ON THE AREA OF CONSTANT MEAN CURVATURE DISCS AND ANNULI WITH CIRCULAR BOUNDARIES

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Abstract

It is still an open question whether a constant mean curvature (CMC) disc which is bounded by a circle is necessarily a spherical cap or a flat disc. The authors together with López [1] recently showed that the only stable CMC discs which are bounded by a circle are spherical caps. In this paper we derive lower bounds for the area of constant mean curvature discs and annuli with circular boundaries in 3-dimensional space forms.

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1 Introduction

A natural question to ask [3] is whether a compact constant mean curvature (CMC) surface which is bounded by a circle is necessarily a spherical cap or a flat disc. A CMC surface with circular boundary is the mathematical model of a soap bubble which has its boundary on a round hoop, and the surfaces we almost always observe are spherical caps, so that it is natural to ask if these are the only solutions. In [6] Kapouleas gave a negative answer to this question by showing that for each $g > 2$ there are infinitely many such surfaces of genus g. However the original question remains open if one requires in addition that the surface has genus zero or that it is embedded. In the genus zero case the authors together with López [1] recently showed that the only *stable* CMC surfaces of disc type which are bounded by a circle are spherical caps. In this paper we will use the Faber-Krahn inequality to convert this stability result into a lower bound for the area of a

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non-spherical CMC disc type surface in the Riemannian space form $M^3(c)$ which is bounded by a circle of radius r. If Σ is such a surface, we obtain (see Theorem 3) that its area $A(\Sigma)$ satisfies

$$A(\Sigma) \geq \max\{A_c(r,H), \tilde{A}_c(r,H)\},$$

where

$$A_c(r,H) := \frac{4\pi}{H^2+c} - \frac{\pi}{H^2+c}\sqrt{1-r^2(H^2+c)},$$

and

$$\tilde{A}_c(r,H) := \frac{2\pi}{H^2+c} + \frac{2\pi}{H^2+c}\sqrt{1-r^2(H^2+c)}.$$

In the case of CMC discs in Euclidean 3-space bounded by a circle of radius r López and Montiel [10] obtained by a different method the lower bound

$$A(\Sigma) \geq \tilde{A}_0(r,H) = \frac{2\pi}{H^2}(1+\sqrt{1-r^2H^2}),$$

where $0 < H^2 \leq 1/r^2$. Our result represents a strict improvement when $5/9r^2 < H^2 \leq 1/r^2$. Indeed, an easy computation shows that $A_c(r,H) = \tilde{A}_c(r,H)$ precisely when $r^2(H^2+c) = 5/9$, and in that case $A_c(r,H) = \tilde{A}_c(r,H) = 6\pi r^2$. Moreover,

$$\tilde{A}_c(r,H) \geq A_c(r,H) \quad \text{when} \quad 0 < H^2 + c \leq \frac{5}{9r^2},$$

whereas

$$A_c(r,H) \geq \tilde{A}_c(r,H) \quad \text{when} \quad \frac{5}{9r^2} \leq H^2 + c \leq \frac{1}{r^2}.$$

Therefore, the lower bound $A_c(r,H)$ is better than $\tilde{A}_c(r,H)$ when the constant mean curvature H takes values

$$\frac{5}{9r^2} < H^2 + c \leq \frac{1}{r^2},$$

and in the case where H^2 attains its maximum value, $H^2 + c = 1/r^2$, the value of $A_c(r,H)$ is exactly the area $4\pi r^2$ of a entire sphere with the same mean curvature. In contrast, observe that when $H^2 + c = 1/r^2$, the value of $\tilde{A}_c(r,H)$ is just the area $2\pi r^2$ of half a sphere with the same mean curvature.

A similar idea is used to give a lower bound for the area of a nonzero CMC annulus in the Euclidean space $\mathbf{E}^3$ which is bounded by circles of radii r_1, r_2 which project to concentric circles in a plane. If Σ is such a surface which is not a surface of revolution, we obtain (see Theorem 4) that its area $A(\Sigma)$ satisfies

$$A(\Sigma) \geq B(r_1,r_2,H,P) := \frac{2\pi}{H^2} - \frac{\pi}{H^2}\sum_{j=1,2}\left(1-(Hr_j+\frac{P}{r_j})^2\right)^{1/2},$$

where P is a "flux parameter" defined below (see equation (18)). As a consequence, we can show that for certain values (r_1,r_2,H,P) Delaunay surfaces are the unique area minimizers with these given parameters.

