Sigma \longrightarrow \mathbf{C}$ be a holomorphic function on Σ such that $\oint_C \varphi dz = 0$ for every circle $C \subset \Sigma$. Then the map $\chi : \Sigma \longrightarrow \mathbb{E}^4_1$ given by

$$\chi(z) = (u(z), z, u(z)), \tag{24}$$

where u is the harmonic map defined by

$$u(z) = \mathrm{Re} \int_{z_0}^{z} \varphi dz,$$

defines a stationary immersion with $K \equiv 0$ and $K^\perp \equiv 0$. Conversely, every stationary immersion in $\mathbb{E}^4_1$ with $K \equiv 0$ and $K^\perp \equiv 0$ can be represented by (24). Observe that, in contrast to the Riemannian case, if φ is not constant, then (24) defines a flat stationary immersion in Minkowski 4-space which is not totally geodesic. ∎

Next, starting with a minimal immersion in Euclidean 3-space $\mathbb{E}^3$ we construct a one parameter family of non-isometric stationary immersions in $\mathbb{E}^4_1$. Let (g, ω) be the Weierstrass data of a minimal immersion

$$\chi : \Sigma \longrightarrow \mathbb{E}^3.$$

Recall that g and ω are respectively a meromorphic function and a holomorphic 1-form on Σ, having the property that poles of g of order m correspond to zeros of ω of order $2m$. Then, for each fixed complex number $a \neq 0$ the 1-forms defined by

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(1 - ag^2)\omega,\\ \alpha_2 &= \frac{i}{2}(1 + ag^2)\omega,\\ \alpha_3 &= \frac{1}{2}(1 + a)g\omega,\\ \alpha_4 &= \frac{1}{2}(1 - a)g\omega,\end{aligned}$$

are holomorphic on Σ and satisfy (20). Moreover, if

$$\arg a \neq \pi, \tag{25}$$

then

$$|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 - |\varphi_4|^2 = \frac{1}{2}|f|^2|1 + a|g|^2|^2 > 0, \tag{26}$$

where $\omega = fdz$ locally. Observe that the condition relating poles of g and zeros of ω must hold in order to have (26) and each α_k holomorphic. Furthemore, if

$$\oint_C \omega = 0, \qquad \oint_C g\omega = 0 \qquad \text{and} \qquad \oint_C g^2\omega = 0 \tag{27}$$

hold for every circle $C \subset \Sigma$, then the holomorphic 1-forms α_k's do not have real periods. Hence, for each a we obtain a stationary immersion

$$\chi_a : \Sigma \longrightarrow \mathbb{E}^4_1,$$

given by

$$\chi_a(z) = \mathrm{Re} \int_{z_0}^{z} \frac{1}{2}(1 - ag^2, i(1 + ag^2), (1 + a)g, (1 - a)g)\omega. \tag{28}$$

We know from (23) and (26) that the metric induced on Σ by χ_a is locally given by $ds_a^2 = e^{\rho_a}|dz|^2$, where

$$e^{\rho_a} = \frac{1}{4}|f|^2|1 + a|g|^2|^2. \tag{29}$$

Observe that for $a = 1$, we have $\alpha_4 = 0$ and $\chi_1 : \Sigma \longrightarrow \mathbb{E}^3 \subset \mathbb{E}^4_1$ corresponds to the original minimal immersion χ. Writing $ds^2 = ds_1^2$ for its induced metric, it is not difficult to see from (25) and (29) that for each a there exists a positive constant C_a such that

$$ds_a^2 \geq C_a ds^2. \tag{30}$$

Therefore, if the original minimal immersion in $\mathbb{E}^3$ is complete, then each stationary immersion χ_a is complete as well. Note also that if we allow $a = -1$, then $\alpha_3 = 0$ and χ_{-1} corresponds to a generalized maximal immersion of Σ into $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$ whose branch points are those where $|g| = 1$.

Let K_a denote the Gaussian curvature of χ_a. Using the formula (5), where here $\rho = \rho_a$ is determined by (29), we get

$$K_a = -4e^{-\rho_a}\mathrm{Re}\left(\frac{a|g'|^2}{(1 + a|g|^2)^2}\right). \tag{31}$$

Observe that we locally have

$$(\chi_a)_z = \frac{1}{2}(\varphi_1, \ldots, \varphi_4) = \frac{1}{4}f(1 - ag^2, i(1 + ag^2), (1 + a)g, (1 - a)g),$$

so that the holomorphic differential q_a defined by (9) can be written as

$$q_a = \frac{a}{4}(fg')^2 dz^4.$$

Therefore, denoting by $K_a^\perp$ the normal curvature of χ_a we find from (10) that

$$||q_a||^2 = K_a^2 + (K_a^\perp)^2 = 16e^{-2\rho_a}\frac{|a|^2|g'|^4}{|1 + a|g|^2|^4}, \tag{32}$$

and

$$K_a^\perp = \pm 4e^{-\rho_a}\mathrm{Im}\left(\frac{a|g'|^2}{(1 + a|g|^2)^2}\right). \tag{33}$$

Writing now dA_a for the element of area for the metric ds_a^2, we can see from (30) and (32) that

$$||q_a||dA_a \leq \frac{|a|}{C_a}||q_1||dA_1 = \frac{|a|}{C_a}|K|dA,$$

where K and dA denote respectively the Gaussian curvature and the element of area of the original minimal immersion χ. Hence, if χ parametrizes a minimal surface with finite total curvature, then

$$\int_\Sigma \sqrt{K_a^2 + (K_a^\perp)^2} dA_a < \infty$$

for every a.

Example 4.2 The simplest choice that we can make for Σ, g and ω is

$$\Sigma = \mathbf{C}, \qquad g(z) = z, \qquad \text{and} \qquad \omega = dz.$$

This corresponds to Enneper's minimal surface, which is complete and has finite total curvature -4π. Therefore, through χ_a we generate a one parameter family of complete conformal immersions of $\mathbf{C}$ into $\mathbb{E}_1^4$ with zero mean curvature such that

$$\int_{\mathbf{C}} \sqrt{K_a^2 + (K_a^\perp)^2} dA_a < \infty.$$

¿From the expression given for the Gaussian and normal curvatures in (31) and (33), we can see that when

$$\text{Re}a \geq 0 \qquad \text{and} \qquad \text{Im}a \neq 0,$$

then these are complete stationary immersions with $K_a \leq 0$ and $K_a^\perp \not\equiv 0$. On the other hand, if

$$\text{Re}a < 0,$$

then these are complete stationary immersions with $K_a > 0$ off the compact annulus

$$\{z \in \mathbf{C}\ |r_a \leq |z| \leq R_a\},$$

with

$$r_a^2 = -\frac{1}{\text{Re}a} + \frac{|\text{Im}a|}{|a|\text{Re}a} \qquad \text{and} \qquad R_a^2 = -\frac{1}{\text{Re}a} - \frac{|\text{Im}a|}{|a|\text{Re}a}.$$

Example 4.3 Examples with non-trivial topology can be constructed from

$$\Sigma = \mathbf{C} - \{0\}, \qquad g(z) = \frac{z^2(z+1)}{z-1}, \qquad \text{and} \qquad \omega = i\frac{(z-1)^2}{z^4}dz,$$

which are the Weierstrass data of a complete minimal surface in $\mathbb{E}^3$ with finite total curvature -12π [5]. It is straightforward to verify that g and ω satisfy (27), and thus using (28) we generate a one parameter family of complete conformal immersions with zero mean curvature of the punctured plane into $\mathbb{E}_1^4$.

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