Variational Problems which are Quadratic in the Surface Curvatures

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Abstract. We study variational problems for surfaces in Euclidean space for functionals involving curvatures. Particularly, we study both the classical Willmore surfaces and their anisotropic analogue.

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INTRODUCTION.

This paper consists of notes prepared for a course given for the international graduate school entitled Curvature and Variational Modeling in Physics and Biophysics which took place at the Santiago de Compostela in September 2007. The overall theme of these notes can roughly be summarized as “plates, conformal geometry and the Willmore functional".

There are basically two origins to this subject. The first is the theory of elasticity that grew out of Sophie Germain’s original work on the Chladni plates. The other is the subject of conformal differential geometry as developed by Wilhelm Blaschke and his students, particularly Gerd Thomsen.

In the early nineteenth century, the German physicist and amateur violinist Ernst Chladni presented a demonstration in which sand was sprinkled on metal plates. Chladni then played the edge of the plate with his violin bow and a variety of interesting shapes were observed to be formed by the sand. (The interested reader in encouraged to do a web search using “Chladni plates" to locate one of the numerous available videos of this experiment.) Today, we would recognize the patterns formed by the sand to be the nodal lines of non linear vibrations, but at the time such an explanation was not available. The Istitut de France offered a substantial prize for their explanation.

The problem excited the interest of the leading mathematicians of the day including James Bernoulli, Poisson and Lagrange who expressed the opinion that the solution would involve tools which had yet to be developed. After several attempts and much controversy, the French mathematician Sophie Germain was awarded the prize . Remarkably, Germain was a self-taught number theorist with no formal training in analysis. Her work was aided by Lagrange’s encouragement and corrections of her manuscript. It is interesting to note that the plate energy was originally introduced in connection with surfaces with boundary, a subject which has been for the most part neglected by geometers. Stoker, [1], has pointed out that it is noteworthy that the non linear theory of plates appeared in essentially correct form before the linear theory.

Blaschke was highly influenced by the Erlangen Program of Felix Klein which interpreted geometry in terms of the invariants of group actions. For a group G acting on Euclidean space, Blaschke sought to determine the G invariants of an immersed surface. For example, when G is the group of rigid motions, one obtains the 'usual' invariants of a surface such as its mean and Gaussian curvature. One could also take other choices of G, for example the group of projective transformations, the special linear group or the group of conformal transformations. In the latter case, one obtains the conformal differential geometry.

After determining the invariant tensors and the integrability relations between them, special classes of surfaces would need to be selected for deeper study. Wisely, Blaschke turned his attention to those surfaces which were critica of the simplest conformally invariant variational problem which he called "conformal minimal surfaces". Today, these surfaces are known as Willmore surfaces, after the English mathematician Tom Willmore who reintroduced them in the nineteen sixties. Blaschke realized that the fact that these surfaces arose from a variational problem with a large symmetry group would manifest itself in special characteristics of their geometry. He and his collaborators obtained many interesting and deep results some of which were rediscovered in the nineteen eighties. The subject fell into obscurity for many years, probably because the the high order of the equations involved made it difficult to obtain an interesting variety of examples.

The notes begin with a discussion of conformal geometry using Blaschke's point of view. This is based on the use of the conformal Gauss map which involves using Lorentzian geometry to express the conformal geometry of surfaces in three dimensional space. The effect is to lower the order of the equation for a Willmore surface to second order. After discussing important results of Bryant, we discuss the our previous work on stability of Willmore surfaces and Willmore surfaces with boundary. The second half of these notes concerns anisotropic surface energies.

CONFORMAL GEOMETRY AND WILLMORE SURFACES.

The space of spheres.

From now on $\langle , \rangle$ will denote the usual inner product on $\mathbf{R}^n$. We denote the pseudometric on the five the five dimensional Minkowski space $\mathbf{R}^5_1$ with a dot. The signature of this pseudometric is $(+,+,+,+,-)$.

We denote the set of non zero null vectors in $\mathbf{R}^5_1$, (i.e. the light cone minus the origin) by $\mathscr{L}$. We let $M^3(c)$ be the unique simply connected space form with constant curvature c which we take to be -1, 0 or $+1$. Note that there are isometric, in particular conformal, embeddings

$$M^3(c) \hookrightarrow \mathscr{L}\,,$$

given as follows. When $c=-1$, we have

$$M^3(-1) = \mathbf{H}^3 = \{X \in \mathbf{R}^4_1 \mid X \cdot X = -1\} \quad \hookrightarrow \quad \mathscr{L}$$

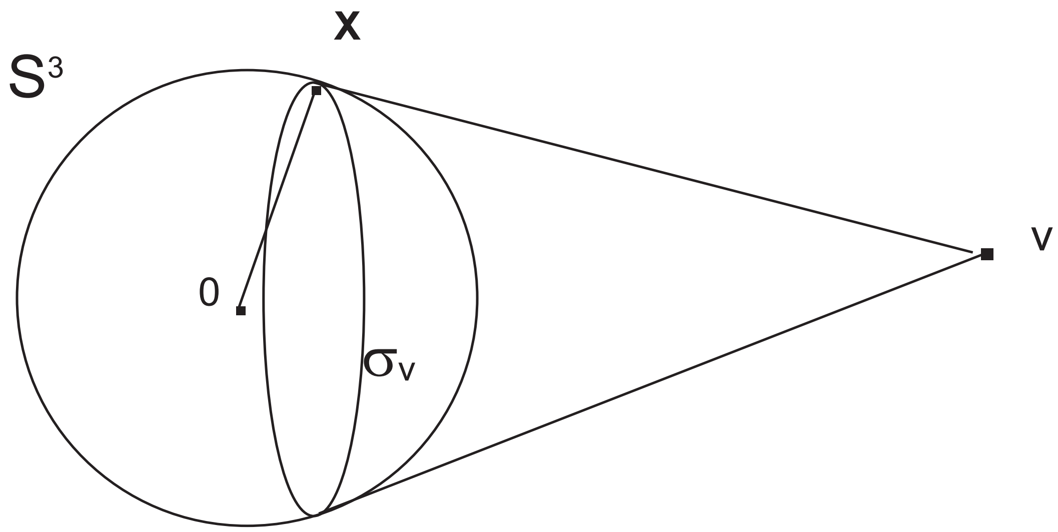

FIGURE 1.

$$X \mapsto (1, X).$$

When $c = 0$, we have

$$\begin{aligned} M^3(0) = \mathbf{R}^3 &\hookrightarrow \mathscr{L} \\ X &\mapsto (X, \frac{\langle X,X\rangle - 1}{2}, \frac{\langle X,X\rangle + 1}{2}). \end{aligned}$$

and when $c = 1$, we have

$$\begin{aligned} M^3(1) = \mathbf{S}^3 = \{X \in \mathbf{R}^4 \mid \langle X,X\rangle = 1\} &\hookrightarrow \mathscr{L} \\ X &\mapsto (X, 1). \end{aligned}$$

Let σ be a non equatorial two dimensional sphere in $\mathbf{S}^3$. We associate to σ the point v in $\mathbf{R}^4$ which is the vertex of the cone tangent to $\mathbf{S}^3$ along σ. This, in fact, defines a bijection between the exterior of $\mathbf{S}^3$ and the non equatorial, unoriented 2-spheres which we write as $v \leftrightarrow \sigma_v$.

In order to include the equatorial spheres, we replace v first with $\underline{v} := (v, 1) \in \mathbf{R}^5_1$ and then consider the projective class $[\underline{v}] \in \mathbb{P}\mathbf{R}^5_1$. Note that $\underline{v} \cdot \underline{v} > 0$ holds and so we have

$$[\underline{v}] \in \{[w] \in \mathbb{P}\mathbf{R}^5_1 \mid w \cdot w > 0\}.$$

As $v \longrightarrow \infty$ along a straight line, we have

$$[(v, 1)] = [(\frac{v}{|v|}, \frac{1}{|v|})] \rightarrow [(\hat{v}, 0)]$$

with $\hat{v} \in \mathbf{S}^3$. In this way we include $\mathbf{S}^3$ itself as the space of unoriented equatorial 2-spheres and we have a bijection:

$$\{[w] \in \mathbb{P}\mathbf{R}^5_1 \mid w \cdot w > 0\} \leftrightarrow \{\text{unoriented } 2-\text{spheres in } \mathbf{S}^3\}.$$

If we want to consider oriented 2-spheres, we take the double cover of the set on the left by the deSitter space which is the unit sphere in $\mathbf{R}^5_1$.

$$\begin{array}{ccc} \mathbf{S}^4_1 & \longrightarrow & \{[w] \in \mathbb{P}\mathbf{R}^5_1 \mid w \cdot w > 0\} \\ Y & \mapsto & [Y] \end{array}$$

The two lifts of a class $[w]$ represent the same sphere with its two orientations.

Now we consider a fixed $Y \in \mathbf{S}^4_1$ and we recover the sphere which it represents. In Figure 1, note that

$$\begin{array}{rcl} X \in \sigma_v & \Leftrightarrow & \langle v - X, X \rangle \rightarrow = 0 \\ & \Leftrightarrow & \langle v, X \rangle - 1 = 0 \\ & \Leftrightarrow & \underline{v} \cdot \underline{X} = 0\,, \end{array}$$

where $\underline{X} = (X, 1)$. This identifies the sphere represented by $Y \in \mathbf{S}^4_1$ with

$$Y^\perp \cap \mathscr{L} \cap \{x_5 = 1\}\,.$$

Next note that $N = (v - x)/|v - x|$ is the unit normal to σ_v in $\mathbf{S}^3$. Solving for v gives

$$v = N + \frac{X}{|v - X|} = N + \frac{X}{\sqrt{v^2 - 1}}\,.$$

Therefore

$$[\underline{v}] = [(N + \frac{X}{\sqrt{v^2 - 1}})] = [\pm(\sqrt{v^2 - 1}\,X + N, \sqrt{v^2 - 1})] = [\pm(hX + N, h)]\,, \qquad (1)$$

where $h = \sqrt{v^2 - 1}$ is the mean curvature of σ_v in $\mathbf{S}^3$. The representatives $\pm(hX + N, h)$ are exactly the ones which lie in $\mathbf{S}^4_1$.

The action of group the $\mathrm{O}(4,1)$ on $\mathbf{R}^5_1$ preserves $\mathscr{L}$ and induces an action on $\mathbf{S}^3 \approx \mathscr{L}/\mathbf{R}^*$. It is well known, [2] that the induced action gives the conformal (Moebius) transformations, i.e. $\mathbb{P}\mathrm{O}(4,1) = \mathrm{Conf}(\mathbf{S}^3)$ For g in $\mathrm{O}(4,1)$ we denote the corresponding Moebius transformation by $[g]$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{g} & \mathscr{L} \\ \downarrow & & \downarrow \\ \mathscr{L}/\mathbf{R}^* \approx \mathbf{S}^3 & \xrightarrow{[g]} & \mathscr{L}/\mathbf{R}^* \approx \mathbf{S}^3 \end{array}\,.$$

Spherical Congruences.

We will now develop the basic extrinsic conformal geometry of an immersed surface. It is most convenient to do this by considering the surface to be immersed in the three dimensional sphere. In this case the calculations are a bit easier than they would be if the surface was in $\mathbf{R}^3$ and the three sphere has the advantage of being compact. We wish to emphasize however that all of the results are just as valid for surfaces in $\mathbf{R}^3$ since three dimensional Euclidean space is conformally equivalent to the three sphere minus one point.

Let Σ be a smooth surface. By an (oriented) *spherical congruence* we mean a smooth map

$$Z : \Sigma \longrightarrow \mathbf{S}^4_1 \,.$$

This is just a two parameter family of oriented spheres. Now consider an immersion

$$X : \Sigma \to \mathbf{S}^3 \,.$$

We will say that X *envelopes* Z if

$$Z \cdot \underline{X} \equiv 0 \tag{2}$$

and

$$Z \cdot d\underline{X} \equiv 0 \tag{3}$$

hold. By the discussion above, this means that the point X lies on the sphere represented by Z to first order.

Proposition 0.1 *Let $X : \Sigma \to \mathbf{S^3}$ be an immersed surface such that the set of umbilic points is nowhere dense. Then there exists a unique spherical congruence Y such that*

(i) X envelopes Y

and

(ii) $Y : (\Sigma, ds^2_X) \to \mathbf{S}^4_1$, is conformal in the sense that $ds^2_Y = B ds^2_X$ where B is a non negative function.

Before proving this we will need to introduce some notation.