2 A flux formula

Let $M^3(c)$ denote the standard model of a simply-connected three-dimensional Riemannian space with constant curvature c, $c \in \{0, 1, -1\}$. That is, if $c = 0$ then $M^3(0) = \mathbf{E}^3$ is the Euclidean space, if $c = 1$ then $M^3(1) = \mathbf{S}^3 \subset \mathbf{E}^4$ is the unit sphere into the Euclidean space $\mathbf{E}^4$,

$$\mathbf{S}^3 = \{x \in \mathbf{E}^4 : \langle x, x \rangle = 1\},$$

and if $c = -1$ then $M^3(-1) = \mathbf{H}^3 \subset \mathbf{E}^4_1$ is the unit hyperbolic space into the Minkowski space $\mathbf{E}^4_1$,

$$\mathbf{H}^3 = \{x \in \mathbf{E}^4_1 : \langle x, x \rangle = -1, x_4 > 0\}.$$

In this section we will derive a very specialized version of the flux or balancing formula for constant mean curvature surfaces in $M^3(c)$, which we will need in the proof of our main results. We refer the reader to [7] for another approach to flux formulas based on conservation laws and momenta for constant mean curvature hypersurfaces in hyperbolic space.

Let us consider first the case where $X : \Sigma \longrightarrow \mathbf{E}^3$ is an immersion of a constant mean curvature surface into the Euclidean space, oriented by its Gauss map ν. Let $a \in \mathbf{E}^3$ be a fixed arbitrary vector and let $a^T \in \mathcal{X}(\Sigma)$ denote its tangent component along X,

$$a^T = a - \langle a, \nu \rangle \nu.$$

It then follows that

$$\nabla_v a^T = \langle a, \nu \rangle A(v)$$

for every $v \in \mathcal{X}(\Sigma)$, where $A = -d\nu$ denotes the shape operator of the surface. Therefore, the divergence of the field a^T on Σ is given by

$$\operatorname{div}(a^T) = \operatorname{tr}(\nabla a^T) = 2H\langle a, \nu \rangle,$$

where H stands for the constant mean curvature of the immersion, so that by the divergence theorem we get

$$2H \int_\Sigma \langle a, \nu \rangle d\Sigma = \oint_{\partial\Sigma} \langle a, n \rangle ds. \tag{1}$$

Here $d\Sigma$ and ds denote, respectively, the area element of Σ (with respect to the induced metric and the chosen orientation) and the induced line element on $\partial\Sigma$, and n denotes the outward pointing unitary conormal vector along $\partial\Sigma$.

On the other hand, let α be the 1-form on Σ given by

$$\alpha(v) = \det(X, v, a) = \langle v \wedge a, X \rangle, \quad v \in \mathcal{X}(\Sigma),$$

which satisfies

$$(\nabla_v \alpha)(w) = \langle v \wedge w, a \rangle + \langle Av, w \rangle \langle X \wedge \nu, a \rangle$$

for every $v, w \in \mathcal{X}(\Sigma)$. From here, it follows that the exterior derivative of α is

$$d\alpha = 2\langle a, \nu \rangle d\Sigma,$$

so that (1) becomes

$$\oint_{\partial\Sigma} \langle a, n\rangle ds = H \oint_{\partial\Sigma} \langle t \wedge a, X\rangle ds, \tag{2}$$

where t stands for the positively oriented unit tangent vector along $\partial\Sigma$.

Consider now the case where the image of the boundary of Σ is a planar Jordan curve Γ. We may assume, without loss of generality, that the plane Π containing Γ passes through the origin and is given by $\Pi = a^{\perp}$, for a unit vector $a \in \mathbf{E}^3$. By choosing the appropriate orientation in Π, we see that $\eta = t \wedge a$ is the outward pointing unitary conormal along Γ in Π, so that from (2) we obtain the following flux formula [8]

$$\oint_{\partial\Sigma} \langle a, n\rangle ds = H \oint_{\partial\Sigma} \langle \eta, X\rangle ds = 2H \ \mathrm{Area}(\Omega), \tag{3}$$

where Ω is the planar domain bounded by Γ in Π.

