Equilibria for anisotropic surface energies with wetting and line tension

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Abstract

We consider the stability of capillary surfaces for non axially symmetric anisotropic surface energies. The energy includes wetting and line tension. We obtain necessary and sufficient conditions for the stability of anisotropic sessile drops.

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1 Introduction

Determining the geometry of interfaces between immiscible materials is an activity of increasing significance. A vast literature exists in relation to a liquid drop which is surrounded by a vapor and in contact with a solid substrate. Gibbs [3] noted that for a drop of a sufficiently small size, the energy contribution of the junction of the three materials, known as the line tension, plays an important role in determining the drop's geometry. This line energy tension is usually taken to be a multiple of the arc length, however for crystals, other notions of line tension appear. An important issue involves the sign of the constant coupling the line energy term in the total energy. Widom [18] shows that a spherical drop S resting on a planer substrate achieves an absolute minimum energy, but only among rotationally symmetric surfaces. Variations of S through non axially symmetric surfaces create an instability [15]. The reference [4] explains that this instability may not preclude the physical stability of S since the wave lengths of the destabilizing variations may fall beneath the length scale at which the entire surface tension model of surface formation is valid.

The present paper may be viewed as an attempt to extend these ideas to non liquid drops, specifically to those where the homogeneous surface tension is replaced by an anisotropic one. Specifically, we take our anisotropic free surface energy to be of the type introduced by Gibbs which sometimes referred to as the Wulff model. The free energy of a surface Σ is

$$\mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma \,,$$

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where γ is a suitable positive function and ν is the surface normal. Energies of this type were shown to determine the equilibrium shape of crystals and were also applied in early investigations into the geometry of liquid crystalline drops [2], [16]. Associated to each such surface energy, there is a unique convex surface W, the Wulff shape, which is the absolute minimizer of $\mathcal{F}$ among all closed surfaces enclosing the same volume V as W. If a closed convex surface W is given, it a simple matter to construct an anisotropic energy functional $\mathcal{F}$ having Wulff shape W. In this paper, will assume that W has a "product form". This amounts to the property that when W is intersected with any horizontal plane, the intersections are curves which are all homothetic to a fixed convex curve Ω and they project to a family of concentric curves. When these curves are circles, W is a surface of revolution. We then regard Ω as the Wulff shape of a one dimensional anisotropic line tension functional $\mathcal{L}$. This definition is a consequence of the demand that the part of the Wulff shape contained between any two horizontal planes is in equilibrium when the line tension, as defined above, and the wetting energy are coupled to the free energy by appropriate constants.

We consider the equilibria for the total energy which includes the free energy $\mathcal{F}$, the wetting energy=a constant times area $\mathcal{A}$ wetted on the horizontal supporting planes and the line tension. Above all, we study the anisotropic Delaunay surfaces which are the equilibria for the free energy all of whose intersections with a horizontal plane are homothetic to Ω. When the line tension is non-negative and the Wulff shape is rotationally symmetric, the Maximum principle is used to show that every equilibrium surface is of this form.

In the case of negative line tension, it is shown that all equilibria obtained from anisotropic Delaunay surfaces, including parts of the Wulff shape, are unstable. However the parts of the Wulff shape between horizontal planes have an interesting property analogous to that found by Widom in the isotropic case; they minimize energy among all surfaces enclosing the same volume and having horizontal cross sections homothetic to Ω.

A sessile drop S is defined to be a part of the Wulff shape having one boundary component on a horizontal plane. As mentioned above, each such surface is in equilibrium for a one parameter continuum of functionals. The parameter τ controls the linear combination of the wetting energy and line tension that appear in the functional. We find a sharp upper bound for the value of τ_0 at which S looses stability, that is, S is stable if and only it $\tau \leq \bar{\tau}$. The quantity $\bar{\tau}$ depends on the contact angle Θ between S and the plane. For $\Theta \approx 0$,(resp. $\Theta \approx \pi$), as $\tau \uparrow \bar{\tau}$ a wetting (resp. drying) transition is shown to occur which generalizes the behavior of liquid sessile drops found in [18].

2 Preliminaries and examples

To begin, consider two parallel (horizontal) planes Π_0, Π_1 in $\mathbf{R}^3$. Denote by Π the union of Π_0 and Π_1. Denote by Δ the closed domain of $\mathbf{R}^3$ bounded by Π. We consider a connected compact smooth surface Σ which is embedded into Δ and has free boundary on Π. Set $C_i := \Sigma \cap \Pi_i$ and denote by D_i the planer domain bounded by C_i. We assume that the compact closed piecewise smooth surface $\Sigma \cup D_0 \cup D_1$ encloses a given fixed volume. The total energy of the surface is taken to be:

$$\mathcal{E} := \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}\,, \tag{1}$$

where

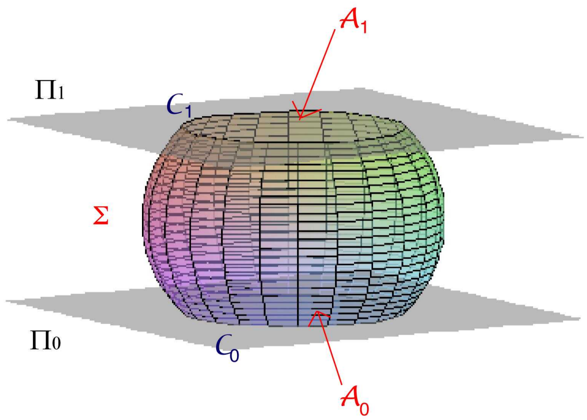


Figure 1: Configuration

- $\mathcal{F} := \int_\Sigma \gamma(\nu)\, d\Sigma$ is an anisotropic surface energy, where $d\Sigma$ is the area element of Σ, ν is the unit normal vector field along Σ, and γ is a function on the unit sphere S^2. We will assume that γ is a smooth function and that the following convexity condition holds. The Cahn-Hoffman field $\chi := \gamma\nu + D\gamma$ defines a map of the two dimensional sphere into $\mathbf{R}^3$. We also assume that the image of this map is a smooth, convex surface. In this case, the surface $W := \chi(S^2)$ is called the *Wulff shape* of the functional.

- $\omega \cdot \mathcal{A} := \omega_0\mathcal{A}_0 + \omega_1\mathcal{A}_1$, where $\mathcal{A}_i$ is the area of D_i. The case $\omega > 0$, (respectively $\omega < 0$), is referred to as lyophobic, (resp. lyophilic), wetting.

- $\tau \cdot \mathcal{L} := \tau_0\mathcal{L}_0 + \tau_1\mathcal{L}_1$, where $\mathcal{L}_i$ is the anisotropic line tension: $\mathcal{L}_i = \mathcal{L}[C_i]$, which will be defined later. The case $\tau > 0$, (respectively $\tau < 0$), is referred to as positive, (resp. negative), line tension.

Such a configuration might arise, for example, if we consider a drop of nematic liquid crystal trapped between two horizontal plates which is surrounded by an isotropic environment. The free surface energy of the drop is the anchoring energy.

Line tension was introduced by Gibbs [3], and is known to play a role an determining a drop's morphology at a small scale (microns). A molecule lying on one of the contact curves C_i should carry an energy which is influenced by the three distinct phases with which it is in contact. If one of these phases, say the material of the drop, is assumed to be anisotropic, the line tension should be anisotropic as well.

A useful class of anisotropic functionals which satisfy the convexity condition can be produced using a smooth norm $||\cdot||$ on $\mathbf{R}^3$. We denote the dual norm by $||\cdot||_*$ and define an anisotropic surface energy for smooth immersed surfaces by

$$\mathcal{F} = \int_\Sigma ||\nu|| \, d\Sigma \,,$$

where ν is the unit surface normal, where $d\Sigma$ is the area element of Σ. In this case, the Wulff shape is just the unit sphere in the dual norm, i.e. $W = \{\chi \in \mathbf{R}^3 \mid ||\chi||_* = 1 \}$. In fact, every anisotropic energy functional which satisfies the convexity condition and whose energy density is even, $\gamma(\nu) = \gamma(-\nu)$ arises in this way. In regard to the application to nematic liquid crystals, the energy density is always taken to have this property.

We will now be more specific about the type of free energy which will be studied. We will consider a Wulff shapes W of *product form*. By this we mean that W can be parameterized

$$\chi(\sigma, t) = (u(\sigma)[\alpha(t), \beta(t)], v(\sigma)) \quad 0 \le \sigma \le \bar{\sigma} \,, 0 \le t \le \bar{t} \,. \tag{2}$$

It is assumed that (u, v) is a smooth, convex, closed planar curve with arc length σ which is symmetric with respect to the v-axis, and (α, β) is a smooth, convex, closed planar curve such that the origin is inside of the domain bounded by this curve.

We are now in a position to define the line tension. The choice of the line tension functional is such that when it is included in the total energy, parts of the Wulff shape bounded by parallel planes are still in equilibrium if the constants ω_i and τ_i are chosen correctly.

The image of the curve (α, β) may be regarded as a planar Wulff shape Ω which defines an anisotropic energy functional for oriented curves C by

$$\mathcal{L}[C] = \int_C \Gamma(N) \, dL \,,$$

where N is the unit normal to the planar, simple closed curve C, and dL is the line element of C. The function Γ is given by the support function $\Gamma(N) = P \cdot N$, where P is the position vector of the curve (α, β) at the point where the normal coincides with N on Ω. The function Γ will depend on the choice of origin in the plane but different choices will give equivalent variational problems.

This definition of the line tension can be justified from a physical point of view. A molecule on the curve C_i is in contact with the (anisotropic) material of the drop and so the line tension should also be anisotropic. Also, the boundary curve only has degrees of freedom within the horizontal plane so only the horizontal component of the anisotropy should affect the curve's shape.

We now will consider the total energy defined by (1).

For suitable choices of the constants ω and τ, the part of the Wulff shape bounded between two horizontal planes will be in equilibrium. In fact, our definition of the line tension is the only one possible which has this property. Let N denote the outward pointing unit normal to the curve C. The first variation of $\mathcal{L}$ can be expressed:

$$\delta\mathcal{L} = \int \lambda \delta C \cdot N \, dL \,,$$

where λ is the *anisotropic curvature* of the curve C with respect to the inward pointing normal, that is, $\lambda = \kappa/m$, where κ, m are the curvatures of C, (α, β), respectively, at points where the normal agree. The first variation of $\mathcal{A}$ is given by

$$\delta\mathcal{A} = \int_C \delta C \cdot N \, dL \,.$$

Under rescalings of the curve $C \to rC$, we have $\lambda \to \lambda/r$.