Let $X : \Sigma \to \mathbf{S}^3$ be a sufficiently smooth immersion of an oriented surface. The orientation will be determined by specifying a normal map $N : \Sigma \to \mathbf{S}^3$. The induced metric, ds^2_X induces a conformal structure on Σ in a natural way and we can, locally near an arbitrary $p \in \Sigma$, introduce a complex coordinate z and write

$$ds^2_X =: e^{\mu} \, |dz|^2 \,.$$

Similarly, we can express the second fundamental form as

$$-\langle dN(\cdot), dX(\cdot) \rangle =: \Re\{\phi dz^2 + h e^{\mu} dz d\bar{z}\} \,.$$

In the formula above, h is the mean curvature of the immersion and the quantity ϕdz^2 is an invariantly defined quadratic differential called the *Hopf differential*. The *Frenet equations* for the immersion are then given by

$$X_{zz} = \mu_z X_z + (\phi/2) N \tag{4}$$

$$X_{z\bar{z}} = h(e^{\mu}/2) N - (e^{\mu}/2) X \tag{5}$$

$$N_z = -h X_z - \phi e^{-\mu} X_{\bar{z}} \tag{6}$$

The integrability equations for this system are the equations of Gauss

$$|\phi|^2 e^{-2\mu} = h^2 + 1 - 2e^{-\mu}\partial_z \partial_{\bar{z}}\mu = h^2 + 1 - K\,, \tag{7}$$

and Codazzi

$$\phi_{\bar{z}} = e^{\mu} h_z. \tag{8}$$

Proof of Proposition (0.1). We define Y by

$$Y := h\underline{X} + (N,0). \tag{9}$$

Computations using the Frenet equations yield

$$Y_z = h_z \underline{X} - \frac{\phi e^{-\mu}}{2}(X_{\bar{z}},0) \tag{10}$$

from the last two equations it follow that X envelopes Y. Note also from (10) that

$$\langle Y_z, Y_z\rangle = 0, \quad \langle Y_z, Y_{\bar{z}}\rangle = |\phi|^2 e^{-\mu}/2. \tag{11}$$

The first equation means that Y is conformal and the second together with the Gauss equation implies

$$ds_Y^2 = (h^2 + 1 - K)ds_X^2 \tag{12}$$

and so (ii) holds also.

We now prove uniqueness. The equations in (2), (3) can also be written $Z \cdot \underline{X} \equiv 0, \langle Z, d\underline{X}\rangle \equiv 0$. Since X is an immersion this forces Z to lie in the 2-dimensional subspace of $\mathbf{R}^5_1$ which contains $\underline{X}$ and Y. It therefore follows that the most general spherical congruence is of the form $Z = Y + \lambda \underline{X}$ for some function λ. A computation using (10) then gives $Z_z = (h_z + \lambda_z)\underline{X} - \phi e^{-\mu}/2(X_{\bar{z}},0) + \lambda(X_z,0)$ so that $Z_z \cdot Z_z = -\phi\lambda$. Since the set of umbilics is assumed to be nowhere dense, we see that Z is conformal if and only if $\lambda \equiv 0$. **q.e.d.**.

From the expression (12) and the formula (1) we see that at each $p \in \Sigma$, $Y(p)$ represents the 2-sphere having the same normal and mean curvature as the immersion at p. In classical differential geometry this sphere was called the *central sphere*. The map Y was reintroduced by Bryant , [3], who called the *conformal Gauss map*. We will see that the map Y encodes the conformal invariants of the immersion X.

Proposition 0.2 *The conformal Gauss map is conformally invariant in the following sense. Suppose* $X : \Sigma \to \mathbf{S}^3 \approx \mathscr{L}/\mathbf{R}^*$ *is as above,* $g \in \mathrm{O}(4,1)$ *and* $[g]$ *is the corresponding map in* $\mathrm{IPO}(4,1)$*. Then if* $X' := [g]X$*, the conformal Gauss map* Y' *of* X' *is given by* $Y' = gY$*.*

Proof. This follows easily from the uniqueness statement of the previous proposition. If $X' := [g]X$ then gY satisfies the conditions (2), (3) for the immersion X'. **q.e.d.**

The proposition shows that any geometric invariant of Y as a surface in $\mathbf{S}^4_1$ is a conformal invariant of X in $\mathbf{S}^3$. In particular, since the metrics induced by X and Y satisfy (12), we have that

$$W[X] = \int_\Sigma h^2 + 1 - K\, d\Sigma_X = \int_\Sigma d\Sigma_Y = \text{Area}[Y]$$

is conformally invariant. The functional W is essentially the Willmore functional of the surface X. If we consider the immersion $X' := \pi X$ where π is the stereographic projection, then

$$\int_\Sigma h^2 + 1 - K\, d\Sigma_X = \int_\Sigma H^2 - K' d\Sigma_{X'}\,,$$

where H and K' are respectively the mean and Gaussian curvature of X'. The integral of the curvature is, of course, a topological invariant if the surface is closed or if the boundary is kept fixed to first order.

The expression for the metric induced by Y given above shows that its singularities coincide exactly with the umbilics of the immersion X. Let U denote the set of umbilics for the immersion X. We will next give several important results of Bryant, [3].

Proposition 0.3 *The immersion X defines a Willmore surface if and only if $Y|_{\Sigma \setminus U}$ has zero mean curvature in $\mathbf{S}^4_1$.*

Proof. The sufficiency is clear. The necessity is slightly surprising since there are many more variations of the map Y than there are variations through conformal Gauss maps. A straightforward computation shows however that the mean curvature field of Y is given by

$$\mathscr{H} = \frac{1}{2}(\Delta_Y Y + 2Y) = \frac{1}{2}(\Delta_Y h + 2h)\underline{X}. \tag{13}$$

If X_ε is a variation of X supported in $\Sigma \setminus U$ and Y_ε are the corresponding conformal Gauss maps then

$$\partial_\varepsilon \text{Area}(Y_\varepsilon)_{\varepsilon=0} = -2\int_\Sigma \mathscr{H} \cdot \partial_\varepsilon (Y_\varepsilon)_{\varepsilon=0}\rangle d\Sigma_Y. \tag{14}$$

Letting "dot" denote $\partial_\varepsilon(\cdot)_{\varepsilon=0}$, we have $\dot{Y} = \dot{h}\underline{X} + h\dot{\underline{X}} + (\dot{N}, 0)$ and so

$$\begin{aligned}
\partial_\varepsilon \text{Area}(Y_\varepsilon)_{\varepsilon=0} &= -\int_\Sigma (\Delta_Y h + 2h)\underline{X} \cdot (\dot{h}\underline{X} + h\dot{\underline{X}} + (tN, 0))d\Sigma \\
&= -\int_\Sigma (\Delta_Y h + 2h) X \cdot \dot{N} d\Sigma_Y \\
&= \int_\Sigma (\Delta_Y h + 2h) N \cdot \dot{X} d\Sigma_Y \\
&= \int_\Sigma (\Delta_X h + 2h(h^2 + 1 - K)) N \cdot \dot{X} d\Sigma_X.
\end{aligned}$$

The result follows since $\langle N, \dot{X} \rangle$ is a smooth arbitrary function on Σ.

Isothermicity

We will consider an immersed surface in $\mathbf{S}^3$. Away from the umbilic points, the conformal Gauss map is a space-like immersion and so the induced metric on the normal bundle is Lorentzian. Recall that $\underline{X}$ defines a null section of the normal bundle $\perp Y$ of Y. We can choose a second normal section Z with

$$\underline{X} \cdot Z \equiv 1\,.$$

Let $D^{\perp}$ denote the connection in the normal bundle $\perp Y$ of Y in $\mathbf{S}^4_1$. Using (4), and working away from umbilics of the immersion X, one easily computes

$$D^{\perp}_z \underline{X} = (\phi_{\bar{z}}/\phi)\underline{X}, \qquad D^{\perp}_{\bar{z}} \underline{X} = (\bar{\phi}_z/\bar{\phi})\underline{X}. \tag{15}$$

and

$$D^{\perp}_z Z = -(\phi_{\bar{z}}/\phi)Z, \qquad D^{\perp}_{\bar{z}} Z = -(\bar{\phi}_z/\bar{\phi})Z. \tag{16}$$

We introduce a Lorentz "orthonormal"frame for $\perp Y$,

$$e_3 := (1/\sqrt{2})(\underline{X}+Z), \qquad e_4 := (1/\sqrt{2})(\underline{X}-Z)$$

for which the connection 1-form and curvature are defined by

$$de_3 =: \omega_{34} e_4, \qquad d\omega_{34} =: K^{\perp} d\Sigma.$$

We have, using (15) and (16),

$$\omega_{34} = (\bar{\phi}_z/\bar{\phi})dz + (\phi_{\bar{z}}/\phi)d\bar{z}$$

from which we obtain

$$\begin{aligned} K^{\perp} d\Sigma : &= d\omega_{34} = ((\phi_{\bar{z}}/\phi)_z - (\bar{\phi}_z/\bar{\phi})_{\bar{z}} dz \wedge \bar{z} \\ &= (\Delta_Y \arg\phi) d\Sigma. \end{aligned}$$

Recall that a surface in $\mathbf{S}^3$ is called *isothermic* if, away from its umbilic points, the lines of curvature are given by the zero sets of a pair of locally defined conjugate harmonic functions u and v. Using these harmonic functions as an isothermal coordinates on the surface, forming the complex coordinate $z = u + iv$ and recalling that the lines of curvature are given by

$$\Im(\phi dz^2) = 0$$

one sees that isothermicity is equivalent to the local existence of a complex coordinate so that the coefficient of the Hopf differential is real valued. This clearly is the same as the condition that $\arg\phi$ is harmonic and hence we obtain that the isothermic surfaces are those surfaces such that the curvature of the normal bundle of the conformal Gauss map vanishes identically.

A surface $X : \Sigma \to M^3(c)$ is a Willmore surface if and only if it satisfies $\Delta_X H + 2H(H^2 + c - K) = 0$. Here H is the mean curvature of X in $M^3(c)$. It is clear from this that any minimal surface ($H \equiv 0$) is a Willmore surface as is any conformal image of such a surface. It is well known that any minimal surface in a space form is isothermic. We state without proof the following theorem of G. Thomen obtained in 1923.

Theorem 0.1 *(Thomsen[4]) Let $X : \Sigma \to \mathbf{S}^3$ be a Willmore immersion without umbilics. Then the surface is isothermic if and only if there exists a conformal map $\Psi : \mathbf{S}^3 \to M^3(c)$ such that $\Psi \circ X$ is minimal.*

Thomsen's original statement did not exclude umbilics and indeed a minimal surface with umbilics is conformal to a Willmore surface in $\mathbf{S}^3$. However for the necessity part of the theorem they must be excluded as the following example of K. Voss shows. Consider an elastic curve γ in the plane which is not a line and whose curvature vanishes at some point p. The cylinder $C := \gamma \times \mathbf{R}$ is a Willmore surface in $\mathbf{R}^3$ which is isothermic and the line $\{p\} \times \mathbf{R}$ consists entirely of umbilic points. Since the set of umbilics is invariant under ambient conformal maps and since umbilics in a minimal surface in $M^3(c)$ are isolated, we see that C cannot be extrinsically conformal to a minimal surface.

We now turn out attention to an important class of isothermic Willmore surface, those which are conformal to minimal surfaces in Euclidean space. We follow the work of Bryant , [3].

Proposition 0.4 *Assume that $X : \Sigma \to \mathbf{S}^3$ is a Willmore surface. Then the quartic form defined locally by*

$$Q = (Y_{zz} \cdot Y_{zz})dz^4 =: Q^{(4,0)}dz^4 \,,$$

is holomorphic.

