

Anisotropic Capillary Surfaces

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Abstract

This is a summary of recent work concerning capillary surfaces for anisotropic surface energies.

Keywords : anisotropic, capillary, stability

1 Introduction

Surfaces of constant mean curvature have been a central topic in low dimensional Differential Geometry for hundreds of years. Geometers are fond of justifying the attention that these surfaces receive by stating that they model a real life situation, namely the interface between two immiscible materials, such as water and air, in the absence of external forces such as gravity. Liquid water is a material in a “disordered” state; its constituent atoms have no internal structure. One may ask what happens if water is replaced by a more structured material. What will the shape of the interface be?

For a homogeneous liquid, the liquid to air interface is modeled by assigning to it a surface tension which is proportional to its area. The interface surface is in equilibrium when it assumes a shape which is a critical point for the area subject to the constraint that the volume is preserved. There are almost always some boundary conditions imposed. If the liquid is replaced by a more structured material, then it is necessary to take into account the direction of the surface at each point when defining the surface energy. Such a surface energy is said to be *anisotropic*.

An anisotropic surface energy can be defined by specifying a fixed closed convex surface W called the *Wulff shape*. Let $N : W \rightarrow S^2$ be the Gauss map of W which we will assume to be a diffeomorphism. We will denote the inverse of N by χ . Then the function on S^2 defined by $F(\mathbf{v}) = \langle \chi(\mathbf{v}), \mathbf{v} \rangle$ is just the support function of W pulled back to S^2 . We define an energy for a smooth, oriented immersed surface $X : \Sigma \rightarrow \mathbf{R}^3$ by

$$\mathcal{F}[X] = \int_{\Sigma} F(\mathbf{v}) d\Sigma, \quad (1)$$

where \mathbf{v} is the Gauss map of Σ . Such an energy is sometimes called a constant coefficient elliptic parametric functional. Additional conditions on the functional \mathcal{F} will be imposed by imposing restriction on the Wulff shape W .

We will consider here a fixed volume of a material which we will simply refer to as a “drop”. The drop will be assumed to lie between two fixed horizontal plates Π_i , $i = 0, 1$. In the case where the drop has non empty boundary, we assume that the boundary is constrained to lie on one or both of the plates. We will refer to this type of configuration as a *trapped drop*.

Just as there is an energy cost assigned to the interface between the drop and the surrounding air, there is a cost assigned interface between the drop and the plates called the *wetting energy* \mathcal{W} . For simplicity, we take the wetting energy to be of the form

$$\mathcal{W} = \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1, \quad (2)$$

where \mathcal{A}_i is the area of that part of the drop which is in contact with the plate Π_i and ω_i are *wetting constants*. In practice, the ω_i 's depend only on the materials involved. The cases $\omega_1, \omega_2 > 0$, (respectively

$\omega_1, \omega_2 < 0$), are referred to as *lyophobic*, (respectively *lyophilic*), wetting since the material of the drop will try to avoid, (respectively, be attracted to), the plates when energy is minimized.

The total energy is thus

$$\mathcal{E} := \mathcal{F} + \mathcal{W}, \quad (3)$$

and the condition which characterizes equilibrium surfaces is that $\mathcal{E}[X]$ be stationary among all embedded surfaces enclosing a fixed volume V which lie in the region B between the two plates. A sufficiently smooth equilibrium surface will be called a *capillary surface*.

The main problem we will consider below is the following. If we regard the energy functional \mathcal{F} , the volume V , the wetting constants ω_i and the distance h separating the plates as prescribed “initial conditions”, can we uniquely determine the capillary surface? Such a question may arise as part of a design problem. We may want to join together parallel plates with a material, e.g. solder, and exact information about the shape of the bridge may be required. On the other hand, a drop may be observed and we may want to use its shape to make statements about what forces are at work in determining its shape.