Obviously, Wulff's Theorem implies that Ω, or any rescaling of Ω, is in equilibrium for the functional $\mathcal{L}_\Omega$ with the area within the curve constrained to be a constant and so $\lambda \equiv$ const. on Ω. Calculation shows that $\lambda_\Omega \equiv 1$.

The set of admissible variations will consist of those $\delta X = \xi + \psi\nu$, where ξ is the tangential component, which satisfy

$$\int_\Sigma \psi = 0\,, \quad \text{and} \quad \delta X \cdot E_3 = 0 \text{ on } \partial\Sigma\,. \tag{3}$$

The first equation insures that, to first order, the variations preserves the volume enclosed by the surface. Using the formulas above and the fact that $N = (-1)^i X_L \times E_3$ on C_i, we obtain the first variation formula,

$$\delta\mathcal{E} = -\int_\Sigma \Lambda\psi\, d\Sigma + \sum_{i=0,1} \oint_{C_i} \delta X \cdot (X_L \times (\chi + (-1)^i[\omega_i + \tau_i\lambda]E_3)\, dL\,. \tag{4}$$

In order to have an equilibrium, we must have that the vector $(X_L \times (\chi + (-1)^i[\omega_i + \tau_i\lambda]E_3))$ is parallel to E_3 along the boundary, since the variation field is constrained to be perpendicular to E_3 on the boundary. From this we obtain that $\chi + (-1)^i[\omega_i + \tau_i\lambda]E_3$ must be perpendicular to E_3 on $\partial\Sigma$.

Equilibria for the constrained free boundary problem are therefore characterized by the two conditions

$$\Lambda \equiv \text{constant}, \ \text{ in } \Sigma\,, \tag{5}$$

$$\chi \cdot E_3 = (-1)^{i+1}(\omega_i + \tau_i\lambda)\,, \text{ on } \partial\Sigma\,. \tag{6}$$

Solutions of this problem will be referred to as *capillary surfaces*. When the line tension is zero and the energy is axially symmetric, the Maximum Principle implies that all capillary surfaces are surfaces of revolution. The same conclusion holds for axially symmetric energies and non negative line tension (Theorem 6.1).

For $\hat{W}$, the part of the Wulff shape bounded by two horizontal planes $v_0 \le v \le v_1$, the plane curves given by the intersections of $\hat{W}$ with these planes are $u(v_i)(\alpha,\beta)$, $u(v_i) > 0$. We have $\lambda \equiv 1/u(v_i)$ on the curve C_i and the boundary condition for equilibrium is

$$v_i + (-1)^i(\omega_i + \tau_i/u(v_i)) \equiv 0 \quad \text{on } C_i, \quad i = 0, 1\,. \tag{7}$$

Define $\omega_i^* = \omega_i + \tau_i/u(v_i)$. We assume $\tau_i \le 0$, which means $\omega_i \ge \omega_i^*$, i.e. the line tension is non positive. The value of ω_i^* is chosen so that the part $\hat{W}$ of W with $\chi \cdot E_3 = (-1)^{i+1}v_i$ on C_i is in equilibrium for the functional $\mathcal{F} + \omega_0^*\mathcal{A}_0 + \omega_1^*\mathcal{A}_1$.

We now briefly recall a class of examples which generalize the Delaunay surfaces which are the classical constant mean curvature surfaces of revolution. We seek surfaces Σ with $\Lambda \equiv$ constant with a parameterization the form $X = (x(s)[\alpha(t), \beta(t)], z(s))$. The curve (x, z) will be called the generating curve of the surface. The normal to Σ and the normal to W agree at points where $z_x = v_u$ holds. With this identification, in [11], we proved that the generating curve must satisfy

$$2ux + \Lambda x^2 = c\,, \quad z = \int x_u\, dv\,. \tag{8}$$

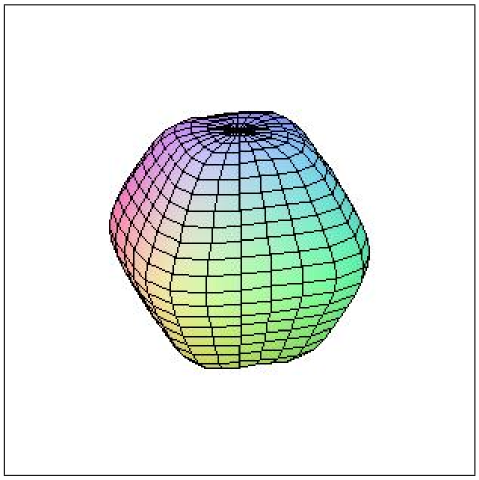


Figure 2: Wulff shape of product form

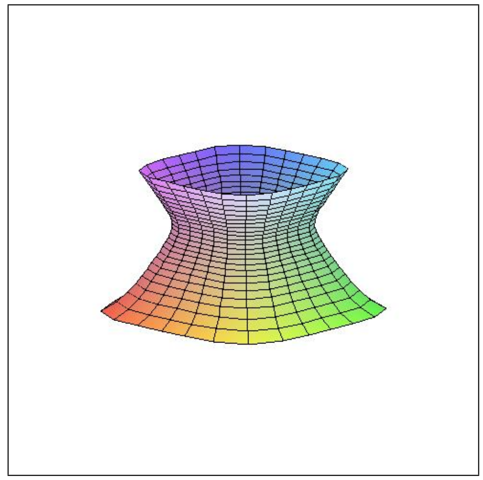


Figure 3: Anisotropic catenoid.

These conditions are independent of the cross sections (α, β). The solutions fall into six types: planes ($\Lambda = 0$, $c = 0$), rescalings of W ($\Lambda < 0$, $c = 0$), cylinders ($\Lambda < 0$, $\sqrt{-\Lambda c} = \bar{u} := \max u$), catenoids ($\Lambda = 0$, $c \neq 0$), unduloids ($\Lambda < 0$, $0 < c < \bar{u}^2/(-\Lambda)$) and nodoids ($\Lambda < 0$, $c < 0$). Here we have normalized the orientation so that $\Lambda \leq 0$ holds. The unduloid is an embedded periodic surface while the nodoid is a periodic surface with self-intersections. The nodoid is the only one of the last four types of surfaces which has points where the tangent plane is horizontal. This will play a crucial role in the stability analysis.

For the case where the functional is axially symmetric and $\tau_i \geq 0$ holds, we show, by using the Maximum Principle, that any capillary surfaces embedded in the region between the planes is, a priori, a Delaunay surface (Theorem 6.1). Later we will develop a version of Schwarz symmetrization which will indicate that when $\tau_i \geq 0$ holds, any minimizer for the capillary problem should be this type of surface also (§4).

For the part of an anisotropic Delaunay surface between the planes Π_i, equilibrium equation (6) reads

$$\chi \cdot E_3 = (-1)^{i+1}(\omega_i + \frac{\tau_i}{x_i}) \text{ , on } \partial\Sigma \text{ ,} \tag{9}$$

since the anisotropic curvature of a curve $R(\alpha, \beta)$ is $1/R$. If this equation is satisfied on both planes, we will refer to the surface as a capillary Delaunay surface. Of course we can always start with the part $\hat{\Sigma}$ of a Delaunay surface between two planes and simply define constants τ_i, ω_i such that (9) holds. The surface $\hat{\Sigma}$ will then be a Delaunay capillary surface for (1).

In the next section we will begin to study the stability of capillary surfaces. Note that for a part of a Delaunay surface between two planes, the values of ω_i^* are determined by the surface.

3 Second variation

In this section, we give the second variation formula of the energy and a criterion for the stability, for a capillary anisotropic Delaunay surface for a Wulff shape of product form.

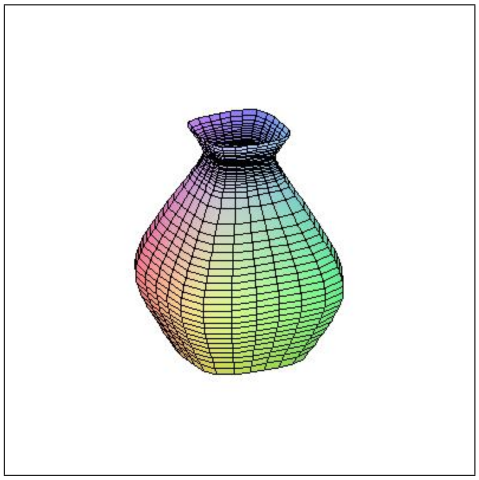


Figure 4: Anisotropic unduloid

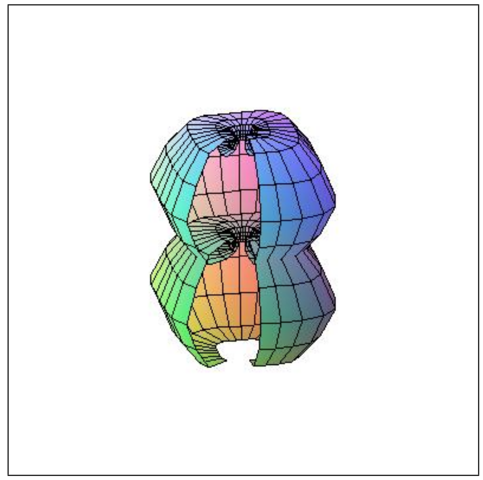


Figure 5: Anisotropic nodoid

Let $X : (\Sigma, \partial\Sigma) \to (\mathbf{R}^3, \Pi)$ be a capillary Delaunay surface. Then, for a variation $X_\epsilon : (\Sigma, \partial\Sigma) \to (\mathbf{R}^3, \Pi)$, $X_\epsilon = X + \epsilon(\xi + \psi\nu) + \mathcal{O}(\epsilon^2)$, of X, the second variation of the energy

$$\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}$$

is given by

$$\delta^2\mathcal{E} = -\int_\Sigma \psi J[\psi]\, d\Sigma + \oint_{\partial\Sigma} \psi B_\tau[\psi]\, dL =: I_X[\psi]\,, \tag{10}$$

where J and B_τ will be defined below, and the formula (10) itself will be proved in §10. First, J is the Jacobi operator given by

$$J[\psi] = \mathrm{Div}[A\nabla\psi] + \langle A \cdot d\nu, d\nu\rangle\psi\,. \tag{11}$$

For a surface of revolution, this is expressed as follows.

$$J[\psi] = x^{-1}\{(\mu_1^{-1}x\psi_s)_s + \mu_2^{-1}x^{-1}\psi_{\theta\theta}\} + \{\mu_1^{-1}k_1^2 + \mu_2^{-1}k_2^2\}\psi\,. \tag{12}$$

Here the μ_i's are the principal curvatures of the Wulff shape with respect to the inward pointing normal while the k_i's are the principal curvatures of Σ. For details, we refer the reader to [9].