Proof. The assertion is that for any local coordinate $\partial_{\bar{z}}Q^{(4,0)} = 0$. Note that, using (13), $\mathscr{H} \equiv 0$ can be expressed

$$Y_{z\bar{z}} + (Y_z \cdot Y_{\bar{z}})Y = 0 \,.$$

Using this to replace $Y_{z\bar{z}}$ below, we obtain

$$\begin{aligned} \partial_{\bar{z}}(Y_{zz} \cdot Y_{zz}) &= 2Y_{z\bar{z}z} \cdot Y_{zz} \\ &= -2(Y_z \cdot Y_{\bar{z}})_z Y \cdot Y_{zz} - 2(Y_z \cdot Y_{\bar{z}})Y_z \cdot Y_{zz} \\ &= 2(Y_z \cdot Y_{\bar{z}})_z Y_z \cdot Y_z - 2(Y_z \cdot Y_{\bar{z}})Y_z \cdot Y_{zz} \\ &= 0 \,, \end{aligned}$$

since $Y_z \cdot Y_z \equiv 0$ since Y is conformal and $2Y_z \cdot Y_{zz} = \partial_z(Y_z \cdot Y_z)$. **q.e.d.**

Lemma 0.1 *Let $X : \Sigma \to M^3(c)$ be a class C^4 immersion of a Willmore surface. Then the surface is real analytic.*

Proof. This is a consequence of elliptic regularity. Since the result is local, we can assume, without loss of generality, that the surface is given as a graph of a C^4 function u over a domain in the plane. The equation $\Delta H + 2H(H^2 - K) = 0$ can be expressed as a fourth order quasilinear, elliptic equation $E(Du, D^2u, .., D^4u) = 0$. The operator E depends real analytically on all the derivatives $D^k u$, $k = 1..4$. Standard results, [5], then imply that the solution is real analytic. **q.e.d.**

Proposition 0.5 *If $X : \Sigma \to \mathbf{S}^3$ is an immersion of a Willmore surface on which $Q \equiv 0$ holds, then the surface is either part of a round sphere or there exists a conformal transformation $f : \mathbf{S}^3 \to \mathbf{R}^3$ such that $f \circ X$ is a minimal immersion.*

Sketch of the Proof. Assume that the surface is not totally umbilic. By the lemma, there is an open, dense set of non umbilic points and we restrict our attention to this set.

We introduce a frame for the normal bundle $\{\underline{X}, Z\}$ as above. The superscript $\cdot^{\perp}$ will mean that the quantity is projected onto the normal bundle $\perp Y$.

We can write $Y_{zz}^{\perp} =: \alpha X + \phi Z$, and $Q = 2\alpha\phi \equiv 0$ implies that $\alpha \equiv 0$, (since $\phi = 0$ characterizes umbilics). Thus, $Y_{zz}^{\perp} =: \phi Z$. This gives

$$D_z^T Z \cdot Y_z = -Z \cdot Y_{zz} = 0\,,$$

and

$$D_z^T Z \cdot Y_{\bar{z}} = -Z \cdot Y_{z\bar{z}} = 0\,.$$

The second equation follows from the harmonicity of Y. These equations tell us that $D^T Z \equiv 0$. If $Z_1 := e^a Z$ for a smooth function a, then clearly $D^T Z_1 \equiv 0$ also.

A local calculation which will be omitted, shows that the intrinsic norm of the quartic form Q with respect to the metric ds_Y^2 is given by

$$||Q||^2 = 16 e^{-4\rho} |Q^{(4,0)}|^2 = (1 - K_Y)^2 + (K^{\perp})^2\,. \tag{17}$$

Therefor $Q \equiv 0$ implies that $K^{\perp} \equiv 0$ holds, and thus it is possible to find a new frame $\{X_1 := e^{-a}\underline{X}, Z_1 := e^a Z\}$ which is parallel in the normal bundle of Y. In particular $D^{\perp} Z_1 \equiv 0$. Since $D^T Z_1 \equiv 0$, it follows that there exists a constant null vector in $C \in \mathbf{R}_1^5$ with $C \equiv Z_1$ along Y. This can be seen by differentiating $E \cdot Z_1$ for any constant vector E.

There exists $g \in \mathrm{O}(4,1)$ such that $gC = (0,0,0,1,1)$. Then, by Proposition 0.2, $[g]\bar{X} =: \bar{X}_1$ is an immersion whose conformal Gauss may Y_1 satisfies $Y_1 \cdot gC \equiv 0$.

Theorem 0.2 *If $X : \Sigma \to \mathbf{S}^3$ be an immersion of a closed Willmore surface of genus zero then the conclusion of Proposition (0.5) holds.*

Proof. By the Uniformization Theorem, Σ is conformal to the complex plane $\mathbf{C}$ with a point adjoined at infinity. Using the usual complex coordinate in the plane, we write $Q = q(z)dz^4$ with q holomorphic in $\mathbf{C}$.

The behavior of Q at ∞ is determined by making the change of coordinate $\zeta = 1/z$ and studying the behavior of

$$Q = q(1/\zeta)[d(1/\zeta)]^4 = \frac{q(1/\zeta)}{\zeta^8} d\zeta^4\,,$$

at $\zeta = 0$. Since Q is assumed to be holomorphic at ∞ also, $q(1/\zeta)\zeta^{-8}$ must be holomorphic at $\zeta = 0$. Thus $q(1/\zeta) = q(z) \to 0$ as $z \to \infty$. In particular, q is bounded in $\mathbf{C}$ and so, by Liouville's Theorem, must reduce to a constant. The constant must be zero because of the limit discussed above.

This shows that any holomorphic quartic form on a genus zero Riemann surface vanishes identically and the result follows from the previous proposition. **q.e.d.**

The previous result also follows easily from the Riemann Roch Theorem.

Stability

It is explained above that the first variation of area for the conformal Gauss map vanishes exactly when the surface is a Willmore surface. What happens for the second variation?

The Clifford torus $T = S^1 \times \mathbf{S}^1 \to \mathbf{S}^3$ is the conjectured minimum of W among all genus one surfaces. It has been shown by Weiner, [8], that the Clifford torus is stable as a Willmore surface. The conformal Gauss map of this surface is an isometric, minimal embedding $Y : T \to \mathbf{S}^3 \subset \mathbf{S}^4_1$. It is well known that all closed minimal surfaces in $\mathbf{S}^3$ are unstable for the area functional and so the second variation of area for the conformal Gauss map of this surface is negative for some variation. This example shows that in order to use the conformal Gauss map to study the second variation of the Willmore functional, it is necessary to identify which variations of this map arise as variations through conformal Gauss maps. The material above is based on [6].

Let

$$Y : \Sigma \to \mathbf{S}^4_1$$

be an immersion of a torus. We assume that the induced metric is spacelike and that the mean curvature vector of the immersion is identically zero. Since the immersion is spacelike, the induced metric induces a conformal structure on Σ and we let z be a complex coordinate for this structure. It is well known that when the mean curvature vanishes, the quartic differential defined by

$$Q := Y_{zz} \cdot Y_{zz} dz^4 =: Q^{(4,0)} dz^4,$$

is holomorphic. By lifting the coefficient $Q^{(4,0)}$ to the complex plane and applying Liouville's theorem, one sees that $Q^{(4,0,)}$ must be a constant. We will assume that this constant is not zero. By the discussion in the previous section, this means that the torus is not the conformal image of a minimal surface in $\mathbf{R}^3$. By a simple change of the variable z, we can assume that

$$Q^{(4,0)} \equiv 1\,.$$

Since the immersion is spacelike, locally we can introduce a frame A, B for the normal bundle of Y, $\perp Y$, such that

$$A \cdot A = 0 = B \cdot B, \qquad A \cdot B = 1. \tag{18}$$

The Frenet equations for Y can then be expressed:

$$Y_{zz} \;=\; \rho_z Y_z + \frac{e^{-i\psi}}{\sqrt{2}} A + \frac{e^{i\psi}}{\sqrt{2}} B$$

$$
\begin{aligned}
Y_{z\bar{z}} &= \frac{-e^{\rho}}{2} Y \\
A_z &= -\sqrt{2} e^{-\rho} e^{i\psi} Y_{\bar{z}} - i\psi_z A \\
B_z &= -\sqrt{2} e^{-\rho} e^{-i\psi} Y_{\bar{z}} + i\psi_z B
\end{aligned}
$$

The integrability conditions for these equations are, respectively, the Gauss and Ricci equations:

$$
\Delta_0 \rho - 8e^{-\rho} \cos 2\psi + 2e^{\rho} = 0 \tag{19}
$$

$$
\Delta_0 \psi + 4e^{-\rho} \sin 2\psi = 0, \tag{20}
$$

where $\Delta_0 = 4\partial_z \partial_{\bar{z}}$. This means that the curvatures of the tangent and normal bundles are given by,

$$
1 - K_Y = 4e^{-2\rho} \cos 2\psi,
$$

$$
K^{\perp} = -4e^{-2\rho} \sin 2\psi.
$$

We now introduce the Jacobi operator of the immersion Y. Since the mean curvature is zero, the second variation of area for a variation $Y_{\varepsilon} = Y + \varepsilon\xi + O(\varepsilon^2)$, $\xi \in \Gamma(\perp Y)$ can be expressed

$$
\delta_{\xi}^2 \text{Area}(Y) = -\int_{\Sigma} \xi \cdot J[\xi] d\Sigma. \tag{21}
$$

Here it is assumed that ξ has compact support. The operator J is given by, [7],

$$
J[\xi] = \Delta^{\perp}\xi + 2\xi + \mathscr{B}\xi,
$$

where

$$
\Delta^{\perp}\xi = \sum_i (D_i^{\perp} D_i^{\perp} - D_{\nabla_i e_i}^{\perp})\xi,
$$

is the rough Laplacian in the normal bundle and $\mathscr{B}$ is an endomorphism of the normal bundle defined for sections by,

$$
(\mathscr{B}\xi) \cdot \eta = (D^T \xi) \cdot (D^T \eta), \qquad \xi, \eta \in \Gamma(\perp Y),
$$

If we express the normal section ξ in terms of our frame,

$$
\xi = \alpha A + \beta B,
$$

then,

$$
\Delta^{\perp} A = |\nabla\psi|^2 A, \qquad \Delta^{\perp} B = |\nabla\psi|^2 B
$$

$$
\mathscr{B} A = 4e^{-2\rho}((\cos 2\psi) A + B)
$$

$$\mathscr{B}B = 4e^{-2\rho}(A + (\cos 2\psi)B)$$

and the Jacobi operator is given by,

$$\begin{aligned} J[\xi] &= [\Delta\alpha + \alpha(|\nabla\psi|^2 + 2 + \mathscr{B}A\cdot B) + 2(D^{\perp}_{\nabla\alpha}A\cdot B) + \beta(\mathscr{B}B\cdot B)]A \\ &\quad +[\Delta\beta + \beta(|\nabla\psi|^2 + 2 + \mathscr{B}A\cdot B) + 2(D^{\perp}_{\nabla\beta}B\cdot A) + \alpha(\mathscr{B}A\cdot A)]B \\ &= [\Delta\alpha + \alpha(|\nabla\psi|^2 + 2 + 4e^{-2\rho}\cos 2\psi) + 4e^{-\rho}i(\alpha_z\psi_{\bar z} - \alpha_{\bar z}\psi_z) + \beta 4e^{-2\rho}]A \\ &\quad +[\Delta\beta + \beta(|\nabla\psi|^2 + 2 + 4e^{-2\rho}\cos 2\psi) + 4e^{-\rho}i(\psi_z\beta_{\bar z} - \beta_z\psi_{\bar z}) + \alpha 4e^{-2\rho}]B. \end{aligned}$$

Define self-adjoint and skew adjoint operators acting on smooth functions by,

$$L[u] := \Delta u + u(|\nabla\psi|^2 + 2 + \mathscr{B}A\cdot B) = \Delta u + u(|\nabla\psi|^2 + 2 + 4e^{-2\rho}\cos 2\psi).$$

$$\Lambda[u] := 2(D^{\perp}_{\nabla u}B\cdot A) = 4e^{-\rho}i(\psi_z u_{\bar z} - u_z\psi_{\bar z}) = 2\langle \mathscr{J}(\nabla u), \nabla\psi\rangle,$$

where $\mathscr{J}$, denotes the almost complex structure, i.e. rotation by 90° in the tangent space. Then

$$J[\xi] = [(L-\Lambda)[\alpha] + \beta(\mathscr{B}B\cdot B)]A + [(L+\Lambda)[\beta] + \alpha(\mathscr{B}A\cdot A)]B$$

Proposition 0.6 *Let $X:\Sigma\to\mathbf{S}^3$ be a Willmore torus with conformal Gauss map Y. Let $\xi\in\Gamma(\perp Y)$. Then there exists a variation $X_\varepsilon:(-c,c)\times\Sigma\to\mathbf{S}^3$, with $X_0 = X$ whose conformal Gauss maps Y_ε satisfy*

$$\partial_\varepsilon(Y_\varepsilon)^{\perp}_{\varepsilon=0} = \xi,$$

if and only if

$$J(\xi)\cdot\underline{X} \equiv 0, \tag{22}$$

holds.

Proof. Assume that $\mathscr{H}\equiv 0$, i.e. that X is a Willmore surface. Consider a one parameter variation X_ε of X and let Y_ε be the corresponding conformal Gauss maps. By (13), we have

$$2\delta\mathscr{H} = \delta\big((\Delta_Y h + 2h)\underline{X}\big) = [\delta(\Delta_Y h + 2h)]\underline{X}. \tag{23}$$

Using (21),one can easily deduce that

$$J[\xi] = 2\delta\mathscr{H},$$

where $\xi := (\delta Y)^{\perp}$. From this and (23), the sufficiency follows.