2 Characterization of equilibria-the first variation

For an immersed surface $X : \Sigma \rightarrow \mathbf{R}^3$, we consider a variation

$$X_\varepsilon = X + \varepsilon\psi\nu + O(\varepsilon^2)$$

where ψ is a smooth function with compact support. The first variation

$$\delta\mathcal{F}[X] := \partial_\varepsilon\mathcal{F}[X_\varepsilon]_{\varepsilon=0} =: - \int_\Sigma \psi\Lambda d\Sigma,$$

defines the *anisotropic mean curvature* function Λ . Since the first variation of the enclosed volume is given by

$$\delta V = \int_\Sigma \psi d\Sigma,$$

the equation $\Lambda \equiv \text{constant}$ characterizes surfaces which are in equilibrium for the anisotropic energy when the volume is held fixed. Note that with this definition Λ is equal to twice the usual mean curvature when $F \equiv 1$. The case $F \equiv 1$ will be referred to as the *isotropic case*.

Recall that we have an embedding $\chi : S^2 \rightarrow W$ which is just the inverse of the Gauss map of W . For a smooth oriented surface, $X : \Sigma \rightarrow \mathbf{R}^3$ we write $\chi(\nu)$ for the composition of these two maps. We will restrict ourselves from now on to the case where the energy density $F(\nu)$ is rotationally invariant, i.e. $F = F(\nu_3)$. This is equivalent to the assumption that the Wulff shape W is a surface of revolution.

Proposition 1 *An immersion $X : (\Sigma, \partial\Sigma) \rightarrow (B, \partial B)$ is a critical point of \mathcal{E} if and only if there holds:*

$$\Lambda \equiv \Lambda_0, \quad \text{in } \Sigma, \quad (4)$$

for some constant Λ_0 and

$$\langle \chi, E_3 \rangle \equiv -(-1)^i \omega_i, \quad \text{on } C_i := X(\partial\Sigma) \cap \Pi_i, \quad i = 0, 1. \quad (5)$$

Note that if one of the wetting constants ω_i is outside of the range of the vertical coordinate of W , then there is no solution which intersects the plane Π_i . For an embedding $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_0 \cup \Pi_1)$ with outward pointing unit normal ν , the *contact angle* of X with Π_i at $X(\zeta) \in \Pi_i$ ($\zeta \in \partial\Sigma$) is defined as the angle $\vartheta \in [0, \pi]$, between $\nu(\zeta)$ and $(-1)^i(0, 0, 1)$. The condition (5) implies that the contact angle is constant, ϑ_i , on each boundary component of a capillary surface.

Under the assumption that W is a uniformly convex surface, the equation $\Lambda \equiv \text{constant}$ (or more generally the equation for prescribed Λ) is absolutely elliptic in the sense of [4]. As a consequence, a Maximum Principle analogous to the one for constant mean curvature surfaces holds. If we assume also that W is a surface of revolution, then this Maximum Principle can be applied to show the following.

Theorem 1 *If X is either a closed embedded surface of constant anisotropic mean curvature for a rotationally symmetric $F = F(\nu_3)$ or an embedded capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (B, \Pi_1 \cup \Pi_2)$ for $F = F(\nu_3)$, then $X(\Sigma)$ is rotationally symmetric with respect to a vertical line and its genus is zero. In particular, if X is closed, then $X(\Sigma)$ is, up to translation and homothety, the Wulff shape.*

The surfaces of revolution with constant anisotropic mean curvature are called *anisotropic Delaunay surfaces*. These surfaces are remarkably similar in structure to the classical Delaunay surfaces. They were extensively studied in the references [5] through [9]. The generating curve $(x(s), z(s))$, where s is the arc length parameterization, satisfies an equation of the form,

$$\frac{2z_s x}{\mu_2} + \Lambda x^2 = c \quad (6)$$

where $\mu_2 = \mu_2(x_s)$ is one of the principal curvatures of the Wulff shape and c is a constant which is defined as a flux related to vertical translation. In figures 3 through 6 a Wulff shape and various forms of anisotropic Delaunay surfaces are displayed. The generating curve of the Wulff shape in Figure 3 is obtained from the Gielis formula, [3],

$$r = \{ |\cos(m\theta/4)|^{n_2} + |\sin(m\theta/4)|^{n_3} \}^{-n_1},$$

with $(m, n_1, n_2, n_3) = (3, 3.2, 4, 4)$.