The boundary operator B_τ is defined as follows. First define B by

$$B[\psi] = -\delta\chi \cdot n = A(\nabla\psi + \frac{\nu_3}{n_3}\psi d\nu(n)) \cdot n\,. \tag{13}$$

For a surface of revolution, this becomes

$$B[\psi] = -(-1)^i\mu_1^{-1}(\psi_s - (z''/z')\psi)\,. \tag{14}$$

The quantities which depend on χ should be evaluated at points of W where (6) holds. Let m be the (positive) curvature of the one dimensional Wulff shape Γ. The boundary operator B_τ is

$$B_\tau[\psi] = B[\psi] - \frac{\tau}{n_3}\big[(\frac{1}{m}(\frac{\psi}{n_3})_L)_L + \frac{\kappa^2}{mn_3}\psi\big]\,. \tag{15}$$

For symmetric variations, we have the simpler form

$$B_\tau[\psi] = B[\psi] - \frac{\tau\psi}{n_3^2 x^2 \Gamma(N)}\,.$$

Consider the spectral problem:

$$J[\psi] + e\psi = 0 \text{ on } \Sigma, \qquad B_\tau[\psi] = 0 \text{ on } \partial\Sigma\,. \tag{16}$$

We order the eigenvalues be $e_1 < e_2 \leq ....$

Because of the constraint, it is not possible to determine the stability purely in terms of the spectrum of the problem (16). However, we have the following criterion.

Proposition 3.1 *(i) If $e_1 \geq 0$, then the surface is stable.*
(ii) Assume that $e_1 < 0 \leq e_2$ holds. Assume that there exists a solution of the problem

$$J[\phi] = 1 \quad \text{in} \quad \Sigma, \qquad B_\tau[\phi] = 0 \quad \text{on} \quad \partial\Sigma\,. \tag{17}$$

Then, the surface is stable if and only if

$$\int_\Sigma \phi\, d\Sigma \geq 0$$

holds. If no solution of (17) exists, the surface is unstable.
(iii) If $e_2 < 0$, then the surface is unstable.

Remark 3.1 We note that when $e_1 < 0 < e_2$ holds, a solution to (17) always exists by the Fredholm Alternative.

Remark 3.2 Except for the case when $e_2 = 0$, a criterion of the stability for a more general variational problem with constraint was first obtained by Maddocks [12], and was first applied in the case of constant mean curvature by Vogel [17], which were essentially similar to Proposition 3.1. The result for the case when $e_2 = 0$ was essentially proved in Koiso [7].

4 Schwarz symmetrization

In this section, we prove that, for the anisotropic surface energy whose Wulff shape is of product form, a Schwarz type symmetrization works for certain class of surfaces embedded into Δ. Also, we will point out that this means that the stability analysis, for the case where the line tension is nonnegative, is reduced to "symmetric variations".

Let $|\cdot|_H$ and $|\cdot|_V$ be smooth norms on $\mathbf{R}^2$ which we will refer to as the horizontal and vertical norm. It is assumed that the vertical norm is invariant under reflection through the coordinate axes so that

$$|(x,y)|_V = |(|x|,|y|)|_V \tag{18}$$

holds for all $(x,y) \in \mathbf{R}^2$. We define a norm $||\cdot||$ on $\mathbf{R}^3$ by $||(x_1,x_2,x_3)|| := \big|(|(x_1,x_2)|_H, x_3)\big|_V$. For a smooth oriented surface, we define the elliptic functional

$$\mathcal{F}[X] := \int_\Sigma ||\nu||\, d\Sigma\,.$$

The Wulff shape corresponding to this functional is

$$W := \{\chi \mid ||\chi||_* = 1\} ,$$

where $||\chi||_* := \big|(|(\chi_1, \chi_2)|_{H*}, \chi_3)\big|_{V*}$ is the dual norm.

Let $(\alpha(t), \beta(t))$ parameterize the curve $|x|_{H*} = 1$ and consider a surface parameterized by

$$X(t,z) = (r(t,z)\alpha(t), r(t,z)\beta(t), z),\ 0 \le t \le T\ , z_0 \le z \le z_1\ , \tag{19}$$

where $r(t,z)$ is a smooth positive function. Define a symmetrized surface $Y(t,z) := (R(z)\alpha(t), R(z)\beta(t), z)$ by requiring that the areas enclosed by each level curves $z =$ constant agree for the surfaces X and Y.

Theorem 4.1 *The symmetrization diminishes the anisotropic energy, that is,* $\mathcal{F}[Y] \le \mathcal{F}[X]$ *and* $\mathcal{L}[Y] \le \mathcal{L}[X]$ *hold.*

Proof. First we will show that $\mathcal{F}[Y] \le \mathcal{F}[X]$ holds.

We have

$$\mathcal{F}[X] = \int_{z_0}^{z_1} \frac{d\mathcal{F}[X]}{dz}\, dz\ , \tag{20}$$

where

$$\frac{d\mathcal{F}[X]}{dz} = \int_0^T ||X_t \times X_z||\, dt = \int_0^T \big|(|((r\beta)_t, -(r\alpha)_t)|_H, -rr_z(\alpha\beta_t - \alpha_t\beta))\big|_V\, dt\ . \tag{21}$$

By the definition of Y, we have

$$R^2(z) \int_0^T (\alpha\beta_t - \alpha_t\beta)\, dt = \int_0^T r^2(t,z)(\alpha\beta_t - \alpha_t\beta)\, dt\ , \forall\, z_0 \le z \le z_1\ . \tag{22}$$

The one dimensional version of Wulff's Theorem implies

$$R(z) \int_0^T |(\beta_t, -\alpha_t)|_H\, dt \le \int_0^T |((r\beta)_t, -(r\alpha)_t)|_H\, dt\ . \tag{23}$$

Note that it is possible to choose the parameter t so that

$$|(\beta_t, -\alpha_t)|_H \equiv 1 \equiv (\alpha\beta_t - \alpha_t\beta)\ . \tag{24}$$

To see this, let Ω^* denote the one dimensional Wulff shape given by $|x|_{H*} = 1$. For each $(\alpha, \beta) \in \Omega^*$, there is a unique vector $(\alpha, \beta)^*$ in the dual 'sphere' $\Omega := \{|x|_H = 1\}$, satisfying the two conditions

$$|(\alpha,\beta)^*|_H = 1\ , \quad \langle(\alpha,\beta), (\alpha,\beta)^*\rangle = 1\ .$$

Let s be the arc length parameter on Ω^* and let $\sigma = \alpha\beta_s - \beta\alpha_s$. It is easily checked that

$$(\alpha,\beta)^* = \frac{(\beta_s, -\alpha_s)}{\sigma}\ .$$

Changing coordinate by $dt/ds = \sigma$ gives (24).

Using this coordinate, (21) becomes

$$\begin{aligned}\frac{d\mathcal{F}[X]}{dz} &= \int_0^T \big|(|((r\beta)_t, -(r\alpha)_t)|_H, -rr_z)\big|_V \, dt \\ &\geq \big|\,\big(\int_0^T |((r\beta)_t, -(r\alpha)_t)|_H \, dt \,, \int_0^T -rr_z \, dt\big)\,\big|_V \end{aligned} \tag{25}$$

by the triangle inequality. Similarly,

$$\frac{d\mathcal{F}[Y]}{dz} = \int_0^T |(|((R\beta)_t, -(R\alpha)_t)|_H, -RR_z)|_V \, dt = T\big|(R, -RR_z)\big|_V = \big|(TR, -TRR_z)\big|_V \, . \tag{26}$$

Using (22), we obtain

$$RR_z T = \int_0^T rr_z \, dt \, .$$

Combining the previous line with (18), (25), (23), and (26), gives $d\mathcal{F}[Y]/dz \leq d\mathcal{F}[X]/dz$. If we regard $\mathcal{F}[X]$ in (20) as a function of the upper limit of integration z_1 and we write $\mathcal{F}[X] = \mathcal{F}[X](z_1)$, then clearly when $z_1 = z_0$, we have $\mathcal{F}[X](z_0) = 0 = \mathcal{F}[Y](z_0)$ and so $d\mathcal{F}[Y]/dz \leq d\mathcal{F}[X]/dz$ implies $\mathcal{F}[Y] \leq \mathcal{F}[X]$.

We will apply this to the second variation. Let Σ be a compact part of a generalized anisotropic Delaunay surface which occurs as a capillary surface. We also assume that the normal vector is nowhere vertical on the closed surface, i.e. $1 - \nu_3^2 > 0$ holds on $\bar{\Sigma}$. Then Σ can be parameterized as in (19) with $r(t, z) = r(z)$. We consider a volume preserving deformation $X_\epsilon = X + \epsilon(\delta X) + \mathcal{O}(\epsilon^2)$ which preserves the free boundary condition. We then replace each surface X_ϵ by its symmetrization $\tilde{X}_\epsilon$. For $\epsilon \approx 0$ each surface $\tilde{X}_\epsilon$ also admits a parameterization as above. Also the new deformation is volume preserving and preserves the boundary conditions. Since symmetrization decreases both the surface energy and the line tension $\mathcal{L}$ and preserves the wetting energy. Therefore, if $\tau_i \geq 0$ holds, we have $\partial^2\mathcal{E}[\tilde{X}_\epsilon] \geq 0$ implies $\partial^2\mathcal{E}[X_\epsilon] \geq 0$. Therefore the surface is stable if and only if the second variation is non negative for symmetric, volume preserving variations.

5 Reduction to the axially symmetric case

In this section, we consider the second variation of energy for a capillary Delaunay surface. The aim is to show that if we restrict to symmetric variations, i.e. variations through anisotropic Delaunay surfaces, then will show that when we consider the problem (16), can reduced to computing the spectrum in the axially symmetric case.