Now let ξ be a smooth section of the normal bundle of Y satisfying (22). We can locally choose a framing $\{A,B\}$ for $\perp Y$ satisfying (18) with $A =: e^{\tau}\underline{X}$ for a smooth function τ. Write $\xi = \alpha A + \beta B$. Then previous calculations given above, $J[\xi]\cdot\underline{X} = J[\xi]\cdot A = 0$ implies

$$(L+\Lambda)[\beta] + \alpha(\mathscr{B}A\cdot A) = 0.$$

Note that $(\mathscr{B}A\cdot A) = 4e^{-2\rho}$ so this means that α *is determined by* β if (22) holds.

Consider the variation of X given by $X_\varepsilon = \cos(\varepsilon(-e^{-\tau}\beta))X + \sin(\varepsilon(-e^{-\tau}\beta))N$. Note

$$-e^{-\tau}\beta = \langle \dot{X}, N\rangle = -\langle X, \dot{N}\rangle = -\underline{X}\cdot(\dot{h}\underline{X} + h\dot{\underline{X}} + \dot{N}) = -\underline{X}\cdot\dot{Y} = -e^{-\tau}A\cdot\dot{Y}\,.$$

By the first part of the proof, the section of $\perp Y$ given by $\dot{Y}^\perp$ must satisfy $J[\dot{Y}^\perp]\cdot\underline{X}\equiv 0$. Therefore, $\dot{Y}^\perp \equiv \xi$, i.e. ξ arises as from a variation through conformal Gauss maps. **q.e.d.**

Theorem 0.3 *Let $X:\Sigma\to\mathbf{S}^3$ be as above. Then X is a stable Willmore immersion if and only if*

$$-\int_\Sigma \xi\cdot J(\xi)\,d\Sigma_Y$$

holds for all $\xi\in\Gamma(\perp Y)$ satisfying

$$J(\xi)\cdot\underline{X}\equiv 0\,.$$

In a general Lorentzian 4-manifold, spacelike zero mean curvature surfaces are neither local maxima nor minima for the area functional under compactly supported variations of the surface. Let $\mathscr{N}$ be a 4-dimensional Lorentzian manifold which satisfies the *null convergence condition*

$$\mathrm{Ricci}_{\mathscr{N}}(Z,Z)\geq 0,\quad \forall Z\in T\mathscr{N}\ s.t.\,Z\cdot Z = 0.$$

In [9], it was shown that if $\Sigma\to\mathscr{N}$ is a spacelike, zero mean curvature immersion, then Σ is locally stable with respect to compactly supported variations through surfaces with null mean curvature. It is somewhat surprising that surfaces with null mean curvature arise in relativity theory where they are called *marginally trapped surfaces*.

Proposition 0.7 *Suppose the eigenvalue problem*

$$(L+\Lambda)f + \lambda f(\mathscr{B}B\cdot B) = 0$$

has an eigenvalue satisfying $|\lambda|<1$. (Note that the eigenvalues of $L+\Lambda$ are, in general, complex). Then, the surface is not stable.

If $J\xi\cdot A\equiv 0$, then $(L+\Lambda)[\beta] = -\alpha(\mathscr{B}A\cdot A)$, and so

$$\begin{aligned}
-(\xi,J\xi) &= -\int(\xi\cdot A)(J\xi\cdot B)d\Sigma \\
&= -\int\beta((L-\Lambda)[\alpha] + \beta(\mathscr{B}B\cdot B)d\Sigma \\
&= -\int\alpha(L+\Lambda)[\beta] + \beta^2(\mathscr{B}B\cdot B)d\Sigma \\
&= \int(\alpha^2-\beta^2)(\mathscr{B}B\cdot B)d\Sigma.
\end{aligned}$$

Here we have used that $(\mathscr{B}A \cdot A) = (\mathscr{B}B \cdot B)$ and that $L+\Lambda$ is the adjoint of $L-\Lambda$. If f is as above, then we have

$$(L+\Lambda)\Re(f) = -\Re(\lambda f)(\mathscr{B}B \cdot B)$$

$$(L+\Lambda)\Im(f) = -\Im(\lambda f)(\mathscr{B}B \cdot B).$$

If we apply the above, with $\beta_1 = \Re(f), \beta_2 = \Im(f)$ and sum the results, we obtain

$$\int [(\alpha_1^2 - \beta_1^2) + (\alpha_2^2 - \beta_2^2)](\mathscr{B}B \cdot B) d\Sigma = \int (|\lambda|^2 - 1)|f|^2 (\mathscr{B}B \cdot B) d\Sigma.$$

If $|\lambda| < 1$ holds then second variation for at least one of the fields determined by β_1 and β_2 must be negative.**q.e.d.**

Remark When $X : \Sigma \to \mathbf{S}^3$ is a minimal surface, then $K^\perp \equiv 0$ and $\Lambda = 0$. The eigenvalue problem in the proposition becomes,

$$\Delta_X f + (3 - 2K_X) f = -\lambda f,$$

where the subscript X indicates the quantities are those computed using the metric induced by X. The condition for instability $-1 < \lambda < 1$ can be interpreted by saying that the stability operator for the area functional $\Delta_X + (4 - 2K_X)$, has an eigenvalue in the interval $(-2, 0)$. This condition was found by Weiner in [8].

Theorem 0.4 *If the norm of the quartic differential Q satisfies*

$$||Q||^2 \geq 1\,, \tag{24}$$

with strict inequality at at least one point, then the surface is unstable.

Proof. We take $\beta := \sin\psi$, to obtain using (20). Then, $J[\xi] \cdot A \equiv 0$ becomes,

$$-4\alpha e^{-2\rho} = -4e^{-2\rho}(\sin\psi \cos 2\psi - \sin 2\psi \cos\psi) + 2\sin\psi.$$

Using a double angle formula, this gives

$$\alpha = \sin\psi(1 - \frac{1}{2}e^{2\rho}).$$

Therefor $\alpha^2 - \beta^2 = \sin^2\psi[(1 - \frac{1}{2}e^{2\rho})^2 - 1]$ which is non positive if and only if $e^{2\rho} < 4$ holds. However with the normalization used above, we have $||Q|| = 4e^{-2\rho}$ and so the result follows. **q.e.d.**

Using the variation given above, the condition that the second variation is negative can be written

$$0 > \int_\Sigma (1 - \frac{1-K}{||Q||})(\frac{1}{||Q||} - 1)\, d\Sigma.$$

This is an open condition which is considerably more general than the condition (24).

The first examples of non isothermic Willmore tori were given by Pinkall [10]. These tori are the lifts of elastic curves $\gamma : \mathbf{S}^1 \to \mathbf{S}^2$ under the Hopf fibration $\pi : \mathbf{S}^3 \to \mathbf{S}^2$. Such surfaces are flat and we can introduce coordinates on T so that the metric has the form $dS_X^2 = ds^2 + dt^2$ and the mean curvature $h = h(t)$ satisfies the Euler-Lagrange equation

$$h'' + 2h(h^2 + 1) = 0\,. \tag{25}$$

Therefore, we have

$$(h')^2 + (h^2 + 1)^2 \equiv \text{constant} =: c + 1, \quad c \geq 0, . \tag{26}$$

with $c = 0$ only in the case when $h \equiv 0$. The only time when this occurs is when $\gamma(S^1)$ is an equator and T is the Clifford torus.

The Hopf differentials of these tori are given by $\phi = h - i$. From this it follows that, except for the Clifford torus, the surfaces are not isothermic and, in particular, they are not conformally related to a minimal surface in any $M^3(c)$. A lengthy but straightforward calculation using (25), (26) and (17) shows that $Q = (-1/4)(c+1)d(t+is)^4$ and the norm of Q with respect to dS_Y^2 is

$$||Q||^2 = \frac{(c+1)^2}{(h^2+1)^4} = \Big(\frac{(h')^2}{(h^2+1)^2} + 1\Big)^2\,,$$

using (26). So $||Q|| \geq 1$ holds with equality only at the critical points of h and it follows from the previous theorem that, except for the Clifford torus, all these tori are unstable. The stability of the Clifford torus was shown by Weiner [8]. We wish to point out that the instability of these surfaces was first shown by Langer and Singer, [11], by showing the instability of the elastic curve γ. However, our proof only uses the S^1 symmetry of the surface to verify the inequality $||Q|| \geq 1$.

BOUNDARY VALUE PROBLEMS

In this section we will treat uniqueness theorems for several boundary value problems for Willmore surfaces using geometric methods. Although the Willmore functional, or more generally, the energy functional of an elastic plate, was introduced in connection with a bordered surface, this subject appears not to have been widely treated in the mathematics literature. Some exceptions are the papers of Nitsche, [12], [13].

We will first treat the simplest such problem we can formulate, that of a Willmore surface bounded by a round circle. In order to obtain some intuition about what to expect, we first consider the linearized version of this problem which is the problem, [24], of finding a biharmonic function on, say, the unit disc D which vanishes on the circle. This problem is clearly not well posed. For any harmonic function h on D, we can solve the Dirichlet problem $\Delta u = h$, in D with $u \equiv 0$ on ∂D and thus obtain infinitely many biharmonic graphs over D with circular boundaries. It is not reasonable to expect to do better in the non linear case so we need to specify a second boundary condition which will make the problem well-posed. A natural one to chose is to require that the surface intersect the plane of the boundary in a constant angle. Then, both the condition that the surface be an extremal and the two boundary conditions will be in some sense conformally invariant.

Theorem 0.5 *Let D denote the unit disc and let $X : (D, \partial D) \to (\mathbf{R}^3, S^1)$ be a $C^4(\bar{D})$ immersion of a Willmore surface. Assume that the surface intersects the plane $\{x_3 = 0\}$ in a constant angle along ∂D. Then the image $X(D)$ is a spherical cap or a flat disc.*

Let $X : (D, \partial D) \to (\mathbf{R}^3, S^1)$ be any $C^1(\bar{D})$ immersion. We orient ∂D in the usual way (counterclockwise) and let t and n respectively denote the the tangent and outward pointing conormal defined along the boundary. We let ν the unit normal field on D such that $n \times t = \nu$ holds on ∂D.

Let E_3 denote the vertical unit vector in $\mathbf{R}^3$. If we define an angle γ by $\cos\gamma := \langle X, \nu \rangle$ then it is easy to check that

$$\langle X, \nu \rangle = \cos\gamma = -\langle n, E_3 \rangle \tag{27}$$

$$\langle X, n \rangle = \sin\gamma = \langle \nu, E_3 \rangle \tag{28}$$

hold.

We will use the conformal Gauss map of the surface to derive important integral identities for the surface. We recall that in the case of a surface in $\mathbf{R}^3$, this map is expressed,

$$Y = H(X, \frac{X^2 - 1}{2}, \frac{X^2 + 1}{2}) + (\nu, \langle X, \nu \rangle, \langle X, \nu \rangle) \tag{29}$$

Away from umbilics in Σ, Y defines a conformal spacelike immersion from (Σ, ds_X^2) into $\mathbf{S}_1^4$. Recall that a surface is a Willmore surface if and only if Y defines a zero mean curvature immersion on Σ minus the umbilic set. In this case we have that Y satisfies

$$\Delta_X Y + 2(H^2 - K)Y = 0 \tag{30}$$

on Σ. As a consequence we have the following lemma.

Lemma 0.2 *(Flux formula) Let $X : \Sigma \to \mathbf{R}^3$ be a Willmore immersion and let $Y = (Y_1, \ldots, Y_5)$ be its conformal Gauss map. Then for $1 \le \alpha < \beta \le 5$, the forms defined by*

$$\omega_{\alpha\beta} := Y_\alpha * dY_\beta - Y_\beta * dY_\alpha$$

are closed.

Proof. Compute

$$d\omega_{\alpha\beta} = (Y_\alpha \Delta Y_\beta - Y_\beta \Delta Y_\alpha) * 1 = 0$$

by (30). **q.e.d.**

We next specialize the flux formula to a Willmore surface with circular boundary. We will let k_n denote the normal curvature of the boundary curve.