Let us consider now the case of an immersion $X : \Sigma \longrightarrow M^3(c) \subset \mathbf{E}^4_q$ of a constant mean curvature surface into the unit sphere $M^3(1) = \mathbf{S}^3 \subset \mathbf{E}^4_0 = \mathbf{E}^4$ or into the unit hyperbolic space $M^3(-1) = \mathbf{H}^3 \subset \mathbf{E}^4_1$. Let $a \in \mathbf{E}^4_q$ be a fixed arbitrary vector and let $a^T \in \mathcal{X}(\Sigma)$ denote its tangent component along X,

$$a^T = a - \langle a, \nu\rangle\nu - c\langle a, X\rangle X.$$

In this case we have that $\nabla_v a^T = \langle a, \nu\rangle A(v) - c\langle a, X\rangle v$ for every $v \in \mathcal{X}(\Sigma)$, and the divergence of a^T on Σ is given by

$$\mathrm{div}(a^T) = 2H\langle a, \nu\rangle - 2c\langle a, X\rangle.$$

Therefore, if we choose $b \in \mathbf{E}^4_q$ another fixed arbitrary vector, we obtain

$$\mathrm{div}(\langle b, X\rangle a^T - \langle a, X\rangle b^T) = 2H(\langle a, \nu\rangle\langle b, X\rangle - \langle b, \nu\rangle\langle a, X\rangle),$$

and using the divergence theorem

$$2H \int_{\Sigma} (\langle a, \nu\rangle\langle b, X\rangle - \langle b, \nu\rangle\langle a, X\rangle) d\Sigma = \oint_{\partial\Sigma} (\langle a, n\rangle\langle b, X\rangle - \langle b, n\rangle\langle a, X\rangle) ds. \tag{4}$$

On the other hand, let α be now the 1-form on Σ given by

$$\alpha(v) = \det(X, v, a, b) = \langle X \wedge v \wedge a, b\rangle,$$

which satisfies

$$(\nabla_v \alpha)(w) = \langle v \wedge w \wedge a, b\rangle + \langle Av, w\rangle\langle X \wedge \nu \wedge a, b\rangle.$$

Therefore, the exterior derivative of α is

$$d\alpha = 2c(\langle a, \nu\rangle\langle b, X\rangle - \langle b, \nu\rangle\langle a, X\rangle) d\Sigma,$$

and (4) becomes

$$\oint_{\partial\Sigma} (\langle a, n\rangle\langle b, X\rangle - \langle b, n\rangle\langle a, X\rangle) ds = cH \oint_{\partial\Sigma} \langle X \wedge t \wedge a, b\rangle ds. \tag{5}$$

3 Constant mean curvature discs

Throughout this section we will consider the case where the surface Σ is a topological disc with circular boundary. In the Euclidean case ($c = 0$) we may assume that this boundary is the round circle given by

$$\mathbf{S}^1(r) = \{x \in \mathbf{E}^3 : x_3 = 0, x_1^2 + x_2^2 = r^2\}$$

with $r > 0$. In the spherical case ($c = 1$) we may assume, by rotating $\mathbf{S}^3$ in $\mathbf{E}^4$ if necessary, that this boundary is

$$\mathbf{S}^1(r) = \{x \in \mathbf{S}^3 \subset \mathbf{E}^4 : x_3 = 0, x_4^2 = 1 - r^2, x_1^2 + x_2^2 = r^2\}$$

with $0 < r \leq 1$. Finally, in the hyperbolic case ($c = -1$), up to a rigid motion of $\mathbf{H}^3$ in $\mathbf{E}_1^4$, we may assume that $X(\partial\Sigma)$ is the circle

$$\mathbf{S}^1(r) = \{x \in \mathbf{H}^3 \subset \mathbf{E}_1^4 : x_3 = 0, x_4^2 = 1 + r^2, x_1^2 + x_2^2 = r^2\},$$

with $r > 0$.

When $c = 0$, a direct computation gives

$$\langle a, n\rangle = \langle E_3, n\rangle = rk_n,$$

where $a = E_3 = (0, 0, 1)$ and k_n stands for the normal curvature of the boundary. In that case, the flux formula (3) implies

$$\oint_{\partial\Sigma} k_n ds = 2\pi H r. \tag{6}$$

On the other hand, when $c = \pm 1$, the boundary circle is described by the equations

$$x_3 = 0, x_4^2 = 1 - cr^2,$$

so that choosing $a = E_3 = (0, 0, 1, 0)$ and $b = E_4 = (0, 0, 0, 1)$, a straightforward computation gives that along the boundary

$$\langle a, n\rangle\langle b, X\rangle = rk_n, \quad \langle b, n\rangle\langle a, X\rangle = 0, \quad \langle X \wedge t \wedge a, b\rangle = r.$$