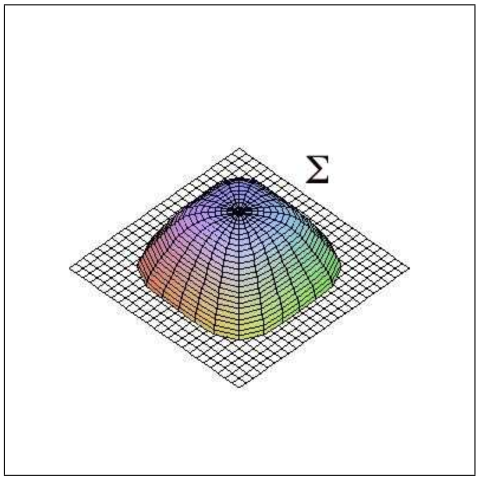
Again, we consider the Wulff shape of the form (2) and consider a second surface of the same form:

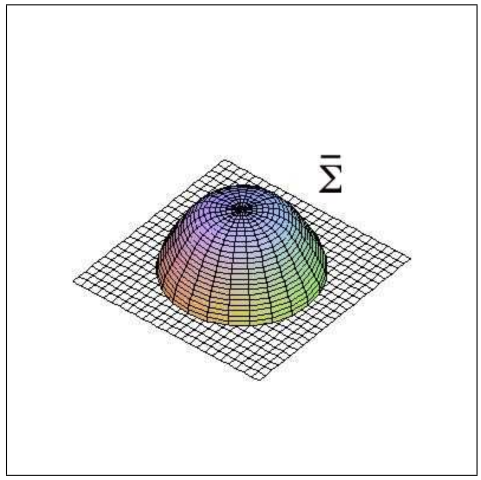
$$X(s, t) = (x(s)[\alpha(t), \beta(t)], z(s)) \, .$$

We will call a variation of X of the form

$$X_\epsilon(s, t) = ((x + \epsilon\delta x)(s)[\alpha(t), \beta(t)], (z + \epsilon\delta z)(s))$$

a *symmetric variation*. The field $\delta X = \partial_\epsilon(X_\epsilon)_{\epsilon=0}$ will be called a symmetric variation field.





The normal to the immersion X is given by

$$\frac{X_t \times X_s}{|X_t \times X_s|} = \nu = [z_s^2(\alpha_t^2 + \beta_t^2) + x_s^2(\alpha\beta_t - \beta\alpha_t)^2]^{-1/2}(z_s\beta_t, -z_s\alpha_t, -x_s(\alpha\beta_t - \beta\alpha_t)) \,, \tag{27}$$

and the area form is given by

$$d\Sigma = x[z_s^2(\alpha_t^2 + \beta_t^2) + x_s^2(\alpha\beta_t - \beta\alpha_t)^2]^{1/2} ds dt \,. \tag{28}$$

It is easily checked that the normal component of a symmetric variation is given by

$$\psi := \nu \cdot \partial_\epsilon(X_\epsilon) = P(s,t)\bar{\psi} \tag{29}$$

where

$$P(s,t) := \frac{(\alpha\beta_t - \beta\alpha_t)}{[z_s^2(\alpha_t^2 + \beta_t^2) + x_s^2(\alpha\beta_t - \beta\alpha_t)^2]^{1/2}} \,, \qquad \bar{\psi} := z_s\delta x - x_s\delta z \,. \tag{30}$$

Note that $\bar{\psi}$ is just the normal component of the variation field for the family of immersions $\bar{X}_\epsilon := ((x + \epsilon\delta x)e^{it}, z + \epsilon\delta z)$. In particular $\bar{\psi} = \bar{\psi}(s)$.

We will now show that *a Delaunay surface $X(s,t)$, $s_0 \le s \le s_1$ is stable for the free energy plus wetting if and only if the surface $\bar{X}(s,\theta) := (x(s)e^{i\theta}, z(s))$, $s_0 \le s \le s_1$ is stable for the functional whose Wulff shape has circular cross sections.*

The anisotropic mean curvature is given by $\Lambda = -u_x - u/x$, which is independent of t. It follows that for symmetric variations, $\delta\Lambda = J[\psi]$ is independent of t also. (Here we are using that $\Lambda \equiv$ constant.) Therefore, with the obvious notation,

$$J[\psi] = \bar{J}\,[\bar{\psi}] \,.$$

It then follows from (28) and (29), that

$$-\int_\Sigma \psi \, d\Sigma = \frac{\text{area}(\Omega)}{\pi} \int \bar{\psi} d\bar{\Sigma} \,, \qquad -\int_\Sigma \psi J[\psi] \, d\Sigma = -\frac{\text{area}(\Omega)}{\pi} \int_\Sigma \bar{\psi}\bar{J}[\bar{\psi}] \, d\bar{\Sigma} \,. \tag{31}$$

Therefore, there exists a volume preserving symmetric variation for X with negative second variation if and only if there exists a volume preserving variation of $\bar{X}$ with negative second variation.

We now consider the boundary operator B. For any admissible variation, i.e. $E_3 \cdot (\xi + \psi\nu) \equiv 0$, we have

$$B[\psi] := A(\nabla\psi + \frac{\nu_3}{n_3}\psi d\nu(n)) \cdot n \,,$$

can be written as

$$n_3 B[\psi] = -\delta\chi \cdot E_3 \,.$$

Since $\chi \cdot E_3$ is independent of t, we obtain

$$n_3 B[\psi] = \bar{n}_3 \bar{B}[\bar{\psi}] \,.$$

For convenience, we assume that t is the coordinate along Ω which was used in the previous section so that $\alpha\beta_t - \alpha_t\beta \equiv 1$ holds. Note that this implies that if ξ is the arc length coordinate on Ω, then

$$dt = \Gamma\, d\xi = \Gamma(\alpha_t^2 + \beta_t^2)^{1/2} dt \,, \tag{32}$$

where Γ is the support function of Ω. Since the boundary curves $\mathcal{C}_i$ have the form, $t \to (x(\alpha(t), \beta(t)), \text{constant})$, we have $dL = x d\xi$.

We compute from (27)

$$n_3^2 = 1 - \nu_3^2 = 1 - x_s^2 P^2 = z_s^2(\alpha_t^2 + \beta_t^2)P^2 = \frac{\bar{n}_3^2 P^2}{\Gamma^2} \,.$$

It is clear that n_3 and $\bar{n}_3$ have the same sign. Combining these formulas, we get

$$\psi B[\psi]\, dL = (\bar{\psi}P)(\frac{\bar{n}_3}{n_3})\bar{B}[\bar{\psi}]\, dL = \bar{\psi}\bar{B}[\bar{\psi}] x\Gamma d\xi \,.$$

This gives

$$\int_{\partial\Sigma} \psi B[\psi]\, dL = \frac{\text{area}(\Omega)}{\pi} \int_{\partial\Sigma} \bar{\psi}\bar{B}[\bar{\psi}]\, d\bar{L} \,. \tag{33}$$

Similarly, we transform the last integral in (51),

$$\frac{\psi^2}{x^2 n_3^2 \Gamma} dL = \frac{(\bar{\psi}^2 P^2)}{x^2\Gamma\bar{n}_3^2}(\frac{\bar{n}_3^2}{n_3^2}) dL = \frac{\bar{\psi}^2}{x^2\Gamma\bar{n}_3^2}\, dL = \frac{\bar{\psi}^2 x\Gamma}{x^2\bar{n}_3^2}\, d\xi \,.$$

Therefore

$$\oint_{\partial\Sigma} \frac{\psi^2}{x^2 n_3^2 \Gamma} dL = \frac{\text{area}(\Omega)}{\pi} \oint_{\partial\Sigma} \frac{\bar{\psi}^2}{x^2\bar{n}_3^2}\, d\bar{L}. \tag{34}$$

By combining (31), (33) and (34), we have

Theorem 5.1 *For symmetric variations,*

$$\delta^2\mathcal{E} = \frac{\text{area}(\Omega)}{\pi}\delta^2\bar{\mathcal{E}},$$

and consequently, the surfaces Σ and $\bar{\Sigma}$ are simultaneously stable or unstable with respect to symmetric variations.

6 A uniqueness theorem

Theorem 6.1 *Let* $\mathcal{F} = \int \gamma(\nu_3)\, d\Sigma$ *be a rotationally invariant anisotropic surface energy with vertical rotation axis, and let* Σ *be a compact embedded capillary surface for* $\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}$ *with non negative line tension* $\tau_i \geq 0$*. We assume that the support surface* Π *is either a single horizontal plane, or the union of two horizontal planes. We also assume that* Σ *is embedded into the closed region bounded by* Π*. Then* Σ *is part of an anisotropic Delaunay surface, i.e.* Σ *is rotationally symmetric.*

Proof. We will prove the assertion for the case that the support surface is the union of two horizontal planes Π_0, Π_1. For the case where the support surface is a single plane, a similar proof works.

Also, for simplicity, we assume that the interior of Σ does not touch $\Pi_0 \cup \Pi_1$. In the case where the interior of Σ touches $\Pi_0 \cup \Pi_1$, a slight modification of the proof works.

We denote by Ω the closed domain of $\mathbf{R}^3$ bounded by Π_0 and Π_1.

It is sufficient to prove that the surface Σ is symmetric with respect to a plane $x =$ constant.

Set

$$C_i := \Sigma \cap \Pi_i, \qquad C := C_0 \cup C_1$$

$(i = 0, 1)$.

Denote by D_i the inside of C_i in Π_i. And set $D := D_0 \cup D_1$. Since $\Sigma^* := \Sigma \cup D$ is a 2-dimensional connected closed topological submanifold of $\mathbf{R}^3$, it bounds a bounded domain G of Ω by virtue of the Alexander duality theorem.

For any real number a, denote the plane $\{x = a\}$ by P_a. Set $a_0 := \min\{x \mid (x, y, z) \in \Sigma\}$.

Define a subset H_a of Σ by

$$H_a := \{(x, y, z) \in \Sigma \mid x < a\}$$

for $a > a_0$. Then H_a is not empty. Denote by $\tilde{H}_a$ the reflection of H_a with respect to P_a. If $a - a_0$ is sufficiently small, $\tilde{H}_a$ is contained in $G \cup D$. Set

$$c = \sup\{b \in \mathbf{R} \mid \tilde{H}_a \subset G \cup D,\ \forall a \in (a_0, b)\}.$$

It is easy to see that we have the following three cases.