Lemma 0.3 *Let Σ be a compact, oriented surface with boundary homeomorphic to a circle and let $X : (\Sigma, \partial\Sigma) \to (\mathbf{R}^3, S^1)$ be a $C^4(\bar{\Sigma})$ Willmore immersion. Define a (not necessarily constant) angle γ by (27). Then the following hold:*

$$\int_{\partial\Sigma} k_n - H\, ds = 0 \tag{31}$$

$$\int_{\partial\Sigma} \cos(\gamma)\partial_n H + \sin(\gamma)H(H-k_n)\, ds = 0 \tag{32}$$

$$\int_{\partial\Sigma} \sin(\gamma)\partial_n H - \cos(\gamma)H(H-k_n)\, ds = 0 \tag{33}$$

*Proof.*We will use the following

$$\partial_n \nu_3 = \langle d\nu(n), E_3\rangle = \langle n, E_3\rangle\langle n, d\nu(n)\rangle = \langle n, E_3\rangle(k_n - 2H) \tag{34}$$

and

$$\partial_n\langle X,\nu\rangle = \langle X, d\nu(n)\rangle = \langle X,n\rangle\langle n, d\nu(n)\rangle = \langle X,n\rangle(k_n - 2H) \tag{35}$$

Note that $Y|_{\partial\Sigma} = H(x_1,x_2,0,0,1) + (\nu_1,\nu_2,\sin\gamma,\cos\gamma,\cos\gamma)$. Using this we compute on ∂D

$$\begin{aligned}
\omega_{34} &= \sin\gamma * d(H(\frac{X^2-1}{2})) - \cos\gamma * d(Hx_3+\nu_3)\\
&= [\sin\gamma(H\langle X,n\rangle + \langle X, d\nu(n)\rangle - \cos\gamma(H\langle E_3,n\rangle + \langle E_3, d\nu(n)\rangle)]ds\\
&= [\sin\gamma(H\sin\gamma - \sin\gamma(2H-k_n)) + \cos\gamma(H(\cos\gamma) + \cos\gamma(k_n-2H))]ds\\
&= [k_n - H]ds.
\end{aligned}$$

Next note that $y_5 = y_4 + H$ holds and so

$$\begin{aligned}
\omega_{45} &= y_4 * dH - H * dy_4\\
&= \cos\gamma * dH - H * d(H\frac{X^2-1}{2} + \langle X,\nu\rangle)\\
&= [\cos\gamma\,\partial_n H - H(H\langle X,n\rangle + \partial_n\langle X,\nu\rangle)]ds\\
&= [\cos\gamma\,\partial_n H - H(H\langle X,n\rangle + \langle X,n\rangle(k_n-2H))]ds\\
&= [\cos\gamma\,\partial_n H - H\sin\gamma(H + (k_n-2H))]ds\\
&= [\cos\gamma\,\partial_n H + H\sin\gamma(H-k_n)]ds
\end{aligned}$$

Again using $y_5 = y_4 + H$ we have

$$\begin{aligned}
\omega_{35} &= \sin\gamma * dH - H * dy_3 + \omega_{34}\\
&= \sin\gamma * dH - H * (Hx_3 + \nu_3) + \omega_{34}\\
&= [\sin\gamma\,\partial_n H - H(H\langle n,E_3\rangle + \partial_n \nu_3)]ds + \omega_{34}\\
&= [\sin\gamma\,\partial_n H + H\cos\gamma(H + (k_n-2H))]ds + \omega_{34}\\
&= [\sin\gamma\,\partial_n H - H\cos\gamma(H-k_n)]ds + \omega_{34}.
\end{aligned}$$

The result then follows by applying Lemma ((0.3)).**q.e.d.**

Proof of Theorem 0.5. If the surface intersects the plane $\{x_3 = 0\}$ in a constant angle, then $\gamma \equiv$ const. and we easily obtain

$$\int_{\partial\Sigma} \partial_n H\, ds = 0 \tag{36}$$

and

$$\int_{\partial\Sigma} H(H - k_n)\, ds = 0 \tag{37}$$

from (32) and (33).

Using a classical theorem of Joachimsthal we have that since the angle between the surface and the plane is constant along the boundary, the boundary is a line of curvature. Letting "prime" denote differentiation with respect to arc length on ∂D we have

$$\text{const.} = \langle X, \nu \rangle = -\langle X'', \nu \rangle = \langle X', \nu' \rangle = -k_n$$

and so k_n is constant on ∂D.

Let $k_1 := k_n$ and $k_2 := 2H - k_n$ denote the principal curvatures on ∂D. Then (31) and (37) yield

$$\int_{\partial D} k_1 - k_2\, ds = 0 \tag{38}$$

and

$$\int_{\partial D} k_1^2 - k_2^2\, ds = 0 \tag{39}$$

By Hölder's inequality, we have for $j = 1, 2$

$$\left| \int_{\partial D} k_j\, ds \right| \leq \int_{\partial D} |k_j|\, ds \leq \left(\int_{\partial D} k_j^2\, ds\right)^{1/2} (2\pi)^{1/2} \tag{40}$$

However, since $k_1 =$ const. we have equality in (40) for $j = 1$ and hence also for $j = 2$ by (38) and (39). We can then conclude from the necessary condition for equality in Hölder's inequality, that $k_2 \equiv$ const. also. In particular $h \equiv$ const. holds on the boundary and by (38), every boundary point is an umbilic.

Let II_Y denote the second fundamental form of the map Y. As shown by Bryant [3], the form

$$Q := II_Y^{(4,0)} \tag{41}$$

defines a holomorphic quartic differential on any Willmore surface. A calculation shows that Q is given locally in terms of a complex coordinate on D by $Q =: Q^{(4,0)}\, dz^4$ where

$$Q^{(4,0)} = \begin{cases} (\phi^2/4)(H^2 + \Delta \log \phi), & \text{if } \phi \neq 0; \\ -\phi_z H_z, & \text{if } \phi = 0. \end{cases}$$

Here ϕdz^2 is the Hopf differential which is the $(2,0)$ part of the second fundamental form of X. We recall that the Codazzi equations on D take the form

$$\phi_{\bar{z}} = e^{\mu} H_z \tag{42}$$

where $e^{\mu}|dz|^2$ is the local expression of the metric.

We will need only the second part of the formula for $Q^{(4,0)}$ which can be obtained as follows. Write (30) in the form $Y = H\underline{X} + (\nu, q, q)$, where $q = \langle X, \nu \rangle$. Differentiation gives $Y_z = H\underline{X}_z - \phi e^{-\mu}\underline{X}_{\bar{z}}$. At a point where $\phi = 0$, differentiating again gives $Y_{zz} = H_{zz}\underline{X} + H_z\underline{X}_z - \phi_z e^{\mu}\underline{X}_{\bar{z}}$. Therefore, at the point in question $Q^{(4,0)} = Y_{zz} \cdot Y_{zz} = -\phi_z H_z$ as claimed.

Since the zeros of ϕ correspond to umbilics, we have $\phi \equiv 0$ on ∂D. Letting $z = re^{i\theta}$ denote the usual coordinate on D, we have on ∂D

$$0 = \partial_\theta \phi = i(z\phi_z - \bar{z}\phi_{\bar{z}}) \tag{43}$$

and

$$0 = \partial_\theta H = i(zH_z - \bar{z}H_{\bar{z}}). \tag{44}$$

Therefore on ∂D we have, using (42), (43), and (44)

$$-q = \phi_z H_z = \bar{z}z\phi_z H_z = (\bar{z})^2\phi_{\bar{z}} H_z = (\bar{z})^2 e^{\mu} H_z H_z = (\bar{z})^3 e^{\mu} z H_z H_z = (\bar{z})^4 e^{\mu} H_z H_{\bar{z}}.$$

It then follows that the function $z^4 q$ is holomorphic on the disc and is real valued on the boundary. By the maximum principle it follows that $z^4 q \equiv a$ for some real constant a. Since $q = a/z^4$ is holomorphic in D it then follows that $a = 0$ and so $q \equiv 0$ holds in D. It then follows from results of [3] that either $X(D)$ is, after a conformal transformation, a minimal immersion or $X(D)$ is part of a sphere. Since the set of umbilics is a conformal invariant, we would in the first case obtain a minimal surface in $\mathbf{R}^3$ having a boundary component made up entirely of umbilics. It follows then that the surface is a flat disc since its Hopf differential vanishes identically. The only remaining possibility is that the surface is a spherical cap. **q.e.d.**

A example of Babich and Bobenko, [14], see Figure 2, has the property that it cuts a horizontal plane in a constant angle along a circle of *umbilics* of the surface. This surface is a topological disc but contains one singularity. In fact, the surface is self-similar near the point which corresponds to the origin in the figure.

By using the Implicit Function Theorem, it is possible to produce infinitely many dic-type Willmore surfaces which are bounded by a circle, [15].

FREE BOUNDARY

We will consider here a more general bending energy functional

$$E = \int_\Sigma ak_1^2 + 2bk_1k_2 + ak_2^2 \, d\Sigma\,, \tag{45}$$

where $x, y \mapsto ax^2 + 2bxy + ay^2$ is a positive semi- definite quadratic form with constant coefficients. The energy E can easily be expressed:

$$E = \int_\Sigma \alpha H^2 + \beta K \, d\Sigma\,,$$

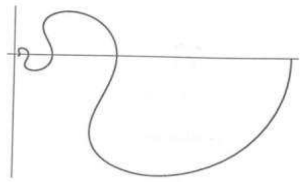

FIGURE 2. Generating curve of the Babich-Bobenko example.

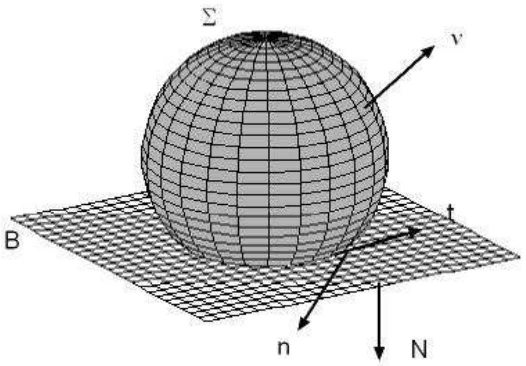

FIGURE 3.

with $a = \alpha/4$, $2b = (\alpha/2) + \beta$. The functional E is variationally equivalent to the Willmore functional if $\partial\Sigma = \emptyset$ or if the boundary is kept fixed to first order. For the free boundary problem we consider now, the problems for various choices of the constants are inequivalent. Their critica will, in general, satisfy different boundary conditions although they will satisfy the same equation in the interior of the surface.

We consider a domain $\Omega \subset \mathbf{R}^3$ with smooth boundary S. it is not required that $\Omega \cup S$ be compact. We consider the critical points of E among all immersions of compact surfaces $X : \Sigma \to \Omega$ such that $X(\partial\Sigma) \subset S$. Let t be the unit tangent vector to $\partial\Sigma$, let n be the unit conormal to $\partial\Sigma$, let ν be the unit normal to Σ such that $\nu = n \times t$ on the boundary and let N denote the unit normal to S which points out of Ω. Let (σ_{ij}) be the components of the second fundamental form of Σ with respect to a frame which agrees with $\{n,t\}$ on $\partial\Sigma$. (See Figure (3).)

We express a variation of X as $X_\varepsilon = X + \varepsilon\dot{X} + \mathscr{O}(\varepsilon^2)$ where $\dot{X} =: \xi + \eta\nu$ with ξ tangent to Σ. A calculation then shows

$$\delta E = \int_\Sigma \eta(\Delta H + 2H(H^2 - K))\, d\Sigma \tag{46}$$

$$+ \oint_{\partial\Sigma} (\alpha H + \beta\sigma_{22})\eta_n - (\alpha H_n - \beta(\sigma_{12})_t)\eta + (\alpha H^2 + \beta K)\langle\xi, n\rangle\, ds\,. \tag{47}$$

The only constraint on the variations in that $\langle\dot{X}, N\rangle \equiv 0$ on $\partial\Sigma$.