Wulff's Theorem states that if Σ is a closed, embedded surface in \mathbf{R}^3 enclosing a volume V and if rW denotes a surface homothetic to the Wulff shape enclosing the same volume, then $\mathcal{F}[rW] \leq \mathcal{F}[\Sigma]$ holds. The following theorem of Winterbottom shows that suitable parts of rW minimize a free boundary problem. Let W_1 be a region in W which is included in the closed domain B bounded by two horizontal planes Π_j , $j = 0, 1$, with Π_0 being the lower of the two.

Theorem 2 *Assume that both of $\Pi_0 \cap W$ and $\Pi_1 \cap W$ are circles. Let Π'_i , $i = 0, 1$ be any pair of horizontal planes with Π'_0 , being the lower one. Consider the following energy functional for embedded surfaces with free boundary on $\Pi'_0 \cup \Pi'_1$ (or without boundary):*

$$\mathcal{E}' = \mathcal{F} + \omega_0 \mathcal{A}'_0 + \omega_1 \mathcal{A}'_1,$$

where \mathcal{A}'_j is the area of the planar region in Π'_j bounded by the boundary components of the considered surface. Assume that S is such an surface and $S \cup (\Pi'_0 \cup \Pi'_1)$ encloses the same volume as W_1 . Then there holds

$$\mathcal{E}(W_1) \leq \mathcal{E}'(S).$$

2.1 Stability

Figure 1 shows values of the volume for two families of capillary surfaces for the area functional. All of the surfaces in both families have the same height and contact angles so there is, in general, no possibility for uniqueness without an additional assumption. A natural condition to impose is that of stability since unstable surfaces will never be observed except as transition states. Here stability means that the second variation of the anisotropic energy is non negative for all volume preserving variations of the surface which preserve the boundary conditions. Stability of trapped drops for the area functional has been extensively studied in [2], [10], [11], [14], [15], [16].

Let X be a capillary surface for the energy functional \mathcal{E} . Let D^2F be the Hessian operator of F on S^2 and I is the identity endomorphism field. These may be considered as endomorphism fields on Σ using parallel translation in \mathbf{R}^3 and we let

$$A := (D^2F + F1)|_{\nu}.$$

The eigenvalue of A will be denoted by $1/\mu_i$, $i = 1, 2$. They are the reciprocals of the principal curvatures of the Wulff shape W with respect to the inward pointing normal.

The *Jacobi operator* is the self-adjoint elliptic operator

$$\tilde{L}[\psi] := \operatorname{div}(A\nabla\psi) + \langle Adv, dv \rangle \psi. \quad (7)$$

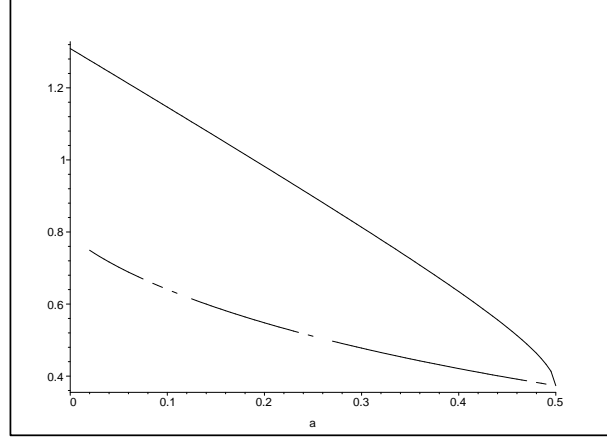


Figure 1: Plot of the volumes of stable (upper) and unstable (lower) unduloids as a function of $a := -\Lambda c$. The height h is 1 and the contact angle is $\pi/4$. The generating curves of the unstable unduloids have exactly one inflection point.