Here, as above, k_n denotes the normal curvature of the boundary. Therefore (5) gives again

$$\oint_{\partial\Sigma} k_n ds = 2\pi H r. \tag{7}$$

Now, let us consider V_c the vector field on $M^3(c)$ given by

$$V_0(x) = E_3 \wedge x$$

when $c = 0$ and

$$V_c(x) = E_3 \wedge E_4 \wedge x$$

when $c = \pm 1$. Observe that the field V_0 corresponds to the one parameter subgroup ϕ_t^0 of rotations of $\mathbf{E}^3$ about the vertical axis, and the field V_1 (respectively, V_{-1}) corresponds to the one parameter subgroup ϕ_t^1 (respectively, ϕ_t^{-1}) of rotations of $\mathbf{E}^4$ (respectively, $\mathbf{E}_1^4$) around the plane passing through the origin and generated by E_3 and E_4. Therefore, if $X : \Sigma \longrightarrow M^3(c)$ is a constant mean curvature immersion, then $X_t = \phi_t^c(X)$ gives a one parameter family of isometric immersions with the same constant mean curvature H, and the Laplacian Δ of the function

$$\psi = \langle \partial_t(X_t)_{t=0}, \nu\rangle = \langle V_c(X), \nu\rangle$$

satisfies

$$\Delta\psi + (|A|^2 + 2c)\psi = 0. \tag{8}$$

Moreover, if $X(\partial\Sigma) \subset \mathbf{S}^1(r)$ then $X_t(\partial\Sigma) \subset \mathbf{S}^1(r)$ for all t and $\psi \equiv 0$ on $\partial\Sigma$.

Let us denote by ds^2 the metric on Σ induced by X and by K its Gaussian curvature, and assume that $H^2 + c > 0$. The traceless self-adjoint endomorphism $T = A - HI_2$ satisfies $|T|^2 = 2(H^2 - K + c) \geq 0$, so that the function

$$u = 2(H^2 + c) - K = (1/2)|T|^2 + (H^2 + c) > 0$$

on Σ and $d\tilde{s}^2 = u ds^2$ defines a metric on Σ which is point to point conformal to the metric ds^2. Observe that in terms of u one gets $|A|^2 + 2c = 2u$. In particular, the Laplacian $\tilde{\Delta}$ of $d\tilde{s}^2$ is given by $\tilde{\Delta} = u^{-1}\Delta$, so that equation (8) can be rewritten as

$$\tilde{\Delta}\psi + 2\psi = 0. \tag{9}$$

We will next state a curvature estimate for the metric $d\tilde{s}^2$ which is due to Ruchert [11]. The proof below was given by Hoffman and Osserman [5].

Lemma 1 *Let $X : \Sigma \longrightarrow M^3(c)$ be an immersion of a surface Σ with constant mean curvature H. Denote by ds^2 the induced metric on Σ and by K its Gaussian curvature. If $H^2 + c > 0$, then the Gaussian curvature $\tilde{K}$ of the metric $d\tilde{s}^2 = u ds^2$, where $u = 2(H^2 + c) - K > 0$, satisfies $\tilde{K} \leq 1$.*

Proof: The result is local so we may assume that the surface is simply connected. According to a result of Lawson [9], the surface (Σ, ds^2) admits an isometric immersion X_0 into the three dimensional Euclidean space with constant mean curvature H_0 satisfying $H_0^2 = H^2 + c$. Clearly we have $d\tilde{s}^2 = (2H_0^2 - K)ds^2$. The last metric appearing on the right is unchanged if we replace the immersion X_0 by rX_0, $r \in \mathbf{R}^*$ and so we may choose r so that the mean curvature of rX_0 is identically one. Let ds_0^2 be the metric induced by rX_0. Then we see by again using Lawson's result that (Σ, ds_0^2) admits a minimal isometric immersion F into $\mathbf{S}^3(1) \subset \mathbf{E}^4$.

Recall from [9] that the map $g : \Sigma \to G_2(\mathbf{E}^4)$ which assigns to $p \in \Sigma$ the normal 2-plane to F in $\mathbf{E}^4$ is a minimal immersion when F is minimal. The metric $d\tilde{s}^2$ is exactly the metric induced by g. Using the Pluecker embedding $\iota : G_2(\mathbf{E}^4) \to \mathbf{S}^5(1)$ one obtains that the composition $\iota \circ g$ is again minimal. The curvature estimate for $d\tilde{s}^2$ then follows from the Gauss equation for $\iota \circ g$.