We find that the Euler-Lagrange equations for the free boundary problem are

$$\Delta H + 2H(H^2 - K) = 0\,, \text{in} \quad \Sigma\,. \tag{48}$$

$$\alpha H + \beta\sigma_{22} = 0\,, \qquad \text{on} \quad \partial\Sigma\,. \tag{49}$$

$$(\alpha H^2 + \beta K)\langle\nu, N\rangle + (\alpha H_n - \beta(\sigma_{12})_t)\langle n, N\rangle = 0 \quad \text{on} \quad \partial\Sigma\,. \tag{50}$$

We now specialize to the case where Ω is the half space $x_3 > 0$ and S is the plane $x_3 = 0$. Let Σ be a solution of (48)-(50). We assume that the surface is in equilibrium and compute the first variation with $\dot{X} = E_3 = \nu_3 + E_3^T$. Even though this variation does not preserve the constraint that $\partial\Sigma$ stays on S, the first variation is nevertheless zero since the variation is a symmetry of the problem. Using (48) and (49), we obtain from (46),

$$0 = \oint_{\partial\Sigma} -(\alpha H_n - \beta(\sigma_{12})_t)\nu_3 + (\alpha H^2 + \beta K)\langle E_3, n\rangle\, ds\,. \tag{51}$$

Note that in (50) we have $(\alpha H^2 + \beta K) \geq 0$ and $\langle n, N\rangle = \langle n, -E_3\rangle \geq 0$ since the surface is contained the upper half-plane. It then follows from (50) that $(\alpha H_n - \beta(\sigma_{12})_t)\nu_3 \geq 0$ must hold. But this means that both the terms in the integrand of the last integral are non positive, so they both must vanish,

$$(\alpha H_n - \beta(\sigma_{12})_t)\nu_3 \equiv 0\,, \tag{52}$$

and

$$(\alpha H^2 + \beta K)\langle E_3, n\rangle \equiv 0\,, \tag{53}$$

on $\partial\Sigma$. Since $\langle n, E_3\rangle^2 + \nu_3^2 \equiv 1$ on $\partial\Sigma$, $\langle n, E_3\rangle$ and ν_3 cannot simultaneously vanish. At boundary points where $\nu_3 \neq 0$, we can multiply (50) by ν_3 and use (52) to obtain $\alpha H^2 + \beta K = 0$. At points where $\nu_3 = 0$, we get the same conclusion from (53) and the remark above. We conclude that $(\alpha H^2 + \beta K) \equiv 0$ holds and by the assumptions on the functional, both principal curvatures vanish identically on the boundary.

We first consider the case $\alpha > 0 = \beta$. From the discussion above, we find that $H \equiv 0 \equiv H_n$ on the boundary. However H satisfies the second order elliptic equation (48) and the only solution of such an equation vanishing to first order on the boundary is $H \equiv 0$, so the surface is a minimal surface. If we next note that the boundary of the surface is contained in the plane, we find by the Maximum Principle that the surface is itself part of the plane.

Another case we will discuss is $\alpha = 1$, $\beta = -1$. This is the conformally invariant functional

$$E = \int_\Sigma H^2 - K\, d\Sigma\,.$$

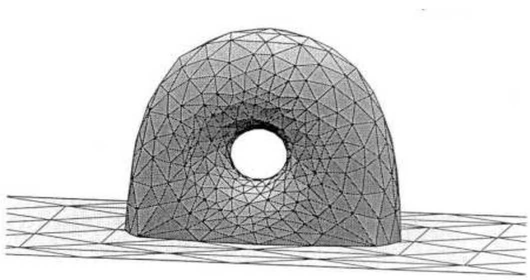

FIGURE 4. A genus one Willmore surface with free boundary on a plane.

In this case we obtain from the argument above that $H^2 - K \equiv 0$ on the boundary of the surface, i.e. the boundary consists entirely of umbilics. If the topology of the surface is restricted to be that of a disc, then the following result holds, [15].

Theorem 0.6 *Let D denote the unit disc and let $X : (D, \partial D) \to (\mathbf{R}^3_+, \mathbf{R}^2)$ be a $C^4(\bar{D})$ immersion which is a solution of (48)-(50) with $\alpha = 1$, $\beta = -1$. Then $X(D)$ is either a spherical cap or flat disc meeting the plane in a constant angle.*

Figure (4) was produced using the Surface Evolver program and indicates that for higher topology there may be other Willmore surfaces with free boundary on a plane.

ANISOTROPIC SURFACE ENERGY

An anisotropic surface energy assigns to a surface an energy whose density at each point depends on the direction of the surface. We will first consider the simplest type of such an energy called a *parametric functional*.

Let $F : S^2 \to \mathbf{R}^3$ be a sufficiently smooth, non negative function. To a smooth, oriented surface $X : \Sigma \to \mathbf{R}^3$ with normal map ν, we assign the value

$$\mathscr{F}[X] := \int_\Sigma F(\nu)\, d\Sigma\,.$$

(We will assume that Σ is at least relatively compact so that the integration makes sense.) This type of functional was introduced by the crystallographer G. Wulff [16] in order to model the shape of a small crystal. It also occurs in connection with nematic liquid crystals,[17], [18]. For example, Virga, [17], [19], has shown that such an energy determines the interface of small nematic liquid crystal drop in an isotropic environment. When the drop is small, the director field will tend to a constant and the surface energy will be of the form given above.

We briefly recall the *Wulff Construction*. Consider the radial plot of F on S^2, that is the map $S^2 \to \mathbf{R}^3$, given by $\nu \mapsto F(\nu) \cdot \nu$. For each $\nu \in S^2$ draw the plane Π perpendicular to ν through the point $F(\nu) \cdot \nu$ and throw away the component of $\mathbf{R}^3 \setminus \Pi$ which does not contain S^2. When this is done at each point of the sphere, what remains is a convex

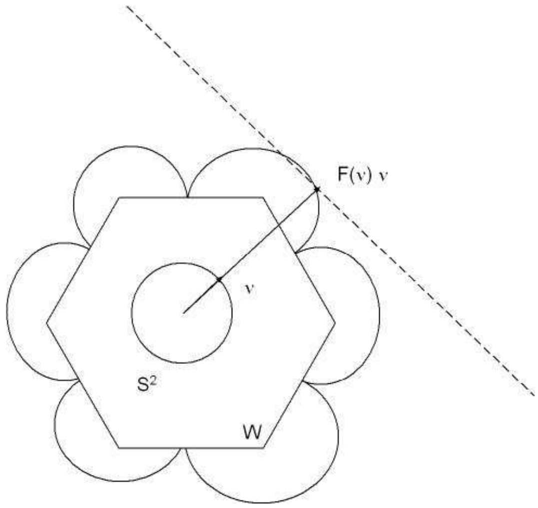

FIGURE 5. The Wulff construction.

region whose boundary will be called the *Wulff shape* or *Wulff crystal*. It wil be denoted by W. (In some references the region itself is given this name.) A result loosely known as *Wulff's Theorem* states that

Theorem 0.7 *W minimizes $\mathscr{F}$ among all closed surfaces enclosing the same volume.*

Thus, W solves the isoperimetric problem for the functional $\mathscr{F}$ and plays the role that the two dimensional sphere plays for the area functional.

We now assume that F is sufficiently smooth. We consider a variation of the surface X given by

$$X_\varepsilon = X + \varepsilon(\xi + \eta\nu) + \mathscr{O}(\varepsilon^2)\,, \tag{54}$$

where ξ is tangent to Σ. A simple calculation shows that $\delta\nu = -\nabla\eta + d\nu(\xi)$. Let DF denote the gradient of F on S^2. Note that $DF(\nu)$ can also be considered as a tangent vector field on Σ.

The first variation of $\mathscr{F}$ is

$$\begin{aligned} \delta\mathscr{F}[X] &= \int_\Sigma \langle DF, -\nabla\eta + d\nu(\xi)\rangle + F(\mathrm{div}\xi - 2H\eta)\, d\Sigma \\ &= \int_\Sigma \eta(\mathrm{div}DF - 2HF\eta) + \langle DF, d\nu(\xi)\rangle + F\mathrm{div}\xi\, d\Sigma \\ &= \int_\Sigma \eta(\mathrm{div}DF - 2HF\eta) + \langle d\nu(DF), \xi\rangle + F\mathrm{div}\xi\, d\Sigma \\ &= \int_\Sigma \eta(\mathrm{div}DF - 2HF\eta)\, d\Sigma + \oint_{\partial\Sigma} \langle DF, n\rangle + F\langle \xi, n\rangle\, ds\,, \end{aligned}$$

using that $d\nu(DF) = \nabla F$ and that $\mathrm{div}(F\xi) = \langle\nabla F, \xi\rangle + F\mathrm{div}\xi$.

If we restrict to compactly supported variations, we see that the the the first variation vanishes exactly when

$$0 \equiv \Lambda := -(\mathrm{div}DF - 2HF)\,. \tag{55}$$

The function Λ is called the *anisotropic mean curvature*. Note that when $F \equiv 1$, Λ is twice the usual mean curvature. Critical points of $\mathscr{F}$ are characterized by the Euler-Lagrange equation $\Lambda = 0$ and more generally, critical points of $\mathscr{F}$ subject to a volume constraint are characterized by the equation $\Lambda \equiv$ constant.

We now will assume that W is a smooth, convex surface. Note that this is much stronger than requiring F to be smooth. If we let μ_j, $j = 1,2$ denote the principle curvatures of W with respect to the inward pointing normal, then the convexity implies $\mu_j < 0$ for $j = 1,2$. In some of our examples, we relax this and only require the non strict inequality.

In order to produce an F so that W is smooth, one can start with any smooth convex surface W. Since W is convex, the normal map N is a diffeomorphism onto S^2. Let Q be the support function of W and define $F := Q \circ N^{-1}$. Then it is not hard to see that W will be the Wulff shape for the functional defined by this choice of F. In what follows, we will use the fact that the normal map is a diffeomorphism to consider all geometric quantities defined on W to also be defined on the sphere S^2.

In this case, we will call $\mathscr{F}$ an *elliptic parametric functional*. There is a more general version in which F may depend on both ν and the position vector X which is useful for inhomogeneous materials but we will not consider this case here. In the literature, the case $F = F(\nu)$ is referred to as a *constant coefficient* elliptic parametric functional.

When W is smooth and uniformly convex, the equation for prescribed anisotropic mean curvature is *absolutely elliptic* in the sense of Hopf, [20] . This means that the linearization of an equation for prescribed Λ is elliptic at any surface, not just a solution surface. It follows that a Maximum Principle analogous to the well known one for constant mean curvature surfaces holds. For example, if two surfaces with the same constant anisotropic mean curvature are in oriented contact at a point p and one of the surfaces lies on one side of the other near p, then the two surfaces must coincide on a neighborhood of p.

For closed surfaces with constant anisotropic mean curvature, we have the following uniqueness result, [21].

Theorem 0.8 *If $X : \Sigma \to \mathbf{R}^3$ is a closed surface with constant anisotropic mean curvature, then the surface is stable if and only if it is the Wulff shape modulo homotheties and those rigid motions of $\mathbf{R}^3$ which are symmetries of W.*

Let D^2F denote the Hessian of F on S^2 and define $A := D^2F + FI$ where I is the identity on each tangent space to S^2. By parallel translation in $\mathbf{R}^3$, $A_{\nu(p)}$ defines a linear transformation on each tangent space $T_p\Sigma$. Although $Ad\nu$ is not, in general, self-adjoint, it nevertheless has real eigenvalues which we denote by λ_i, $i = 1,2$. They will be called *anisotropic principal curvatures*. They satisfy:

$$\Lambda = \lambda_1 + \lambda_2 = -\mathrm{trace}\, A \circ d\nu \,, \tag{56}$$

and

$$\lambda_1\lambda_2(p) = K_\Sigma(p)/K_W(\nu(p)) \tag{57}$$

Here K_W denotes the curvature of the Wulff shape.

The surfaces of revolution with constant anisotropic mean curvature are called *anisotropic Delaunay surfaces*, [22]. It is to be expected that these surfaces exist only

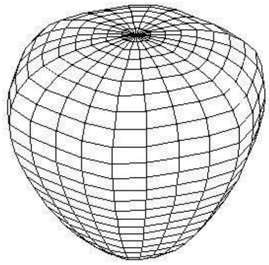

FIGURE 6. A Wulff shape of the form (58)

when the free energy is rotationally invariant, $F(\nu_3)$, i.e. when the Wulff shape is a surface of revolution. There is a generalization to the case when W has a product structure, [23]. We assume that W can be parameterized

$$\chi(\sigma,\tau) = (\sigma,\tau) \mapsto (u(\sigma)\alpha(\tau), u(\sigma)\beta(\tau), v(\sigma))\,, \tag{58}$$

where (u,v) and (α,β) are convex curves. We then look for a surface given by

$$(s,\tau) \mapsto (x(s)\alpha(\tau), x(s)\beta(\tau), z(s))\,,$$

having constant anisotropic mean curvature for the functional defined for the given Wulff shape. It can be shown that this is the case provided the following hold.

$$2ux + \Lambda x^2 = c\,, \tag{59}$$

$$z = \int_{v_0} x_u \, dv\,. \tag{60}$$

In (59), Λ and c are constant. This equation must be solved to obtain $x = x(u)$ whose derivative appears in (60). The orientation of an anisotropic Delaunay surface may be chosen so that $\Lambda \le 0$ holds and then the anisotropic Delaunay surfaces fall into six cases as follows:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane*.
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid*.
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to vertical translation and homothety)*.
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *cylinder of radius* $(\mu_2|_{\nu_3=0}|\Lambda|)^{-1}$.
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid*.
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid*.