We also introduce the *boundary operator*

$$\tilde{\beta}[\Psi] = \begin{cases} \langle n, A\nabla\Psi \rangle - (-1)^i (\cot\vartheta)(k_1/\mu_1)\Psi, & \sin\vartheta_i \neq 0, \\ \Psi, & \sin\vartheta_i = 0 \end{cases}$$

If

$$X_\varepsilon : (-\varepsilon, \varepsilon) \times (\Sigma, \partial\Sigma) \rightarrow (B, \partial B),$$

is a one parameter family of embeddings with $X_0 = X$ and

$$V(X_\varepsilon) \equiv V(X)$$

where $V(\cdot)$ denotes the three dimensional volume enclosed by the surface, then the second variation formula is given by

$$\delta^2\mathcal{E} = - \int_{\Sigma} \Psi \tilde{L}[\Psi] d\Sigma + \int_{\partial\Sigma} \Psi \tilde{\beta}[\Psi] d\tilde{s}. \quad (8)$$

As stated above, the capillary surfaces we will be considering are, a priori, anisotropic Delaunay surfaces. In this case we have the following.

Proposition 2 *Assume that $z'(s) \neq 0$ for all $s \in (s_1, s_2)$. Then, X is stable if and only if X is stable for rotationally symmetric variations.*

The Jacobi operator L when restricted to symmetric functions $\Psi = \Psi(s)$ has the form

$$L[\Psi] = \frac{1}{x} \left(\frac{x\Psi_s}{\mu_1} \right)_s + \left(\frac{k_1^2}{\mu_1} + \frac{k_2^2}{\mu_2} \right) \Psi,$$

while the boundary operator is expressed

$$\beta[\Psi] = \begin{cases} (-1)^{i+1} \mu_1^{-1} \{ \Psi_s - (z_{ss}/z_s) \Psi \} & \sin\vartheta_i \neq 0, \\ \Psi, & \sin\vartheta_i = 0 \end{cases} \quad (9)$$

For a part of an anisotropic Delaunay surface with generating curve $(x(s), z(s))$, $s_1 \leq s \leq s_2$, we let λ_i , $i = 1, 2, 3, \dots$ be the i^{th} eigenvalue of the problem

$$(L + \lambda_i)\Psi(s) = 0, \text{ on } [s_1, s_2], \quad \beta[\Psi(s_i)] = 0, \quad i = 1, 2. \quad (10)$$

When the problem (10) has zero eigenvalues, denote by E the eigenspace of zero eigenvalues, and by E^\perp its orthogonal complement in $L^2([s_1, s_2], x ds)$.

Let

$$C_*^\infty(\Sigma) := \{\psi \in C^\infty(\Sigma) \mid \psi \text{ vanishes at any point in } \partial\Sigma \text{ where } X \text{ is tangent to } \partial B.\}$$

A characterization of stable equilibria in terms of the spectrum for this problem is the following.

Lemma 1 *Assume that $z'(s) \neq 0$ holds for all $s \in (s_1, s_2)$.*

(I) *If $\lambda_1 \geq 0$, then X is stable.*

(II) *If $\lambda_1 < 0 < \lambda_2$, then there exists a uniquely determined function $\phi \in C_*^\infty[s_1, s_2]$ satisfying $L[\phi] = 1$ and $\beta[\phi](s_i) = 0$, $i = 0, 1$, and the following (II-1) and (II-2) hold.*

(II-1) *If $\int_{s_1}^{s_2} \phi x ds \geq 0$, then X is stable.*

(II-2) *If $\int_{s_1}^{s_2} \phi x ds < 0$, then X is unstable.*

(III) *If $\lambda_2 = 0$, then, for an eigenfunction ϕ_2 belonging to 0, the following (III-A) and (III-B) hold:*

(III-A) *If $\int_{s_1}^{s_2} \phi_2 x ds \neq 0$, then X is unstable.*

(III-B) *If $\int_{s_1}^{s_2} \phi_2 x ds = 0$, then there exists a uniquely determined function $\phi \in E^\perp \cap C_*^\infty[s_1, s_2]$ satisfying $L[\phi] = 1$ and $\beta[\phi](s_i) = 0$, $i = 1, 2$ and the following (III-B1) and (III-B2) hold.*

(III-B1) *If $\int_{s_1}^{s_2} \phi x ds \geq 0$, then X is stable.*

(III-B2) *If $\int_{s_1}^{s_2} \phi x ds < 0$, then X is unstable.*

(IV) *If $\lambda_2 < 0$, then X is unstable.*

2.1.1 Lyophobic wetting

We will first impose conditions on the functional \mathcal{F} which will be described via the corresponding Wulff shape W . It will be assumed that

(W1) W is a uniformly convex surface of revolution with vertical rotation axis.