□

We are now ready to state and proof the main result of this section.

Theorem 2 *Let $X : (\Sigma, \partial\Sigma) \longrightarrow (M^3(c), \mathbf{S}^1(r))$ be a smooth immersion of a topological disc with constant mean curvature H, $H^2 + c > 0$. If the image of X is not a spherical cap then the area of Σ with the induced metric satisfies*

$$A(\Sigma) \geq A_c(r, H) := \frac{4\pi}{H^2 + c} - \frac{\pi}{H^2 + c}\sqrt{1 - r^2(H^2 + c)}. \tag{10}$$

Proof: Assume that the surface is not a spherical cap. It was shown in [1] that the normal derivative $\partial_n \psi$ must vanish at least at three points on the boundary of the surface. In fact this derivative must vanish at least at four points on the boundary since ψ must change sign across each nodal curve interior to Σ. It follows easily that the nodal curves of ψ partition Σ into a collection of subdomains which include at least three simply connected ones which we will denote by Ω_j, $j = 1, 2, 3$. Since ψ does not change its sign on each Ω_j, equation (9) implies that the first Dirichlet eigenvalue of the Laplacian $\tilde{\Delta}$ in each Ω_j is exactly 2.

We next apply the Faber-Krahn inequality to $(\Omega_j, d\tilde{s}^2)$. This inequality states that among all simply connected surfaces of area A and with curvature bounded above by a constant κ, the first Dirichlet eigenvalue is minimized by the geodesic disc of area A in the simply connected two dimensional surface of constant curvature κ. It follows from Lemma 1 that we may apply this result with $\kappa = 1$, so that $\tilde{A}(\Omega_j) \geq 2\pi$, and hence we obtain

$$\begin{aligned}
\tilde{A}(\Sigma) &= \int_\Sigma d\tilde{\Sigma} \\
&= \int_\Sigma (2(H^2 + c) - K) d\Sigma \\
&= 2(H^2 + c)\ A(\Sigma) - \int_\Sigma K d\Sigma \\
&\geq \sum_{j=1,2,3} \tilde{A}(\Omega_j) \geq 6\pi,
\end{aligned}$$

that is,

$$A(\Sigma) \geq \frac{3\pi}{H^2 + c} + \frac{1}{2(H^2 + c)} \int_\Sigma K d\Sigma. \tag{11}$$

From the Gauss-Bonnet theorem, we obtain

$$\int_\Sigma K d\Sigma + \oint_{\partial\Sigma} k_g ds = 2\pi,$$

where k_g denotes the geodesic curvature of the boundary of Σ. Combining this with (11) gives

$$A(\Sigma) \geq \frac{4\pi}{H^2 + c} - \frac{1}{2(H^2 + c)} \oint_{\partial\Sigma} k_g ds \geq \frac{4\pi}{H^2 + c} - \frac{1}{2(H^2 + c)} \left| \oint_{\partial\Sigma} k_g ds \right|. \tag{12}$$

Since the boundary of the surface is a circle of radius r, we have that $c + k_g^2 + k_n^2 \equiv 1/r^2$ holds on the boundary, so that by Cauchy-Schwarz inequality

$$\left| \oint_{\partial\Sigma} k_g ds \right| \leq \sqrt{2\pi r} \left(\oint_{\partial\Sigma} k_g^2 ds \right)^{1/2} = \sqrt{2\pi r} \left(2\pi r (\frac{1}{r^2} - c) - \oint_{\partial\Sigma} k_n^2 ds \right)^{1/2}.$$

From the flux formula, using again the fact that the boundary is a circle of radius r (see formulas (6) and (7)), we obtain

$$\left| \oint_{\partial\Sigma} k_n ds \right| = 2\pi |H| r,$$

and so by Cauchy-Schwarz inequality

$$\oint_{\partial\Sigma} k_n^2 ds \geq 2\pi H^2 r.$$

Combining these inequalities gives

$$\left|\oint_{\partial\Sigma} k_g ds\right| \leq 2\pi\sqrt{1-r^2(H^2+c)}, \tag{13}$$

which jointly with (12) gives

$$A(\Sigma) \geq A_c(r,H) := \frac{4\pi}{H^2+c} - \frac{\pi}{H^2+c}\sqrt{1-r^2(H^2+c)}$$

□

López and Montiel [10], using a different approach which is based on an isoperimetric inequality by Barbosa and do Carmo [2], obtained another lower bound for the area of constant mean curvature discs with circular boundary in the Euclidean space. Below, and using the flux formula (7), we will extend their method to the case of constant mean curvature discs with circular boundary in the other space forms, obtaining another lower bound for the area of such surfaces. This will allow us to compare both lower bounds, deciding which one is better depending on the range of H. In fact, we will prove the following result.