(I) There exists some $p \in (\partial H_c \cap \partial \tilde{H}_c) - C$ such that $T_p(\Sigma)$ is perpendicular to P_c.

(II) There exists some $p \in \tilde{H}_c \cap (\Sigma - C)$ such that $\tilde{H}_c$ and $\Sigma - \overline{H_c}$ touch each other from one side at p.

(III) $\partial \tilde{H}_c \cap C \neq \emptyset$.

First, we assume (III). We take a point $p \in \partial \tilde{H}_c \cap C$. We divide the situation into the following two cases:

(III-1) Tangent spaces of $\Sigma - H_c$ and $\overline{\tilde{H}_c}$ at p coincide.

(III-2) Tangent spaces of $\Sigma - H_c$ and $\overline{\tilde{H}_c}$ at p do not coincide.

Assume (III-1). $\Sigma - H_c$ and $\overline{\tilde{H}_c}$ touch each other from one side at p. Since the anisotropic energy density γ is rotationally symmetric with respect to the z-axis, by the Hopf Maximum Principle ([6, p. 159, Corollary 4.6]), $\Sigma - H_c$ and $\tilde{H}_c$ coincide in a neighborhood of p.

Next consider the condition (III-2). We will prove that this case does not occur. We will prove this under the assumption that $p \in \Pi_1$. In the case where $p \in \Pi_0$, the proof is similar. p is the reflection of a uniquely determined point $p_0 \in H_c \cap C_1$ with respect to P_c. We first claim that because of the assumption (III-2), it follows that $\nu_3(p) > \nu_3(p_0)$.

To see this, we denote by $\mathbf{u}$ the unit tangent vector of C_1 at p. Here, we take the orientation of C_1 so that a point goes along C_1 having the interior of C_1 on the left. $\mathbf{u}$ can be represented as

$$\mathbf{u} = (\cos(\theta + \pi/2), \sin(\theta + \pi/2), 0).$$

Denote by $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3)$ the outward-pointing unit normal of $\tilde{H}_c$. The outward-pointing unit normal vectors of Σ, $\tilde{H}_c$ at p are

$$\begin{aligned} \nu(p) &= (\cos\rho\cos\theta, \cos\rho\sin\theta, \sin\rho), \\ \tilde{\nu}(p) &= (\cos\rho_0\cos\theta, \cos\rho_0\sin\theta, \sin\rho_0), \end{aligned}$$

respectively. Here

$$-\frac{\pi}{2} \le \rho, \rho_0 \le \frac{\pi}{2}, \quad \rho \ne \rho_0$$

is satisfied. Denote by $\mathbf{v}$, $\mathbf{v}_0$ the inward-pointing unit normals along $\partial\Sigma$, $\partial\tilde{H}_c$ at p, respectively. Then

$$\begin{aligned} \mathbf{v} &= (\cos(\rho - \pi/2)\cos\theta, \cos(\rho - \pi/2)\sin\theta, \sin(\rho - \pi/2)), \\ \mathbf{v}_0 &= (\cos(\rho_0 - \pi/2)\cos\theta, \cos(\rho_0 - \pi/2)\sin\theta, \sin(\rho_0 - \pi/2)) \end{aligned}$$

holds. Since, in a neighborhood of p,

$$\tilde{H}_c \subset G, \quad \mathbf{v} \ne \mathbf{v}_0$$

holds, we have

$$\rho_0 - \pi/2 < \rho - \pi/2.$$

Hence,

$$\rho_0 < \rho$$

holds. Since $-\frac{\pi}{2} \le \rho, \rho_0 \le \frac{\pi}{2}$ holds, $\sin\rho_0 < \sin\rho$ holds. Therefore,

$$\tilde{\nu}_3(p) < \nu_3(p)$$

holds. Because $\nu_3(p_0) = \tilde{\nu}_3(p)$, we have

$$\nu_3(p_0) < \nu_3(p),$$

proving the claim.

Therefor, it follows that, $\chi_3(p) > \chi_3(p_0)$ holds. This inequality together with (6) implies $\tau_1\lambda(p) > \tau_1\lambda(p_0)$. Since $\tau_1 \ge 0$, $\lambda(p) > \lambda(p_0)$ holds. However, this inequality contradicts the assumption that $\partial\tilde{H}_c \subset \overline{D_1}$ near p.

Now assume (I). Then $\Sigma - H_c$ and $\overline{\tilde{H}_c}$ touch each other from one side at $p \in \partial(\Sigma - H_c) \cap \partial\overline{\tilde{H}_c}$. Since the anisotropic energy density γ is rotationally symmetric with respect to the z-axis, again by the Hopf Maximum Principle, $\Sigma - H_c$ and $\tilde{H}_c$ coincide in a neighborhood of p.

In the case where (II) holds, also by the Hopf Maximum Principle, $\Sigma - H_c$ and $\tilde{H}_c$ coincide in a neighborhood of p.

Consequently, in all cases, we have at least one point $p \in (\Sigma - H_c) \cap \overline{\tilde{H}_c}$ such that $\Sigma - H_c$ and $\overline{\tilde{H}_c}$ touch each other from one side at p, and $\Sigma - H_c$ and $\tilde{H}_c$ coincide in a neighborhood of p.

Now, denote by $\tilde{K}$ the connected component of $\overline{\tilde{H}_c}$ which contains p. Then $\tilde{K}$ is the reflection of a certain component K of $\overline{H_c}$ with respect to P_c. If we use the Hopf Maximum Principle repeatedly, we derive that $\tilde{K} \cap \Sigma$ is an open and closed subset of $\tilde{K}$. Because $\tilde{K}$ is connected, $\tilde{K} \cap \Sigma$ is all of $\tilde{K}$. Hence $\tilde{K}$ is contained in Σ. Therefore $\partial K \cap \Omega^\circ$ coincides with $\partial\tilde{K} \cap \Omega^\circ$, where Ω° is the open domain of $\mathbf{R}^3$ bounded by Π_0 and Π_1. Therefore, $K \cup \tilde{K} \cup \Pi_0 \cup \Pi_1$ determines a 2-dimensional topological manifold K^* without boundary which is contained in $\Sigma^* = \Sigma \cup D$. Therefore, K^* is open and closed in Σ^*. Since Σ^* is connected, K^* coincides with Σ^*. Consequently, we conclude that Σ is symmetric with respect to a plane $x =$ constant. **q.e.d.**

7 Negative line tension

An important result known as Winterbottom's Theorem [19] states that $\hat{W}$ is the absolute minimizer of the the functional $\mathcal{F} + \omega^* \cdot \mathcal{A}$ among all surfaces containing the same volume and having free boundary components on any two horizontal planes, not just Π_i, $i = 0, 1$. We will now give a modest result in this direction for the case of negative anisotropic line tension.

Wulff's Theorem, [1], [2], states that the Wulff shape is the absolute minimzer of the anisotropic energy among all curves (resp. surfaces), enclosing the same area, (resp. volume). This gives

$$\frac{(\mathcal{L}[C])^2}{\mathcal{A}[C]} \geq \frac{(\mathcal{L}[\Omega])^2}{\mathcal{A}[\Omega]} =: 4p_\Omega\ .$$

Let S be a surface which contains the same volume as $\hat{W}$ and is contained between the planes Π_i, $i = 0, 1$. We also assume that S has the same product structure as W, i.e., S can be parameterized in the form $X(s,t) = (x(s)(\alpha(t), \beta(t)), z(s))$. Let S_i be the part of Π_i wetted by S.

$$\begin{aligned}
\mathcal{E}[S] &= (\mathcal{F} + \sum_{i=0,1} \omega_i \mathcal{A}_i)[S] + \sum_{i=0,1} \tau_i \mathcal{L}[\partial S_i] \\
&= (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[S] + \sum_{i=0,1} (\omega_i - \omega_i^*)\mathcal{A}_i[S] + \sum_{i=0,1} \tau_i \mathcal{L}[\partial S_i] \\
&\geq (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[\hat{W}] + \sum_{i=0,1} (\omega_i - \omega_i^*)\mathcal{A}_i[S] + \sum_{i=0,1} \tau_i \mathcal{L}[\partial S_i] \\
&= (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[\hat{W}] + \sum_{i=0,1} (\omega_i - \omega_i^*)\frac{(\mathcal{L}[\partial S_i])^2}{4p_\Omega} + \sum_{i=0,1} \tau_i \mathcal{L}[\partial S_i] \\
&= (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[\hat{W}] + \sum_{i=0,1} \frac{\omega_i - \omega_i^*}{4p_\Omega}\big((\mathcal{L}[\partial S_i])^2 - 4p_\Omega u(v_i)\mathcal{L}[\partial S_i]\big)
\end{aligned}$$

$$
\begin{aligned}
&= (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[\hat{W}] + \sum_{i=0,1} \frac{\omega_i - \omega_i^*}{4p_\Omega} \big([\mathcal{L}[\partial S_i] - 2p_\Omega u(v_i)]^2 - 4p_\Omega^2 (u(v_i))^2\big) \\
&\geq (\mathcal{F} + \sum_{i=0,1} \omega_i^* \mathcal{A}_i)[\hat{W}] - \sum_{i=0,1} (\omega_i - \omega_i^*) p_\Omega (u(v_i))^2 \\
&= \mathcal{E}[\hat{W}] .
\end{aligned}
$$

Winterbottom's Theorem was used to go from the second to the third line. We have shown:

Theorem 7.1 *$\hat{W}$ minimizes the functional $\mathcal{E}$ among all surfaces enclosing the same volume and having cross sections which are rescalings of Ω.*

As with Winterbottom's Theorem, it is not necessary that the comparison surfaces are bounded by the same horizontal planes as $\hat{W}$. Also note that the proof works just as well when the Wulff shape is crystalline, i.e. it is a convex surface of product type which is composed of planer faces.

In contrast to the fact that $\hat{W}$ minimizes energy among surfaces having the same cross sections, we have the following.