(W2) W is symmetric with respect to reflection through the horizontal plane $z = 0$.

(W3) The generating curve of W has non-decreasing curvature (with respect to the inward pointing normal) as a function of arc length on $\{z \geq 0\}$ as one moves in an upward direction.

For example, the parts of the surfaces of revolution generated by rotating the curves $u^{2p} + v^2 = 1$, $p \geq 1$ about a vertical axis have this property if we restrict v to be suitably close to zero. In addition, it will be assumed that $\omega_0 = \omega_1 =: \omega \geq 0$ holds in (2) so that the functional is of the form:

$$\mathcal{E}_\omega = \mathcal{F} + \omega \left(\mathcal{A}_0 + \mathcal{A}_1 \right), \quad (11)$$

The case of the ω_i 's being equal would occur for example, if both supporting planes were made from the same material.

The following result characterizes completely the stable equilibria.

Theorem 3 *Let X be a capillary surface with free boundary on two horizontal planes for the functional \mathcal{E}_ω with $\omega \geq 0$ and with the Wulff shape for the functional satisfying the conditions (W1) through (W3) stated above.*

(i) *If $\omega = 0$, then X is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder which is perpendicular to $\Pi_0 \cup \Pi_1$ and whose height h and radius R satisfy*

$$\frac{\mu_1(0)}{\mu_2(0)} (1/R^2) \leq (\pi/h)^2,$$

where $\mu_i(0)$, $i = 1, 2$, is the value of μ_i along the equator of W .

(ii) *If $\omega > 0$ holds, then X is stable if and only if X is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.*

A capillary surface will be called *spanning* if it intersects both the planes Π_i in a circle of positive radius. We will refer to a compact anisotropic capillary surface having non-empty boundary components only on the plane $z = z_0$, (respectively, $z = z_1$) as a *sessile drop*, (respectively, *pendent drop*). Such a surface is necessarily rotationally invariant, and therefore homothetic to a part of the Wulff shape.

We denote by $V_0(h, \omega)$ the infimum of the volumes of stable spanning anisotropic capillary surfaces having height h and contact angle $\vartheta(\omega)$. V_1 is the volume of the capillary surface which is homothetic to the part of the Wulff shape with contact angle $\vartheta(\omega)$ on the plane Π_i , $i = 0, 1$. Also $\bar{\omega}$ will denote the maximum of the vertical coordinate on the Wulff shape.

Theorem 4 *Assume that the Wulff shape satisfies the conditions (W1) through (W3). If $0 < \omega \leq \bar{\omega}$, then*

$$V_0(h, \omega) > \frac{h^3}{\pi} \left(\frac{2u(\omega)(u(0) - u(\omega))}{\omega^2 - (u(0) - u(\omega))^2} \right) > 0 \quad (12)$$

holds. If $\omega = 0$, then

$$V_0(h, 0) \geq \frac{h^3}{\pi} \left(\frac{\mu_1(0)}{\mu_2(0)} \right) \quad (13)$$

holds, and this inequality is sharp in the sense that there is a stable cylinder which satisfies the equality in (13).

Theorem 5 *We assume (W1) through (W3) stated above.*

[I] Assume $0 \leq \omega < \bar{\omega}$. Then, $V_0(h, \omega) > 0$ holds and,

(i) For volume $V < V_0$, any stable, connected capillary surface for the energy \mathcal{E}_ω with volume V and height h is a sessile or pendent drop.

(ii) For volumes $V \geq V_0$, there exists a unique stable spanning capillary surface for the energy \mathcal{E}_ω with volume V and height h .

[II] Assume $\omega = \bar{\omega}$. Then, any capillary surface for the energy \mathcal{E}_ω is tangent to the supporting planes $\Pi_0 \cup \Pi_1$. $V_0(h, \omega) > 0$ holds, and it coincides with the volume of the closed surface homothetic to the Wulff shape which is tangent to both of Π_0 and Π_1 . And,

(i) For volume $V \leq V_0$, there is no stable capillary surface for the energy \mathcal{E}_ω with volume V and height h .