Theorem 3 *Let* $X : (\Sigma, \partial\Sigma) \longrightarrow (M^3(c), \mathbf{S}^1(r))$ *be a smooth immersion of a disc with constant mean curvature* H, $H^2+c>0$. *If the image of* X *is not a spherical cap then the area of* Σ *with the induced metric satisfies*

$$A(\Sigma) \geq \max\{A_c(r,H), \tilde{A}_c(r,H)\},$$

where

$$A_c(r,H) := \frac{4\pi}{H^2+c} - \frac{\pi}{H^2+c}\sqrt{1-r^2(H^2+c)},$$

and

$$\tilde{A}_c(r,H) := \frac{2\pi}{H^2+c} + \frac{2\pi}{H^2+c}\sqrt{1-r^2(H^2+c)}.$$

Proof: We know that $H^2+c \geq K$ with equality precisely at the umbilical points. Integrating this inequality over Σ and using the Gauss-Bonnet theorem, we obtain

$$A(\Sigma) \geq \frac{1}{H^2+c}\int_\Sigma K d\Sigma = \frac{1}{H^2+c}\left(2\pi - \oint_{\partial\Sigma} k_g ds\right) \geq \frac{2\pi}{H^2+c} - \frac{1}{H^2+c}\left|\oint_{\partial\Sigma} k_g ds\right|,$$

which jointly with (13) gives

$$A(\Sigma) \geq \frac{2\pi}{H^2+c} - \frac{2\pi}{H^2+c}\sqrt{1-r^2(H^2+c)}. \tag{14}$$

Moreover (14) becomes an equality if and only if the immersion is totally umbilical and $X(\Sigma)$ is the small spherical cap.

On the other hand, the Barbosa and do Carmo isoperimetric inequality [2] implies that

$$4\pi^2 r^2 - 2A(\Sigma)\left(2\pi - \int_\Sigma (K-\kappa)^+ d\Sigma\right) + \kappa A(\Sigma)^2 \geq 0$$

for any arbitrary real number κ, with equality if and only if $K \equiv \kappa$. Choosing $\kappa = H^2 + c > 0$ we get

$$(H^2 + c)A(\Sigma)^2 - 4\pi A(\Sigma) + 4\pi^2 r^2 \geq 0,$$

that is, either

$$A(\Sigma) \leq \frac{2\pi}{H^2 + c}\left(1 - \sqrt{1 - r^2(H^2 + c)}\right) \tag{15}$$

or

$$A(\Sigma) \geq \frac{2\pi}{H^2 + c}\left(1 + \sqrt{1 - r^2(H^2 + c)}\right). \tag{16}$$

Inequalities (14), (15) and (16) implies that either

$$A(\Sigma) = \frac{2\pi}{H^2 + c}\left(1 - \sqrt{1 - r^2(H^2 + c)}\right)$$

and $X(\Sigma)$ is the small spherical cap, or

$$A(\Sigma) \geq \tilde{A}_c(r, H) := \frac{2\pi}{H^2 + c}\left(1 + \sqrt{1 - r^2(H^2 + c)}\right),$$

with equality if and only if $X(\Sigma)$ is the big spherical cap. Therefore, when the image of X is not a a spherical cap we conclude from our previous lower bound (10) that

$$A(\Sigma) \geq \max\{A_c(r, H), \tilde{A}_c(r, H)\}.$$

□

4 Constant mean curvature annuli

In this section we will use the same method as that in Theorem 2 in order to derive a lower area bound for a constant mean curvature annulus in Euclidean space which is bounded by circles which project to concentric circles in a plane.