Theorem 7.2 *Let Σ be a capillary Delaunay surface for $\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}$ with $\tau_i < 0$ ($i = 0, 1$). Then Σ is unstable.*

Proof. We begin by identifying $\mathbf{R}^2$ with $\mathbf{C}$ and parameterizing the one dimensional Wulff shape Ω in tangential coordinates by

$$\varphi \mapsto (\Gamma + i\Gamma_\varphi)e^{i\varphi} , \varphi \in S^1 .$$

Here Γ is the support function of Ω considered as a function on S^1. The arc length density of Ω and the curvature of Ω with respect to the inner normal are given by

$$dL_\Omega = (\Gamma + \Gamma_{\varphi\varphi})d\varphi , \quad m = \frac{d\varphi}{dL_\Omega} = \frac{1}{\Gamma + \Gamma_{\varphi\varphi}} (> 0) . \tag{35}$$

If Σ is an embedding of an anisotropic Delaunay surface, we can parameterize Σ as

$$X(s, \varphi) = (x(s)(\Gamma + i\Gamma_\varphi)e^{i\varphi}, z(s)) .$$

We write $d\Sigma =: G(s, \varphi) ds\, d\varphi$.

We denote by Σ_{s_0,s_1} the part of Σ with $s_0 \leq s \leq s_1$. The curve C_0 is assumed to correspond to $s = s_0$. Let

$$\psi = \frac{f(s) \sin M\varphi}{G(s, \varphi)}$$

represent the normal component of a variation field. The function $f(s)$ will be assumed to satisfy $0 \leq f \leq 1$, $f(s_0) = 1$, $f(s_1) = 0$ and $|f_s| \leq 2(s_1 - s_0)^{-1}$. Note that for $M \geq 1$, there holds

$$\int_{\Sigma_{s_0,s_1}} \psi \, d\Sigma = 0 .$$

If we define the operator

$$Q[g] = [\frac{1}{m}(g)_L]_L + \frac{\kappa^2 g}{m} ,$$

then the second variation formula gives

$$\delta^2\mathcal{E} = \int_{\Sigma_{s_0,s_1}} A\nabla\psi\cdot\nabla\psi - \langle A\, d\nu, d\nu\rangle \psi^2\, d\Sigma + \oint_{s=s_0} \psi^2 \frac{\nu_3}{n_3} A\, d\nu(n)\cdot n\, dL - \tau \oint_{s=s_0} \frac{\psi}{n_3} Q[\frac{\psi}{n_3}]\, dL\,. \quad (36)$$

In what follows, we let $a_1, a_2, ...$ be non negative constants which depend only on the local geometry of Σ_{s_0,s_1}, but not on s_0 or s_1. It is clear that

$$A\nabla\psi\cdot\nabla\psi\, d\Sigma \le a_1(\psi_s^2+\psi_\varphi^2)\, ds\, d\varphi \le (a_2 + a_3(f_s)^2 + a_4M^2)\, ds\, d\varphi\,.$$

(a_2 is obtained from bounds on G and its derivatives.) It follows that

$$\int_{\Sigma_{s_0,s_1}} A\nabla\psi\cdot\nabla\psi - \langle Ad\nu, d\nu\rangle\psi^2 d\Sigma + \oint_{s=s_0} \psi^2\frac{\nu_3}{n_3} Ad\nu(n)\cdot n dL \le a_2(s_1-s_0) + 2a_3(s_1-s_0)^{-1} + a_4M^2(s_1-s_0) + a_5. \quad (37)$$

Next, we examine the operator Q. Note that since C_0 and Ω are similar, we have $dL = x\, dL_\Omega$ and the curvature of C_0 is $k = m/x$. It follows from (35), that

$$Q[g] = \frac{1}{x^2(\Gamma_{\varphi\varphi}+\Gamma)}\big(g_{\varphi\varphi}+g\big) = \frac{m}{x^2}\big(g_{\varphi\varphi}+g\big)\,. \quad (38)$$

Since $\psi/n_3 = \sin(M\varphi)[f(s)/(n_3G(s,\varphi)]$, we have

$$\begin{aligned} Q[\psi/n_3] &= \frac{f(s)}{n_3G}Q[\sin(M\varphi)] + f(s)\sin(M\varphi)Q[\frac{1}{n_3G}] + 2(m/x^2)Mf(s)\cos(M\varphi)\partial_\varphi[\frac{1}{n_3G}] \\ &= -\frac{m}{x^2}M^2\frac{\psi}{n_3} + f(s)\sin(M\varphi)Q[\frac{1}{n_3G}] + 2(m/x^2)Mf(s)\cos(M\varphi)\partial_\varphi[\frac{1}{n_3G}]\,. \end{aligned}$$

It then follows, using that $\tau < 0$ and $f \equiv 1$ on C_0, that

$$-\tau\oint_{s=s_0}\frac{\psi}{n_3}Q[\frac{\psi}{n_3}]\, dL \le \tau 2a_6M^2 + a_7 + a_8M\,,$$

with $a_6 > 0$. Combining this with (37), we get

$$\delta^2\mathcal{E} \le a_2(s_1-s_0) + 2a_3(s_1-s_0)^{-1} + a_4M^2(s_1-s_0) + a_5 + \tau 2a_6M^2 + a_7 + a_8M\,. \quad (39)$$

By choosing $s_1 \approx s_0$ so that $a_4(s_1-s_0) + \tau a_6 < 0$ holds and then choosing $M >> 0$ so that $a_2(s_1-s_0) + 2a_3(s_1-s_0)^{-1} + a_5 + \tau a_6 M^2 + a_7 + a_8M < 0$ holds, we get that the second variation is negative. **q.e.d**

One sees from (39) that for instability, the period of oscillation $2\pi/M$ should decrease like $[A_1 + A_2 s^{-1/2}]^{-1}$ where A_1, A_2 are positive constants and s is the arc length of the generating curve.

8 Sessile drops

In this section, we will consider the stability of an anisotropic sessile drop, i. e. a surface S which is a part of a rescaled Wulff shape bounded by one horizontal plane. It is shown below that when the line tension is non negative, these are the only possible solutions with one free boundary. We recall that the boundary condition is

$$\chi \cdot E_3 = -(\omega + \frac{\tau}{x}) \ . \tag{40}$$

In applications, the parameters τ and ω would be determined by the materials involved. We will consider the existence of a drop with these parameters and drop's volume prescribed. Let $\underline{v}$, resp.$\bar{v}$ be respectively the minimum and the maximum of the height function on W.

If such a drop S exists, it is contained in RW for some $R > 0$. We then obtain that the volume is

$$V = \pi R^3 \int_{v}^{\bar{v}} u^2(v)\, dv \ ,$$

We use this to solve for R and write $x = Ru(v)$ in (40). We obtain the condition

$$0 = v + \omega + \frac{\tau}{u(v)} \left(\frac{\pi \int_v^{\bar{v}} u^2(v)\, dv}{V} \right)^{1/3} =: G(v) \ .$$

For some $\xi \in [v, \bar{v}]$, there holds $\int_v^{\bar{v}} u^2\, dv = u^2(\xi)(\bar{v} - v)$ and so

$$\lim_{v \uparrow \bar{v}} \frac{1}{u(v)} \left(\frac{\pi \int_v^{\bar{v}} u(v)^2\, dv}{V} \right)^{1/3} = \lim_{v \uparrow \bar{v}} \left(\frac{u^2(\xi)}{u^2(v)} \frac{(\bar{v} - v)}{(u - u(\bar{v}))} \right)^{1/3} = (-v_u(u = 0))^{1/3} = 0$$

It then follows easily that

$$\lim_{v \uparrow \bar{v}} G(v) = \bar{v} + \omega \ , \quad \lim_{v \downarrow \underline{v}} G(v) = \mathrm{sign}(\tau) \cdot \infty \ .$$

We conclude from this that *if $\tau > 0$ and $\bar{v} + \omega < 0$ holds or $\tau < 0$ and $\bar{v} + \omega > 0$ holds, then there exists a sessile drop for any volume V.*

From the Schwartz symmetrization we get the following.

Proposition 8.1 *Consider the functional $\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}$ with $\tau_i \geq 0, i = 0, 1$. If a minimizer exits for the volume constrained problem with one free boundary component, then the minimizer must be a sessile drop.*

We continue with the case where W is a surface of revolution and we assume that the line tension τ is positive. (In the case where $\tau = 0$, Winterbottom's theorem implies that $\hat{W}$ is stable.) Recall that when the line tension is non negative, then the surface is stable if and only if it is stable with respect to symmetric variations.

Consider the eigenvalue problem

$$(J + \lambda)\psi = 0, \quad \text{in } S, \quad B_\tau[\psi] = 0, \quad \text{on } \partial S \ ,$$

restricted to rotationally invariant functions. The eigenvalues will be enumerated in ascending order $\lambda_1 < \lambda_2 \leq ....$

Proposition 8.2 *(i) $\lambda_1 < 0$ holds.*

(ii) If $\underline{\nu}_3 \geq 0$ holds, then $\lambda_2 \geq 0$.

(iii) If $\underline{\nu}_3 < 0$ holds then, $B_\tau[\nu_3] > 0$ on ∂S implies the surface is unstable.

(iv) If If $\underline{\nu}_3 < 0$ holds then, $B_\tau[\nu_3] \leq 0$ on ∂S implies $\lambda_2 \geq 0$.

Proof. Note that the function z' satisfies $J[z'] = z'/(x^2\mu_2)$ with $B_1[z'] = 0$ on ∂S. From this it it follows easily that $\lambda_1 < 0$ holds.

To show (ii) note that $\nu_3 \geq 0$ on all of S and so the first eigenvalue of J for the Dirichlet problem is non negative. If ψ is an eigenfunction belonging to λ_2, then ψ changes sign and one of its nodal domains has first eigenvalue equal to λ_2. By monotinicity of Dirichlet eigenvalues with respect to the domain, we get $\lambda_2 \geq 0$.

We now show (iii). Write $\nu_3 = \nu_3^+ - \nu_3^-$. Since ν_3 changes sign, we can find a positive constant a such that $g := a\nu_3^+ - \nu_3^-$ satisfies

$$\int_S g \, dS = 0 \, .$$

Since g is piecewise smooth, it is admissible in the second variation formula. Then for this function

$$\delta^2 \mathcal{E} = \oint_{\partial S} \nu_3 B_\tau[\nu_3] \, dL < 0 \, ,$$

proving (iii).