(ii) For volumes $V > V_0$, there exists a unique stable capillary surface for the energy \mathcal{E}_ω with volume V and height h . Moreover, this surface is spanning.

More precise information about the relationship between the volume and the stable equilibria is given by the following.

Theorem 6 *We assume (W1) through (W3) stated above.*

(I) Assume $0 < \omega < \bar{\omega}$. Then,

(i) For volumes $V_0 \leq V < V_1$, there exists a unique stable spanning capillary surface with volume V , height h and contact angle $\vartheta(\omega)$, and the surface is an anisotropic unduloid. For $V = V_0$, this surface has inflection points on the boundary, while, for $V_0 < V < V_1$, it does not have inflection points.

(ii) For $V = V_1$, there exists a unique stable capillary surface with volume V , height h and contact angle $\vartheta(\omega)$, and the surface is homothetic to a part of the Wulff shape.

(iii) For $V_1 < V$, there exists a unique stable capillary surface with volume V , height h and contact angle $\vartheta(\omega)$, and the surface is an anisotropic nodoid.

(II) Assume $\omega = \bar{\omega}$. Then, for $V_0 < V$, there exists a unique stable capillary surface with volume V , height h and contact angle $\vartheta(\omega)$, and the surface is an anisotropic nodoid.

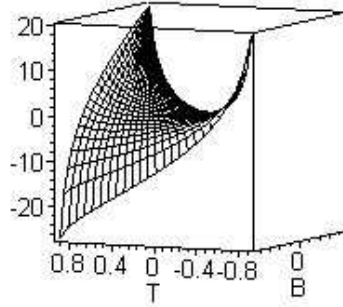


Figure 2: Plot of the expression in (14).

2.1.2 Lyophilic wetting

In the case of lyophilic wetting ($\omega < 0$ in (11)), it is more difficult to characterize the stable equilibria. For lyophilic wetting, capillary surfaces whose generating curves are free of inflection points are not necessarily stable. The situation is complicated by the fact, found by Zhou [16] in the isotropic case, that stability is not a monotonic property for capillary surfaces with wetting, i.e. a stable capillary surface may contain a proper subdomain which is unstable. However, using lemma 1, we can give criteria to determine the stability of any given anisotropic capillary surface by computational methods. We wish to point out that in the isotropic case, the determination of stable capillary surfaces with hydrophilic wetting is also partially based on numerical methods.

The following result generalizes a result of Zhou [16] for the isotropic case. It characterizes stable domains in anisotropic catenoids which lie between horizontal planes.

Theorem 7 *Let D be the domain in an anisotropic catenoid given by $z_1 \leq z \leq z_2$. Then D is stable for the free boundary problem if and only if the expression*

$$9 \left(\int_{z_1}^{z_2} x^2 dz \right)^2 - 5(z_2 - z_1) \int_{z_1}^{z_2} x^4 dz \quad (14)$$

is non negative.

We consider the stability of anisotropic catenoids for the Wulff shape generated by rotating the profile curve $u^4 + v^2 = 1$. A plot of the expression in (14) is shown in Figure 2. It is graphed as a function of the variables T and B which are respectively the values of v at the top and bottom of the domain in the catenoid. The plot shows that there is no general statement about the stability of capillary surfaces whose generating curve is free of inflection points.

The corresponding stability criteria for anisotropic unduloids and nodoids are the following.