Let $\Sigma_{R_1,R_2} = \{z \in \mathbf{C} : 0 < R_1 < |z| < R_2\}$. We denote by $\mathcal{A}_{r_1,r_2}$ the set of over all immersions of *any* Σ_{R_1,R_1} into $\mathbf{E}^3$ such that the boundary circles are mapped onto round circles Γ_1, Γ_2 of radii r_1, r_2 which lie in parallel planes and have there centers on a line perpendicular to the planes of the circles. We may assume, without loss of generality, that the planes are those given by $\{x_3 = 0\}$ and $\{x_3 = h > 0\}$, and the boundaries round circles are

$$\Gamma_1 = \mathbf{S}^1(r_1) = \{x \in \mathbf{E}^3 : x_3 = 0, x_1^2 + x_2^2 = r_1^2\},$$

and

$$\Gamma_2 = \mathbf{S}^1(r_2) = \{x \in \mathbf{E}^3 : x_3 = h, x_1^2 + x_2^2 = r_2^2\}.$$

In that case, choosing $a = E_3 = (0, 0, 1)$ we obtain from (2) the following flux formula

$$\begin{aligned} \oint_{\partial\Sigma} \langle a, n\rangle ds &= \oint_{\Gamma_1} \langle a, n\rangle ds + \oint_{\Gamma_2} \langle a, n\rangle ds = H \oint_{\partial\Sigma} \langle t \wedge a, X\rangle ds \\ &= H \oint_{\Gamma_1} \langle \eta_1, X\rangle ds - H \oint_{\Gamma_2} \langle \eta_2, X\rangle ds \\ &= 2\pi H r_1^2 - 2\pi H r_2^2, \end{aligned} \tag{17}$$

since $\eta_1 = t \wedge a$ is the *outward* pointing unitary conormal along Γ_1 in $\{x_3 = 0\}$, but $-\eta_2 = t \wedge a$ is the *inward* pointing unitary conormal along Γ_2 in $\{x_3 = h\}$ (observe that Σ induces on Γ_2 the opposite orientation to the one induced on Γ_1).

Let us denote by $|\Gamma_j|$ the circle Γ_j *positively oriented.* Then the flux formula (17) implies that

$$\oint_{|\Gamma_1|} \langle a, n\rangle ds - 2\pi H r_1^2 = \oint_{|\Gamma_2|} \langle a, n\rangle ds - 2\pi H r_2^2.$$

This allows us to define for every immersion $X \in \mathcal{A}_{r_1,r_2}$ with constant nonzero mean curvature H the *flux parameter* P by

$$2\pi P := \oint_{|\Gamma_j|} \langle a, n\rangle ds - 2\pi H r_j^2. \tag{18}$$

On the other hand, a direct computation gives now

$$\langle a, n\rangle = r_1 k_n \quad \text{on} \quad \Gamma_1,$$

and

$$\langle a, n\rangle = -r_2 k_n \quad \text{on} \quad \Gamma_2.$$

Therefore, the flux parameter P can also be written as

$$2\pi P = r_j \oint_{\Gamma_j} k_n ds - 2\pi H r_j^2. \tag{19}$$

We are now ready to state our lower bound for the area of a CMC annulus with circular boundaries.

Theorem 4 *Let $X \in \mathcal{A}_{r_1,r_2}$ be an immersion with constant nonzero mean curvature H and with flux parameter P. Assume that X is not an immersion of a Delaunay surface. Then the area of Σ with the induced metric satisfies*

$$A(\Sigma) \geq B(r_1, r_2, H, P) := \frac{2\pi}{H^2} - \frac{\pi}{H^2} \sum_{j=1,2} \left(1 - (Hr_j + \frac{P}{r_j})^2\right)^{1/2}. \tag{20}$$

Remark It is easy to see from the definition of P that the quantity $1 - (r_j H + P/r_j)^2$ is non negative.

The proof of this theorem needs the following.

Lemma 5 *Let $X \in \mathcal{A}_{r_1,r_2}$. Then the normal derivative of the function $\psi := \langle E_3 \wedge X, \nu\rangle$ must vanish at least twice on each boundary component.*

Proof: The formula for the torsion of a space curve γ is

$$\tau = -\frac{\langle \gamma''', \gamma' \wedge \gamma''\rangle}{|\gamma' \wedge \gamma''|^2}.$$

If we assume that γ is a circle in a horizontal plane then one easily obtains

$$0 = \tau = \tau_g + \partial_t \theta$$

where τ_g is the geodesic curvature and θ is defined by $\cos\theta = \langle E_3, \nu \rangle$. It follows that

$$\int_\gamma \tau_g ds = 2\pi N$$

for some integer N. We can find a collar neighborhood of Γ_j which is regular homotopic to a standard cylinder in such a way that Γ_j is setwise fixed throughout the homotopy. Clearly N is invariant under regular homotopies of the surface which fix the boundary setwise and hence by comparison with the cylinder, we get $N = 0$.