Finally, we show (iv). Again let ψ_2 be an eigenfunction belonging to λ_2. The nodal set of ψ is a circle which we denote by $\mathcal{C}$ and we let U denote the region bounded by $\mathcal{C}$ and ∂S. If a_1 is the first eigenvalue of the problem

$$(J + a\lambda)\psi = 0, \quad \text{in } U, \quad B_\tau[\psi] = 0, \quad \text{on } \partial S \, , \psi = 0, \quad \text{on } \mathcal{C} \, ,$$

then $a_1 = \lambda_2$ holds. Since $\nu_3 > 0$ on U, we can write any function ψ satisfying these boundary conditions as $\psi = f\nu_3$ for some function with $f \equiv 0$ on $\mathcal{C}$. We have by straightforward calculations:

$$-\int_U \psi J[\psi] \, dS = -\int_S \zeta \text{div}[\nu_3^2 A \nabla f] \, d\Sigma = \int_U \nu_3^2 \nabla f \cdot A \nabla f \, dS - \oint_{\partial S} f \nu_3^2 A \nabla f \cdot n \, dL \, ,$$

and

$$\oint_{\partial S} \psi B_\tau[\psi] \, dL = \oint_{\partial S} f^2 \nu_3 B_\tau[\nu_3] + \nu_3^2 f A \nabla f \cdot n \, dL \, .$$

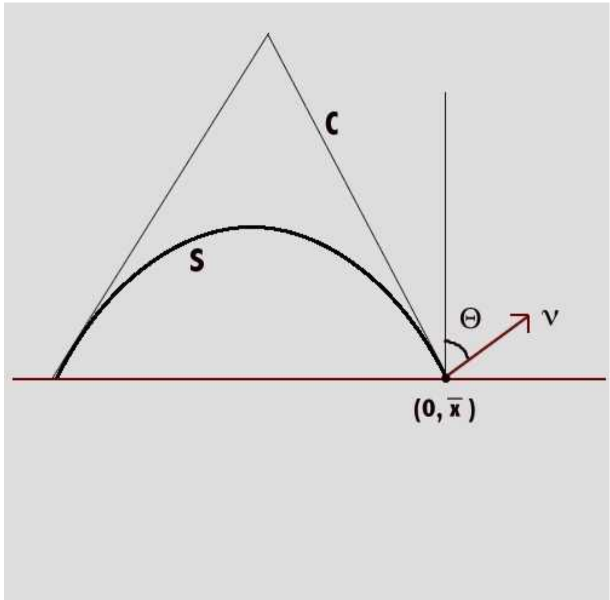
It follows by adding the last two formulas that $a_1 \geq 0$ holds from which it follows that $\lambda_2 \geq 0$ holds. **q.e.d**

Let S be a rotationally invariant sessile drop whose boundary circle lies in the plane $z = \underline{z}$. We will always assume that S is a part of a rescaling of the Wulff shape, i.e. $S \subset RW$ for some $R \in \mathbf{R}$. If the angle Θ between S and the plane $\{z = \underline{z}\}$ is not a right angle, then there is a unique cone C which is tangent to S along ∂S. Let V_C denote the volume of this cone. If $\Theta \in (0, \pi/2)$ holds, then C and S lie on the same side of the plane and by the convexity of W, $V_C > V$ holds. On the other hand, for $\Theta \in (\pi/2, \pi)$, C and S lie on opposite sides of the plane and $V_C < V$ holds.

Theorem 8.1 *Let W be a rotationally symmetric Wulff shape which is symmetric with respect to the plane $z = 0$ and let S be a sessile drop with $\tau \geq 0$.*

Then, S is stable if and only if

$$\tau \leq \frac{3V_C V}{\pi R \underline{x}(V_C - \sigma(\nu_3)V)} \, , \tag{41}$$



holds, where

$$\sigma(\nu_3) := \begin{cases} +1, & \nu_3 > 0, \\ 0, & \nu_3 = 0, \\ -1, & \nu_3 < 0. \end{cases}$$

Proof. We first attempt to solve the equation

$$J[\phi] = -\Lambda\,, \quad B_\tau[\phi] = 0\,, \text{on } \partial S\,. \tag{42}$$

Since $J[q] = -\Lambda$ and since ν_3 is the only rotationally symmetric solution of solution of $J[u] = 0$ defined in all of S, any rotationally symmetric solution must be of the form, $\psi = q + c\nu_3$, where c is chosen so that $B_\tau[q] + cB_\tau[\nu_3] = 0$ on ∂S. We have from (previous paper),

$$B_\tau[\nu_3] = \frac{k_1}{\mu_1 z'} - \frac{\tau\nu_3}{x^2(z')^2} = \frac{-1}{Rz'} - \frac{\tau\nu_3}{x^2(z')^2}\,, \tag{43}$$

One then sees that since $q = xz' - x'z$ and $\nu_3 = -x'$,

$$B_\tau[q] = \frac{k_1 z}{\mu_1 z'} - \frac{\tau q}{x^2(z')^2} = \frac{-z}{Rz'} - \frac{\tau q}{x^2(z')^2} = zB_\tau[\nu_3] - \tau/(xz')\,. \tag{44}$$

Thus, if $B_\tau[\nu_3] = 0$, then $B_\tau[q] \neq 0$ and the equation (42) is unsolvable and the surface is therefore unstable.

From now on we can assume that $B_\tau[\nu_3] \neq 0$ on ∂S and so the solution of (42) is

$$\phi := q - \frac{B_\tau[q]}{B_\tau[\nu_3]}\nu_3\,,$$

From the Divergence Theorem applied to the position vector X and the constant vector E_3,

$$\int_S q\,dS = 3V + \underline{z}\mathcal{A}\,, \quad \int_S \nu_3\,dS = \mathcal{A}\,,$$

and so

$$\int_S \phi\,dS = 3V + \underline{z}\mathcal{A} - \big(\underline{z} - \frac{\tau}{xz' B_\tau[\nu_3]}\big)\mathcal{A} = 3V + \frac{\tau\mathcal{A}}{xz' B_\tau[\nu_3]}\,.$$

Therefore

$$\int_S \phi\,dS \geq 0 \Longleftrightarrow \frac{3V}{\mathcal{A}} \geq \frac{-\tau}{xz' B_\tau[\nu_3]}\,. \tag{45}$$

If $\nu_3 > 0$ holds, then from (43), $B_\tau[\nu_3] < 0$ holds. By using (43) and (45), we see that stability occurs if and only if

$$\frac{Vx^2z'}{R\nu_3} \geq \tau\big(\frac{xz'\mathcal{A}}{3\nu_3} - V\big)\,.$$

Then this case of the result follows since $V_C = xz'\mathcal{A}/(3|\nu_3|)$.

When $\nu_3 = 0$, we get from (43) that $B_\tau[\nu_3] = -1/Rz'$ and and from (ii) of the proposition $\lambda_2 \geq 0$. Therefore from (45), we get that stability occurs if and only if $3\underline{x}V/R\mathcal{A} \geq \tau$ holds.

Finally if $\underline{\nu}_3 < 0$ holds, then $B_\tau[\nu_3] = 0$ if and only if $\tau = -x^2z'/(R\nu_3) = 3V_C/(\pi R\underline{x})$. For this value of τ, the surface is unstable for reasons discussed above. If τ is larger than this value, then $B_\tau[\nu_3] > 0$ holds and instability follows from (iii) of the proposition. If τ is less than this value, (45) and (iv) of the proposition imply that stability occurs if and only if

$$\tau \leq \frac{3xV/(R\mathcal{A})}{1 - 3\nu_3 V/(xz'\mathcal{A})} = \frac{3V_C V}{\pi R\underline{x}(V_C + V)}\,.$$

Since this number is already less than $V_C/(\pi R\underline{x})$, (41) is the necessary and sufficient condition for stability. **q.e.d**

Using the notation of section 5, it is clear that if a Wulff shape W of product form is replaced by the Wulff shape $\bar{W}$ having the same generating curve and circular cross sections, then the wetting energy, line tension and volume all transform according to

$$\mathcal{A} = \frac{\text{area}(\Omega)}{\pi}\bar{\mathcal{L}}\,,\ \mathcal{A} = \frac{\text{area}(\Omega)}{\pi}\bar{\mathcal{L}}\,,\ V = \frac{\text{area}(\Omega)}{\pi}\bar{V}$$

From this we can deduce the following:

Theorem 8.2 *Let S be a sessile drop which is a subset of a rescaling of a Wulff shape of product form. Assume that $0 \leq \tau$ holds. Then S is stable if and only if*

$$\tau \leq \left(\frac{\pi}{\text{area}(\Omega)}\right)\frac{3V_C V}{\pi R\underline{x}(V_C - \sigma(\nu_3)V)}\,, \tag{46}$$

Remarks If S is in equilibrium for $\mathcal{E} = \mathcal{F} + \omega\mathcal{A} + \tau\mathcal{L}$, then RS, $R \in \mathbf{R}^+$ is in equilibrium for $\mathcal{E}' = \mathcal{F} + \omega\mathcal{A} + (R\tau)\mathcal{L}$. Since both sides of all of the conditions of the theorem rescale linearly, we can conclude that S is a *stable* equilibrium for $\mathcal{E}$ if and only if RS is a stable equilibrium for $\mathcal{E}'$.

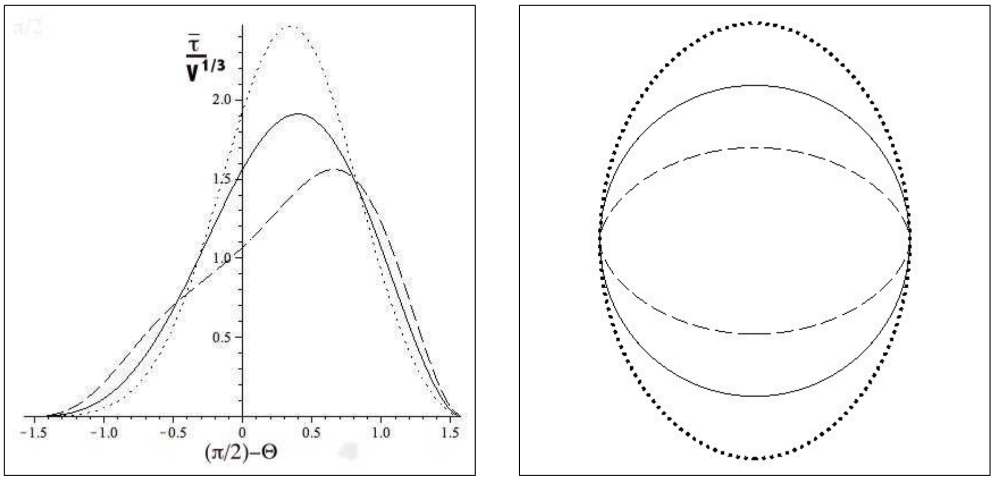


Figure 6: Plot of $\bar{\tau}/\text{vol}^{1/3}$ for Wulff shapes with energy density $\gamma = 1 + e\nu_3^2$. dotted $\leftrightarrow e = 0.4$, solid $\leftrightarrow e = 0$, dashed $\leftrightarrow e = -0.4$. The Wulff shapes are shown on the right.