Here, μ_2 is the principal curvature of the Wulff shape along a circle of latitude. The surfaces in each case above are complete and they have properties similar to those of the corresponding constant mean curvature surfaces. For example, the generating curve of

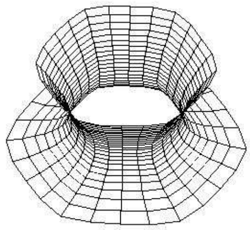

FIGURE 7. Anisotropic catenoid for the energy having Wulff shape given in Figure (6).

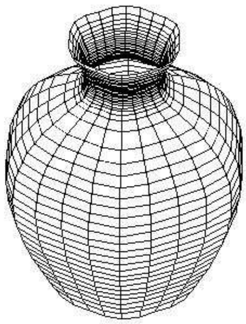

FIGURE 8. An anisotropic unduloid for the energy having Wulff shape given in in Figure (6).

an anisotropic unduloid is an embedded periodic curve, while the generating curve of an anisotropic nodoid is a periodic curve with self-intersections.

By analogy with the bending energy (45), we define an *anisotropic bending energy* by:

$$E_{a,b} := \int_\Sigma a(\lambda_1^2 + \lambda_2^2) + 2b\lambda_1\lambda_2 \, d\Sigma, \quad a > 0\,. \tag{61}$$

Here a and b are constants. When $F \equiv 1$ holds, the λ_i's are just the usual principal curvatures and the definition of E corresponds to the usual bending energy, [24]. These functional are scale invariant but are not, in general, conformally invariant. If the plate is closed ($\partial\Sigma = \emptyset$) or if the boundary values are kept fixed to first order under deformations, then $E_{a,b}$ is variationally equivalent to the anisotropic Willmore functional

$$\Xi[\Sigma] := \int_\Sigma \Lambda^2 \, d\Sigma\,..$$

In particular, $\delta E_{a,b} = 0$ if and only if $\delta\Xi = 0$.

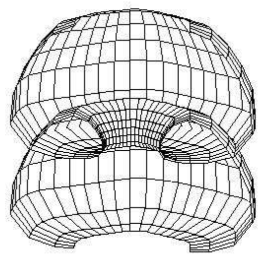

FIGURE 9. An anisotropic nodoid for the energy having Wulff shape given in in Figure (6).

We state the following simple result from [25] and [22].

Theorem 0.9 *Let* $\Sigma \rightarrow \mathbf{R}^3$ *be any immersed topological sphere. Then,*

$$\Xi(\Sigma) \geq \Xi(W) = 4\mathrm{Area}(W)$$

holds with equality in the inequality if and only if Σ *is either a homothety of* W *or* Σ *differs from* W *by a rigid motion which is a symmetry of* W.

Proof The definitions in (56) and (57) give that

$$0 \leq (\lambda_1 - \lambda_2)^2 = \Lambda^2 - 4K_\Sigma/K_W \, .$$

It is known that $\lambda_1 \equiv \lambda_2$ only for surfaces which differ from W by a homothety or a rigid motion which is a symmetry of W, [26]. Let $d\Omega$ denote the usual area form on S^2.

$$\begin{aligned} \Xi(\Sigma) = \int_\Sigma \Lambda^2 \, d\Sigma &\geq \int_\Sigma 4K_\Sigma/K_W \, d\Sigma \\ &= \int_{S^2} 4/K_W \, d\Omega \\ &= 4\mathrm{Area}(W) \\ &= \int_W \Lambda^2 \, dW. \end{aligned}$$

Here we have used the change of variables theorem to equate the first and second lines together with the fact that the Gauss map of a genus zero surface has degree one. The last equality is that the anisotropic mean curvature of W with the outwards orientation is -2. **q.e.d**

The three theorems already stated in this section give a nice picture of how certain results for isotropic functionals can be extended to the anisotropic case.

We consider now the first variation of the functional Ξ. We include the boundary terms for use later. The pointwise variation of the anisotropic mean curvature for the variation (54) is given by

$$\delta\Lambda = L[\eta] + \langle \nabla\Lambda, \xi \rangle \, ,$$

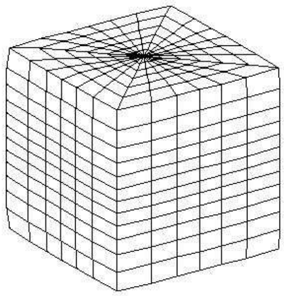

FIGURE 10. A Wulff shape of the form (58)

where L is the self adjoint operator

$$L[\eta] = \mathrm{div} A\nabla\eta + \langle A\cdot d\nu, d\nu\rangle\eta\,.$$

$$\begin{aligned}\delta\Xi &= \int_\Sigma 2\Lambda\delta\Lambda + \Lambda^2(\mathrm{div}\xi - 2H\eta)\,d\Sigma\\ &= \int_\Sigma 2\Lambda(L[\eta] + \langle\nabla\Lambda,\xi\rangle) + \Lambda^2(\mathrm{div}\xi - 2H\eta)\,d\Sigma\\ &= \int_\Sigma 2\eta(L - H\Lambda)[\Lambda]\,d\Sigma + \oint_{\partial\Sigma} 2\Lambda\langle A\nabla\eta, n\rangle - 2\eta\langle A\nabla\eta, n\rangle + \Lambda^2\langle\xi, n\rangle\,ds.\end{aligned}$$

By restricting to compactly supported variations, we obtain the Euler Lagrange equation

$$(L - H\Lambda)[\Lambda] = 0\,. \tag{62}$$

This is a fourth order non linear equation for the surface. Any C^4 surfaces which satisfies this equation will be called an *anisotropic Willmore surface*.

We will now show how to obtain some simple examples of anisotropic Willmore surfaces.

- **Rescalings of W.** It is clear from Theorem (0.9) that any such surface must solve the equation 62.
- **Anisotropic catenoids** , [22]). This is a special case of the construction of anisotropic Delaunay surfaces given above. The Wulff shape is assumed to be given by (58). The anisotropic catenoid is a surface with anisotropic mean curvature $\Lambda \equiv 0$ parameterized as follows. Its profile curve $(x(\sigma), z(\sigma))$ is given by:

  $$x = \frac{c}{2u}\,, \quad z = \frac{-c}{2}\int^\sigma \frac{dv(\sigma)}{u^2}\,.$$

  The catenoid can then be parameterized:

  $$X(\sigma,t) = (x(\sigma)\alpha(t), x(\sigma)\beta(t), z(\sigma))\,, \tag{63}$$

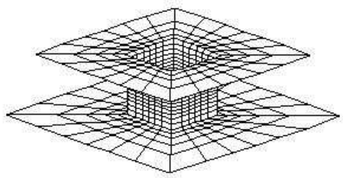

FIGURE 11. Anisotropic catenoid for the energy having Wulff shape given in Figure (10).

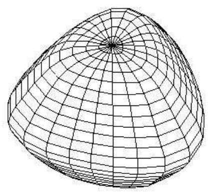

FIGURE 12. A Wulff shape of the form (58)

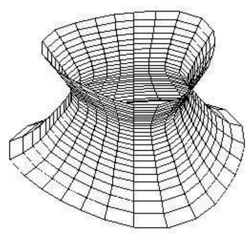

FIGURE 13. Anisotropic catenoid for the energy having Wulff shape given in Figure (12).

where $c \neq 0$ is an arbitrary constant. The catenoid has the property that *sufficiently small pieces of it minimize the anisotropic energy defined by W among all surfaces having the same boundary.*

- **Helicoids.** Let

$$\mathscr{F} = \int F(\nu_3)\, d\Sigma$$

be a rotationally invariant elliptic functional. We will consider surfaces representable as graphs

$$x, y \mapsto (x, y, z(x, y))$$

Let $d^2x = dxdy$ and note that the induced measure is given by $d\Sigma = \nu_3^{-1} d^2x$, so

$$\mathscr{F} = \int F(\nu_3)\nu_3^{-1}\, d^2x$$

If we consider a variation of the function $z : z \to z + \varepsilon \dot{z} + \ldots$, the corresponding change in ν_3 is

$$\dot{\nu}_3 = -\nu_3^3 \langle Dz, D\dot{z} \rangle \,,$$

and so (by a simple calculation)

$$\delta(F\, d\Sigma) = \frac{\nu_3}{\mu_2} \langle Dz, D\dot{z} \rangle\, d^2x \,.$$

Integrating this last expression by parts yields the Euler-Lagrange equation $\Lambda = 0$ is equivalent to

$$\mathrm{Div}(\frac{\nu_3}{\mu_2} Dz) = 0$$

Let (u, v) be the profile curve of the Wulff shape W. The axial coordinate of the Wulff shape is given by,

$$u = \frac{\sqrt{1 - \nu_3^2}}{\mu_2} \,.$$

Using this and the formula $\nu_3 = (1 + |Dz|^2)^{-1/2}$, allows us to rewrite the Euler Lagrange equation as

$$\mathrm{Div}(u \frac{Dz}{|Dz|}) = 0 \,. \tag{64}$$

Here u is the function of ν_3 given above re-expressed as a function of $|Dz|$.

Proposition 0.8 *For any $a \in \mathbf{R}^+$, the (multi-valued) function $z = a\theta := a \arctan(y/x)$ solves (64). In particular, for any rotationally invariant functional, the usual helicoids*

$$x, y \mapsto (x, y, a\theta)$$

have zero anisotropic mean curvature.

Proof. Set $\vec{r} = (x,y)$, $J\vec{r} = (-y,x)$ and $r = \sqrt{x^2+y^2}$. Then

$$D(a\theta) = a\frac{J\vec{r}}{r^2}, \quad \frac{D(a\theta)}{|D(a\theta)|} = \frac{J\vec{r}}{r}.$$

We then have $\nu_2 = (1+|Da\theta|^2)^{-1/2} = (1+a^2/r^2)^{-1/2}$ so that $u(\nu_3) = u((1+a^2/r^2)^{-1/2}) =: U(r)$. Combining these facts, we find that the $\Lambda \equiv 0$ is equivalent to

$$\mathrm{Div}(\frac{U(r)}{r}J\vec{r}) = 0.$$

Note that

$$\mathrm{Div}(\frac{U(r)}{r}J\vec{r}) = \partial_r(\frac{U(r)}{r})\langle Dr, J\vec{r}\rangle + \frac{U(r)}{r}\mathrm{Div}J\vec{r}.$$

However $Dr = \vec{r}/r \perp J\vec{r}$ and $\mathrm{Div}(J\vec{r}) = 0$ so the result follows. **q.e.d**

For more general functionals it is sometimes possible to construct a surface with $\Lambda \equiv 0$ which is a periodic multi-valued graph over the punctured plane.

- **Anisotropic Willmore surfaces of revolution.** We assume that the Wulff shape W is a surface of revolution (with vertical axis) and look for another surface of revolution Σ which satisfies the Euler-Lagrange equation (62). For this we will use the boundary terms in the first variation formula. We take the the variation field δX to be a symmetry of the Lagrangian, Then if Σ is a part of a surface of revolution satisfying (62), we have

$$0 = \delta\Xi = \oint_{\partial\Sigma} 2\Lambda\langle A\nabla\eta, n\rangle - 2\eta\langle A\nabla\Lambda, n\rangle + \Lambda^2\langle\xi, n\rangle\, ds. \tag{65}$$

Assuming that the boundary consists of two horizontal circles C_{Top} and C_{Bottom}. We take first, $\delta X = E_3 = E_3^T + \nu_3\nu$ and then take $\delta X = X = X^T + q\nu$ where q is the support function. These variations fields correspond to vertical translation and homothety which are symmetries of Ξ. For both these variational fields, the integrands in (65) are constant on each of the boundary circles. The integrals are just a constant times the arc length $2\pi x$ and the results are the two equations given below:

$$x_s\frac{2\Lambda_s}{\mu_1} + z_s\Lambda(\frac{k_2}{\mu_2} - \frac{k_1}{\mu_1}) = \frac{C_1}{x}, \tag{66}$$

$$-q\frac{2\Lambda_s}{\mu_1} + \langle X, X_s\rangle\Lambda(\frac{k_2}{\mu_2} - \frac{k_1}{\mu_1}) = \frac{C_2}{x}. \tag{67}$$

Computing the determinant $x_s\langle X, X_s\rangle + qz_s = x$ and inverting this system, yields

$$\frac{2\Lambda_s}{\mu_1} = \frac{1}{x^2}(-C_2 z_S + C_1\langle X, X_s\rangle) \tag{68}$$

$$\Lambda(\frac{k_2}{\mu_2} - \frac{k_1}{\mu_1}) = \frac{1}{x^2}(C_2 x_s + C_1 q) \tag{69}$$

if we make a vertical translation of the generating curve $(x,z) \to (x, z+c)$ changes the support function q according to $q \to q - cx_s =: q_1$. Thus choosing $cc_1 = -c_2$ changes (69) to the simpler

$$\Lambda(\lambda_2 - \lambda_1) = \frac{cq}{x^2}\,, \tag{70}$$

where we have renamed q_1 as q. In the isotropic case $F \equiv 1$, this equation occurs in [27].