Recall (see equation (4) in [1]) that for the function ψ, we have that $\partial_n \psi = -\tau_g$, on each boundary circle so that

$$\int_{\Gamma_j} \partial_n \psi ds = 0$$

and $\partial_n \psi$ must vanish at least at two distinct points on each boundary component.

□

Proof of Theorem 4: If the immersion is not rotationally invariant then the function ψ does not vanish identically. It is known that at each zero of $\partial_n \psi$ on the boundary, a branch of the nodal set $\psi = 0$ must enter into Σ. It follows then that there are at least two simply connected nodal domains for ψ.

The proof proceeds as in the case of a topological disc. In place of (11), we have, since there are now only two nodal domains

$$A(\Sigma) \geq \frac{2\pi}{H^2} + \frac{1}{2H^2}\int_\Sigma K d\Sigma = \frac{2\pi}{H^2} - \frac{1}{2H^2}\oint_{\partial\Sigma} k_g ds, \tag{21}$$

where we have used the Gauss Bonnet theorem

$$\int_\Sigma K d\Sigma + \oint_{\partial\Sigma} k_g ds = 0.$$

Using that $k_n^2 + k_g^2 = 1/r_j^2$ holds on Γ_j, we obtain by Cauchy-Schwarz inequality that

$$\left|\oint_{\partial\Sigma} k_g ds\right| \leq \sum_{j=1,2}\left|\oint_{\Gamma_j} k_g ds\right| \leq \sum_{j=1,2}\sqrt{2\pi r_j}\left(\oint_{\Gamma_j} k_g^2 ds\right)^{1/2} = \sum_{j=1,2}\sqrt{2\pi r_j}\left(\frac{2\pi}{r_j} - \oint_{\Gamma_j} k_n^2 ds\right)^{1/2}.$$

From (19) and using again Cauchy-Schwarz inequality we also know that

$$\oint_{\Gamma_j} k_n^2 ds \geq \frac{1}{2\pi r_j^2}\left(\oint_{\Gamma_j} k_n ds\right)^2 = \frac{2\pi}{r_j}\left(\frac{P}{r_j} + H r_j\right)^2$$

Combining these inequalities gives

$$\left|\oint_{\partial\Sigma} k_g ds\right| \leq 2\pi \sum_{j=1,2}\left(1 - (H r_j + \frac{P}{r_j})^2\right)^{1/2}, \tag{22}$$

which jointly with (21) yields (20).

5 Delaunay Graphs

Let r denote the radial coordinate in the plane. The equation of a Delaunay graph, i.e. a rotationally symmetric graph with constant mean curvature H is

$$\frac{u_r}{\sqrt{1+u_r^2}} = Hr + c/r \tag{23}$$

where c is a constant of integration. We will assume that u is non-negative on some annulus $r_1 \leq r \leq 1$ and that $u \equiv 0$ on $r = 1$. It is easy to check that the flux parameter of the graph is $P = c$. In general, the area of the surface given by (23) over $r_1 \leq r \leq 1$ is given by the elliptic integral

$$D(r_1, 1, H, P) := 2\pi \int_{r_1}^{1} \frac{r dr}{\sqrt{1-(Hr+P/r)^2}}.$$

The case $P = 0$ corresponds to spheres (cylinders also have flux parameter zero). In this case $u(r) = \sqrt{(1/H)^2 - r^2} - \sqrt{(1/H)^2 - 1}$ and the area of the spherical annulus is

$$D(r_1, 1, H, 0) = \frac{2\pi}{H^2}(\sqrt{1 - r_1^2 H^2} - \sqrt{1-H^2}).$$

Note that for $r_1 \approx 1$, we have

$$B(r_1, 1, H, 0) = \frac{2\pi}{H^2} - \frac{\pi}{H^2}(\sqrt{1 - r_1^2 H^2} + \sqrt{1-H^2}) > D(r_1, 1, H, 0).$$

This shows that the spherical annulus is the absolute minimizer of area among all CMC annuli with the same parameters $(r_1, 1, H, 0)$.

Since both B and D depend continuously on P, there is an open set of small values of $|P|$ for which the Delaunay surfaces will minimize area when $r_1 \approx 1$. (e.g. for $r_1 \approx 1$, $P \neq 0$, then $B(r_1, 1, H, P) \to \infty$ as $H \to 0$ while $D(r_1, 1, H, P)$ remains bounded as $H \to 0$).

Espirito-Santo and Ripoll [4] have recently obtained some existence and non-existence results for compact CMC surfaces with boundary in two parallel planes, and which are given as graphs over these planes.

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