Theorem 8 *Let X be an embedded capillary surface for a rotationally symmetric surface energy having contact angle $\vartheta(\omega)$.*

(i) Suppose $\omega < 0$ and X is contained in the negatively part of an anisotropic unduloid. Let $a := -\Lambda c$, with Λ and c given by (6). Then the surface is stable if and only if the following inequality holds.

$$1 \geq \frac{a \left[\int_{-\omega}^{\omega} \frac{u}{\sqrt{u^2-a}} - 1 dv \right] - 2 \left[\int_{-\omega}^{\omega} \frac{u}{\sqrt{u^2-a}} - 1 dv \right]^2 \left(\int_{-\omega}^{\omega} \frac{u}{(u^2-a)^{3/2}} dv \right)^{-1}}{3 \int_{-\omega}^{\omega} (u - \sqrt{u^2-a})^2 \left(\frac{u}{\sqrt{u^2-a}} - 1 \right) dv} := Q(\omega, a). \quad (15)$$

(ii) Suppose $\omega < 0$ and X is contained in the negatively part of an anisotropic nodoid. Let $a := -\Lambda c$, with Λ and c given by (6). Then the surface is stable if and only if the following inequality holds.

$$1 \leq \frac{a \left[\int_{-\omega}^{\omega} \frac{u}{\sqrt{u^2-a}} - 1 dv \right] - 2 \left[\int_{-\omega}^{\omega} \frac{u}{\sqrt{u^2-a}} - 1 dv \right]^2 \left(\int_{-\omega}^{\omega} \frac{u}{(u^2-a)^{3/2}} dv \right)^{-1}}{3 \int_{-\omega}^{\omega} (u - \sqrt{u^2-a})^2 \left(\frac{u}{\sqrt{u^2-a}} - 1 \right) dv} := Q(\omega, a). \quad (16)$$

2.1.3 Future directions-open questions

We have considered here the shape of capillary drops in the absence of any external force. Near the surface of the earth, gravity plays an important role in determining the shape of a large drop while the surface energy is the dominating factor for smaller drops. In order to include this force, a term for the gravitational potential energy

$$\mathcal{G} := \int_{\Sigma} z^2 \nu_3 d\Sigma$$

is adjoined and the total energy becomes

$$\mathcal{E} = \mathcal{F} + \mathcal{W} + \gamma \mathcal{G},$$

where γ is a coupling constant. The corresponding Euler-Lagrange equation is

$$\Lambda = \gamma z + \text{constant}.$$

Two examples are given in figures 7 and 8. What are the stable equilibria for this problem? The one dimensional case was considered in [1]. In [12], Wente gave a partial analysis of the isotropic case.

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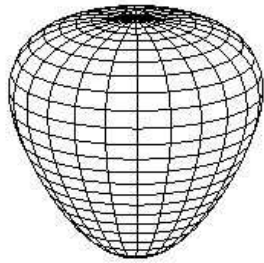


Figure 3: Wulff shape

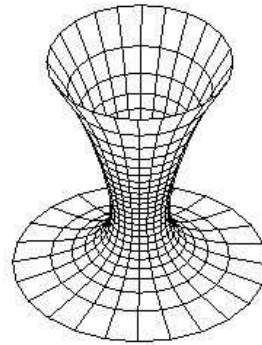


Figure 4: Anisotropic catenoid for the Wulff shape in Figure 3.



Figure 5: Anisotropic unduloid for the Wulff shape in Figure 3.

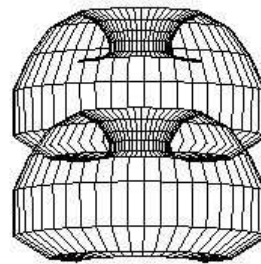


Figure 6: Anisotropic nodoid for the Wulff shape in Figure 3.

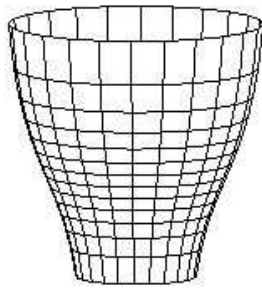


Figure 7: Surface with prescribed anisotropic mean curvature $\Lambda = -3.5z$. The Wulff shape is generated by the curve $u^4 + v^4 = 1$.

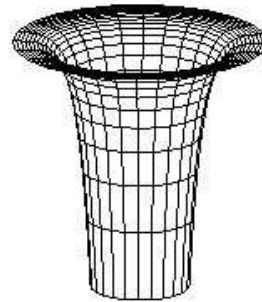


Figure 8: Surface with prescribed anisotropic mean curvature $\Lambda = 2z$. The Wulff shape is generated by the curve $u^4 + v^4 = 1$.