Remarks. (i) In the case of a surface of revolution , the equation (62) becomes,

$$\frac{1}{x}(\frac{x\Lambda_s}{\mu_1})_s + (\frac{k_1^2}{\mu_1} + \frac{k_2^2}{\mu_2} - \Lambda H)\Lambda = 0\,. \tag{71}$$

It can be shown that, except for the case when the surface is a round cylinder, equations (70) implies equation (71) . For this one needs the “Codazzi equation "

$$x^2\Lambda_s = (x^2(\lambda_1 - \lambda_2))_s\,,$$

which is easily derived from (73) and (74) below.

(ii) An anisotropic Willmore surface of revolution with zero or one boundary component is, up to rescaling, the Wulf shape W or a flat disc. To see this, note that in either of these cases we can let one of the boundary circles shrink to a point and we obtain in (68) and (69) that the C_i's are both zero. Thus from (70) $\Lambda(\lambda_2 - \lambda_1) = 0$ holds. If $\Lambda \equiv 0$, is is easy to see that the surface must be a flat disc since the Gaussian curvature is non positive everywhere. If $\lambda_1 \equiv \lambda_2$, then the surface is known, [26], to be homothetic to a part of W.

In order to make these conclusions, it is essential that the surface be sufficiently regular, i.e. of class C^4. For example, one can invert the catenoid to get a closed surface but this surface fails to have the required differentiability.

Denote by $(u(\sigma), v(\sigma))$, $(x(s), z(s))$ the arc length parameterizations of the generating curve of the Wulff shape and of a second surface of revolution Σ respectively. At points where the normals to these two curves agree, we have

$$u_\sigma = x_s, \qquad v_\sigma = z_s\,. \tag{72}$$

We denote the principal curvatures of W (with respect to the outward pointing normal) by μ_i, $i = 1,2$ and we denote the principal curvatures of Σ by k_i, $i = 1,2$. Then $z_{ss} = -k_1 x_s$, $v_{\sigma\sigma} = \mu_1 u_\sigma$. It follows that from (72) that,

$$-\lambda_1 = \frac{z_{ss}}{v_{\sigma\sigma}} = \frac{v_{\sigma s}}{v_{\sigma\sigma}} = \frac{d\sigma}{ds}\,.$$

Then using that $v_z = v_\sigma \sigma_s s_z$, we obtain

$$\lambda_1 := k_1/\mu_1 = -v_z\,. \tag{73}$$

Again, u and x are related by the equations (72). We also have that

$$k_2 = -z_s/x, \qquad \mu_2 = v_\sigma/u\,,$$

so by (72)

$$\lambda_2 = -u/x\,. \tag{74}$$

If we now think of the generating curve of Σ as being given as a graph $x = x(z)$, we can then express the equation (70) as

$$u^2 - x^2 v_z^2 = \frac{c(x - zx_z)}{\sqrt{1+x_z^2}}\,. \tag{75}$$

We can easily obtain from (72) that

$$x_z = u_v\,. \tag{76}$$

Because W is convex, we have that $u_{vv} < 0$ holds globally and thus it is possible to solve $V = V(u_v)$. Using (76) , we can write the generating curve of W in the form $u = u(v)$ and replace u in (75) by $u(V(u_v)) =: U(x_z)$. Likewise, we can write

$$v_z = \frac{\partial V}{\partial u_v}(u_v)\cdot\frac{\partial u_v}{\partial z} = \frac{\partial V}{\partial u_v}(u_v)\cdot x_{zz}\,.$$

Using this to replace v_z in (75) and solving algebraically for the highest order derivative, gives the quasilinear *second order* ODE:

$$x_{zz} = \pm\frac{1}{xV_{u_v}(x_z)}\Big[U^2(x_z) - \frac{c(x-zx_z)}{\sqrt{1+x_z^2}}\Big]^{1/2} \tag{77}$$

for the generating curve $z = z(x)$. For a single example, it is possible that there is a sign change in (77), that is for different parts of the surface, a different choice of the sign must be chosen.

In Figures (14) through (16) we show pieces of several examples. In all cases the Wulff shape W_p is derived from a generating curve given by $u^p + v^p = 1$. We use the notation (p, c, x_i, x_{zi}, s) to denote the solution of (77) for the Wulff shape W_p having initial conditions $x(0) = x_i$, $x_z(0) = x_{zi}$ and where $s = \pm 1$ is used to denote the choice of sign.

- **Cylinders over anisotropic elastica.**
  We consider a 1-dimensional, smooth Wulff shape Ω and the corresponding 1-dimensional anisotropic energy

$$\mathscr{F}[C] = \int_C F(\nu_2)\,ds\,.$$

Here C is a C^2 curve parameterized by arc length and $n := (n_1, n_2) = (z', -x')$ is its normal. Also F is the support function of W considered as a function on the circle via the inverse of the normal map. If $C + \varepsilon\eta(s)n$ is a deformation of C, then

$$\delta\mathscr{F}[C] = -\int_C \lambda\eta\,ds\,,$$

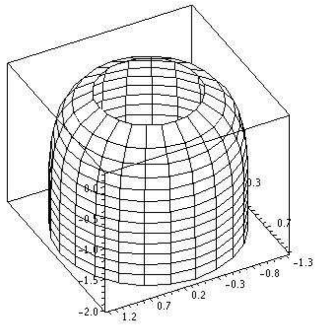

FIGURE 14. A piece of an anisotropic Willmore surface, $(4, .5, 1, -1, -1)$

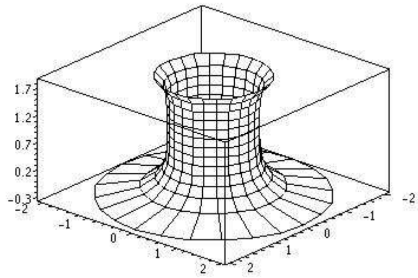

FIGURE 15. A piece of an anisotropic Willmore surface, $(4, .5, 1, -1, 1)$

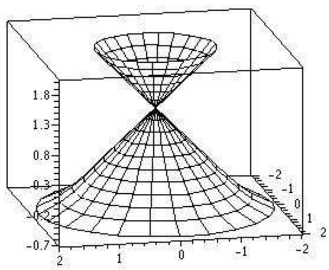

FIGURE 16. A piece of an anisotropic Willmore surface,$(4, 1, 1, -1, 1)$

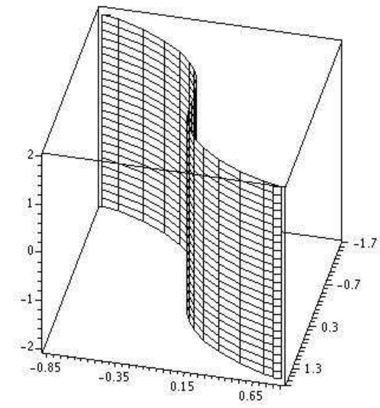

FIGURE 17. An anisotropic Willmore surface which is a cylinder over an anisotropic elastic curve. The Wulff shape is the surface of revolution generated by $u^4 + v^4 = 1$.

where $\lambda = k/\mu$. Here k denotes the curvature of C and

$$1/\mu = (1 - n_2^2)F''(n_2) - n_2 F'(n_2) + F(n_2)\,,$$

is negative the reciprocal of the curvature of W. We introduce another energy for sufficiently smooth curves by

$$\mathscr{E}[C] := \int_C \lambda^2 - \tau\, ds\,,$$

where τ is a real constant. A lengthy calculation gives the Euler Lagrange equation

$$(\frac{\lambda_s}{\mu_1})_s + (\frac{k^2}{2\mu})\lambda + \frac{\tau k}{2} = 0\,. \tag{78}$$

Solutions will be called *generalized anisotropic elastica*. When $\tau = 0$ we will call the curves simply generalized elastic. If now the curve Ω is considered as the generating curve of a surface of revolution W, then any cylinder over an anisotropic elastic curve will be an anisotropic Willmore surface for the energy functional whose Wulff shape is W.

A free boundary problem

We will consider briefly a boundary value problem for the anisotropic bending energy (45). We assume that the Wulff shape is rotationally symmetric ($F = F(\nu_3)$) and uniformly convex. This means that the principal curvatures of W , μ_j, $j = 1,2$ are everywhere positive. We will express (45) as

$$E_{a,b} = \int_\Sigma \alpha\Lambda^2 + \beta K_\Sigma / K_W \, d\Sigma\,.$$

We have discussed above the first variation of the integral of Λ^2 and we use

$$\delta \int_\Sigma K_\Sigma/K_W \, d\Sigma = \oint_{\partial\Sigma} \langle \delta\chi, \nu \times d\chi \rangle \, ds \,,$$

where $\delta\chi = A\delta\nu = A(-\nabla\eta + d\nu(\xi))$.

As in section in the section above where free boundary problems were considered in the isotropic case, we consider a surface in the half plane $x_3 > 0$ having free boundary on the plane $x_3 = 0$. Using the notation of that section, we obtain the first variation formula

$$\begin{aligned} \delta E_{a,b} &= 2\alpha \int_\Sigma 2\eta (L - H\Lambda)[\Lambda] \, d\Sigma + \\ &\oint_{\partial\Sigma} \left(\frac{2\alpha\Lambda + \beta\frac{\sigma_{22}}{\mu_2}}{\mu_1}\right)\eta_n - \left(\frac{2\alpha\Lambda_n}{\mu_1} - \beta\left(\frac{\sigma_{12}}{\mu_1\mu_2}\right)_t\right)\eta + \left(\alpha\Lambda^2 + \beta\frac{K_\Sigma}{K_W}\right)\langle \xi, n \rangle \, ds \,. \end{aligned}$$

Following the same steps as in the isotropic case, we arrive at the equations for equilibrium: We find that the Euler-Lagrange equations for the free boundary problem are

$$(L - H\Lambda)[\Lambda] = 0 \,, \text{in} \quad \Sigma \,. \tag{79}$$

$$2\alpha\Lambda + \beta\frac{\sigma_{22}}{\mu_2} = 0 \,, \qquad \text{on} \quad \partial\Sigma \,. \tag{80}$$

$$\left(\alpha\Lambda^2 + \beta\frac{K_\Sigma}{K_W}\right)\langle \nu, E_3 \rangle + \left(\frac{2\alpha\Lambda_n}{\mu_1} - \beta\left(\frac{\sigma_{12}}{\mu_1\mu_2}\right)_t\right)\langle n, E_3 \rangle = 0 \quad \text{on} \quad \partial\Sigma \,. \tag{81}$$

The discrepancy between the constants here and in the isotropic case is because $\Lambda = 2H$ in the isotropic case. We will make the rather strong assumption that the quadratic form $\mathcal{Q}(\lambda_1, \lambda_2) := a(\lambda_1^2 + \lambda_2^2) + 2b\lambda_1\lambda_2$ which appears in the definition of $E_{a,b}$ is positive definite. Following the same steps as in the isotropic case, we arrive at the conclusion that $\alpha\Lambda^2 + \beta K_\Sigma/K_W \equiv 0$ on the boundary. By using that $\mathcal{Q}$ is positive definite, we conclude that the tensor field $A \cdot d\nu$ vanishes identically on the boundary since both of its eigenvalues vanish there. For the chosen frame on the boundary, we have,

$$A \cdot d\nu = \begin{pmatrix} -\sigma_{11}/\mu_1 & -\sigma_{12}/\mu_1 \\ -\sigma_{12}/\mu_2 & -\sigma_{22}/\mu_2 \end{pmatrix}$$

so we see that the vanishing of $A \cdot d\nu$ implies that the geodesic torsion σ_{12} of the boundary is identically zero.

From (79) and (81), we have that Λ satisfies $(L - H\Lambda)[\Lambda] = 0$ in Σ with $\Lambda \equiv 0 \equiv \Lambda_n$ on $\partial\Sigma$. Since $u \to (L - H\Lambda)[u]$ is a second order elliptic operator, we can conclude that $\Lambda \equiv 0$ in Σ, i.e. the surface is an anisotropic minimal surface. A Maximum Principle similar to the one for minimal surfaces holds also in the anisotropic case, an anisotropic minimal surface lies in the convex hull of its boundary. (Alternatively one can use that a surface with $\Lambda \equiv 0$, has non positive Gaussian curvature.) Therefore, if there is only one boundary component which is a planer curve, the surface must be flat.

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