For a fixed value of the contact angle Θ, we let $\bar{\tau}(\Theta)$ denote the maximum value of τ for which the sessile drop is stable. In Figure 8 we show the plot of the scale invariant quantity $\bar{\tau}/\text{vol}^{1/3}$ versus $(\pi/2) - \Theta$ for various functionals with densities of the form $\gamma = 1 + \epsilon\nu_3^2$.

Note that for $\Theta \approx 0$ and for $\Theta/V^{1/3} \approx \pi$, we have $\bar{\tau} \approx 0$. The first case corresponds to a sessile drop which consists of a small neighborhood of the north pole of W. If we think of this part of W rescaled so as to contain a fixed volume, then on the boundary $\mathcal{A} >> \mathcal{L}$ holds. In this case if τ is too large, the drop diminishes its energy by spreading itself out over the plane as a large, nearly flat shape. This means that a type of wetting transition, similar to that described in [18] occurs.

In the other limiting case $\Theta \approx \pi$, we have $\mathcal{L} >> \mathcal{A}$. In this case, it τ is too large, the drop can diminish its energy by energy by detaching from the plane entirely (drying transition).

9 Other boundary configurations

We consider a functional whose Wulff shape W is of product type. Let Π_V be a vertical plane which passes through the origin and which intersects the (α, β) curve in a right angle. As before, let Π_0 and Π_1 be horizontal planes. Then the part of the Wulff shape $\hat{W}$ bounded by $\Pi_0 \cup \Pi_1 \cup \Pi_V$ is in equilibrium for the functional $\mathcal{E} = \mathcal{F} + \omega_0\mathcal{A}_0 + \omega_1\mathcal{A}_1$ among all surfaces having the same enclosed volume and having free boundary on $\Pi_0 \cup \Pi_1 \cup \Pi_V$. Here $\omega_i := \chi|_{W\cap\Pi_i} \cdot (-1)^{i+1}E_3$. Note that Π_V carries no wetting energy. To see that $\hat{W}$ is in equilibrium, note that along $W \cap \Pi_V$, the position vector χ is contained in the plane Π_V and thus automatically satisfies the boundary condition $\chi \cdot N_V = 0$, where N_V is the normal to Π_V, along this intersection. In face, by Winterbottom's Theorem, $\hat{W}$ is the absolute minimizer of $\mathcal{E}$ for this boundary and the given volume.

We now consider a generalized anisotropic Delaunay surface Σ for the functional $\mathcal{F}$. Observe that if we take a part of the surface Σ which is in equilibrium for the functional $\mathcal{E}$ and we take $\hat{\Sigma}$ to be the part of Σ on one side of Π_V, then $\hat{\Sigma}$ is in equilibrium for $\mathcal{E}$ among all surfaces containing

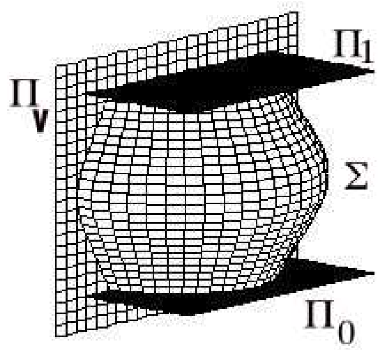


Figure 7: Configuration with a vertical plane added.

the same volume and having free boundary on $\Pi_0 \cup \Pi_1 \cup \Pi_V$. Just note that at points of $\hat{\Sigma} \cap \Pi_V$, the vectors (α', β') and N_V are parallel. The points on W where the normals are perpendicular to N_V are exactly those points in $W \cap \Pi_V$ and thus $\chi \cdot N_V \equiv 0$ on $\hat{\Sigma} \cap \Pi_V$.

If we again assume that the tangent planes to $\hat{\Sigma}$ are nowhere horizontal, then again we need only consider symmetric variations to analyze the stability. In order to do this, we need a minor modification of the symmetrization argument given above.

If we make a deformation of our capillary surface $\hat{\Sigma}$ then we replace each level set of x_3 of the deformed surface by a homothety of the arc of (α, β) between the right angles. The scaling factor is chosen so that the area is the same as the area of the level set of the deformed surface. An application of the one dimensional case of Winterbottom's Theorem shows that this symmetrization diminishes the one dimensional energy of each level set and using a method similar to that given above, one sees that the total energy of the surface is also diminished.

Similar results are valid if the surface is intersected by several vertical planes which are orthogonal to the curve (α, β).

10 Appendix

Let $X : \Sigma \to \mathbf{R}^3$ be a capillary anisotropic Delaunay surface. Here we will compute the second variation of energy for symmetric variations. The point of departure is (4). The variation of the first term is

$$\delta \int_\Sigma -\Lambda\psi \, d\Sigma = -\int_\Sigma \psi J[\psi] \, d\Sigma \, , \tag{47}$$

where

$$J[\psi] := \delta_{\psi\nu}\Lambda \, .$$

is the Jacobi operator (11).

The boundary condition for the capillary problem implies that $\chi + (-1)^i[\omega + \tau\lambda]E_3$ is perpendicular to E_3 on C_i. This means that if we take the first variation of the line integral in (4), the only non zero terms are given by

$$\int_{\partial\Sigma} \delta X \cdot (X_L \times \delta\big[\chi + (-1)^i[\omega + \tau\lambda]E_3)\big] \, dL \, ,$$

since δX and $(\delta X)_L$ are also perpendicular to E_3.

We compute, $\delta\chi = A\delta\nu = A(-\nabla\psi + d\nu(\xi))$. Using $\delta X \cdot E_3 = 0$, we have on $\partial\Sigma$, $\xi \cdot n = -\psi(\nu_3/n_3)n$. Therefore

$$\delta\chi = -A(\nabla\psi + \frac{\psi\nu_3}{n_3}n) + (\xi \cdot X_L)A \, d\nu(X_L) \, . \tag{48}$$

We will also need the first variation of the anisotropic curvature function λ. This will first be done assuming that the variation is symmetric. For a symmetric surface, $\lambda = 1/x_i$ on C_i and therefore $\delta\lambda_i = -(\delta x_i)/x_i^2$. Since

$$\delta X = (-1)^{i+1}(\psi/n_3)N + (\delta X \cdot X_L)X_L = \delta x(\alpha, \beta, 0) = \delta x(\Gamma N + D\Gamma) \, , \tag{49}$$

on C_i, we get $\delta x = (-1)^{i+1}\psi/\Gamma(N)$ and so

$$\delta\lambda = (-1)^i \frac{\psi}{x_i^2 n_3 \Gamma(N)} \quad \text{on } C_i \,. \tag{50}$$

(Recall Γ is the support function of the one dimensional Wulff shape.) We then obtain

$$\begin{aligned}
\oint_{\partial\Sigma} \delta X \cdot (X_L \times \delta\big[\chi + (-1)^i[\omega + \tau\lambda]E_3)\big]\, dL &= \oint \delta X \cdot (X_L \times \delta\chi)\, dL + (-1)^i \tau_i \oint_{\partial\Sigma} (\delta\lambda)\, \delta X \cdot (X_L \times E_3)\, dL \\
&= -\oint \frac{\psi}{n_3} \delta\chi \cdot E_3\, dL - \tau_i \oint_{\partial\Sigma} \frac{\psi^2}{x_i^2 n_3^2 \Gamma(N)}\, dL \\
&= \oint \frac{\psi}{n_3} A(\nabla\psi + \frac{\psi\nu_3}{n_3} d\nu(n)) \cdot E_3\, dL - \tau_i \oint_{\partial\Sigma} \frac{\psi^2}{x_i^2 n_3^2 \Gamma(N)}\, dL \,.
\end{aligned}$$

The second variation of energy is therefore given by

$$\delta^2\mathcal{E} = -\int_\Sigma \psi J[\psi]\, d\Sigma + \oint \psi A(\nabla\psi + \frac{\psi\nu_3}{n_3} d\nu(n)) \cdot n\, dL - \tau_i \oint_{\partial\Sigma} \frac{\psi^2}{x_i^2 n_3^2 \Gamma(N)}\, dL \,. \tag{51}$$

A formula for non symmetic variations can also be obtained. We can obtain from (12) that for a variation of a curve $\delta C := fN + gT$,

$$\delta\lambda = -\big((\frac{f_L}{m})_L + \frac{\kappa^2 f}{m}\big) + g\lambda_L = -\big((\frac{f_L}{m})_L + \frac{\kappa^2 f}{m}\big) \,,$$

since $\lambda \equiv$ constant on each boundary component by (9). The reason for the minus sign is that we have chosen the opposite sign convention for the anisotropic curvature function λ from that chosen for Λ.

For the variations of the boundary curves that we are considering,

$$\delta X = (-1)^{i+1} \frac{\psi}{n_3} N + (\delta X \cdot X_L) X_L \,,$$

so

$$\delta\lambda = (-1)^i \big([\frac{1}{m}(\frac{\psi}{n_3})_L]_L + \frac{\kappa^2\psi}{m n_3}\big) \,. \tag{52}$$

Using this above, yields

$$\delta^2\mathcal{E} = -\int_\Sigma \psi J[\psi]\, d\Sigma + \oint \psi A(\nabla\psi + \frac{\psi\nu_3}{n_3} d\nu(n)) \cdot n\, dL - \tau \oint_{\partial\Sigma} \frac{\psi}{n_3} \big([\frac{1}{m}(\frac{\psi}{n_3})_L]_L + \frac{\kappa^2\psi}{m n_3}\big)\, dL \,. \tag{53}$$

It should be remarked that ψ is not independent of L, except in the case of circular cross sections. The dependence is given by $\psi = \bar{\psi}(s)P(s,t)$ where P is given by (30) and $dt = (\Gamma/x)dL$.

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