

Stability of anisotropic capillary surfaces between two parallel planes

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Abstract

We study the stability of capillary surfaces without gravity for anisotropic free surface energies. For a large class of rotationally symmetric energy functionals, it is shown that the only stable equilibria supported on parallel planes are either cylinders or a part of the Wulff shape.

1 Introduction

The interface between two non mixing materials may often be represented as a surface. The surface forms itself into an equilibrium shape for a potential energy which is determined by the forces acting on it. When the surface is liquid, its surface tension is isotropic, i.e. it does not depend on the direction of the surface normal. For more structured materials an anisotropic surface tension is more appropriate. Here, a free boundary problem for this type of energy is considered.

Let F be a smooth, positive function on S^2 . Then F can be used to define a free anisotropic surface energy \mathcal{F} which assigns to an oriented, immersed surface $X : \Sigma \rightarrow \mathbf{R}^3$ the value

$$\mathcal{F}[X] := \int_{\Sigma} F(\nu) d\Sigma, \quad (1)$$

where $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \rightarrow S^2$ is the Gauss map and $d\Sigma$ is the area form of the induced metric. The property that X is a critical point of \mathcal{F} for all compactly supported volume-preserving variations is characterized by having *constant anisotropic mean curvature*. This definition is a generalization of the idea of constant mean curvature (CMC) which arises from the area functional, that is $F \equiv 1$.

One of the basic problems related to CMC surfaces is the *free boundary problem* of finding the compact surface of least area and enclosing a fixed volume subject to the condition that the boundary is constrained to lie on a given supporting surface. Such problems provide a model for many of the surfaces formed in nature such as a drop resting on or suspended from a horizontal plane in the absence of gravity. They have a long history which dates back to the research of Laplace, Taylor, and Young into estimation of the height to which a fluid rises in a narrow vertical tube [6].

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In the mid eighties, Athanassenas [1] and Vogel [13] independently obtained the uniqueness result for stable, embedded CMC surfaces between two parallel planes, having boundary components constrained to lie on these planes, here a CMC surface X is said to be stable if the second variation of the area is nonnegative for all volume-preserving variations of X which satisfy the boundary condition. Their result states that the only stable embedded critical equilibria with nonempty boundary are short cylinders and hemispheres. The analysis is greatly simplified by the fact that Alexandrov reflection implies that the equilibria are surfaces of revolution so that the fact that the surfaces lie in the region between the planes reduces the problem to consideration of the unduloids.

We will treat the analogous problem in the anisotropic case when area is replaced by an anisotropic energy functional \mathcal{F} which is assumed to be invariant under rotation with axis perpendicular to the supporting planes. Our approach is limited in that a technical condition on the functional is required but is general enough to be applied to many easily recognizable cases which include the CMC case.

Recall that an anisotropic surface energy \mathcal{F} possesses a canonical critical point for the isoperimetric problem of minimizing \mathcal{F} among closed surfaces enclosing a specific three dimensional volume ([12]). This critical point is known as the *Wulff shape* \mathcal{W} . In the special case where $F \equiv 1$, \mathcal{W} is the round sphere of radius 1 with center at the origin.

It will be assumed that

- \mathcal{W} is a smooth, uniformly convex surface of revolution with vertical rotation axis.
- \mathcal{W} is invariant with respect to reflection through the horizontal plane $z = 0$.
- The curvature function of the generating curve of \mathcal{W} with respect to the inward pointing normal is a non-decreasing function of arc length on $\{z \geq 0\}$ as one moves in an upward direction.

Then, we show:

Any stable embedded equilibrium capillary surface between two horizontal planes with nonempty free boundary on these planes is either a sufficiently short cylinder or a suitable part of the Wulff shape.

In fact, our method is general enough to be applicable to surfaces of revolution which may not be contained between the supporting planes. We remark that if the surface is not assumed to lie between the planes then one cannot conclude a priori that it is a surface of revolution. However, such an equilibrium surface of revolution, an *anisotropic nodoid*, always exists. We show:

Any capillary surface which is a part of the anisotropic nodoid is unstable without any additional conditions on the curvature of the Wulff shape.

We remark that it was shown in [11] that any closed, stable surface of nonzero constant anisotropic mean curvature is, up to translations and homotheties, the Wulff shape. Also in [9] we showed that if the functional \mathcal{F} is close to the area functional, then any complete, stable surface of nonzero constant anisotropic mean curvature is, up to translations and homotheties, the Wulff shape.

The paper is organized as follows. Section 2 contains a basic assumption on the energy functional and preliminaries for later use. Section 3 contains derivations of the first and second variation

formulas for anisotropic surface energies with a free boundary condition. Section 4 is a summary of some of our previous results about rotation surfaces with constant anisotropic mean curvature which we use in Sections 3 and 5. Section 5 contains the main results of the paper including the instability of anisotropic unduloids and nodoids. The last section is a summary of known results about eigenvalues for self-adjoint problems which were used in the previous section.

Anisotropic capillary surfaces for surfaces with wetting energy can, of course, also be considered. The stability of equilibria for this type of problem is currently being investigated by the authors and the results will appear elsewhere in the future.

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2 Basic assumption and preliminaries

From now on, we will assume a *convexity condition* on the energy functional formulated as follows: Denote by DF and D^2F the gradient and Hessian of F on S^2 . Then we assume that at each point in S^2 the matrix $D^2F + F1$ is positive definite. The functional appearing in (1) is called an elliptic parametric functional (cf. [5], [7]).

The convexity condition implies that the embedding $\chi : S^2 \rightarrow \mathbf{R}^3$ defined by $\chi(\nu) = DF(\nu) + F\nu$ defines a smooth, convex surface in \mathbf{R}^3 ([12]). The image \mathcal{W} of χ is called the Wulff shape. We will call also χ the Wulff shape.

Let $X : \Sigma \rightarrow \mathbf{R}^3$ be an immersion of a two dimensional oriented smooth manifold Σ with Gauss map ν . Denote by H the mean curvature of X . The Euler-Lagrange equation for the functional \mathcal{F} for compactly supported volume-preserving variations is

$$\operatorname{div}_{\Sigma} DF - 2HF = \operatorname{trace}_{\Sigma} A d\nu = \text{constant} \quad (2)$$

(cf. Section 3), where DF is considered as a smooth tangent vector field along X by parallel translation in \mathbf{R}^3 , and A is defined as

$$A := D^2F + F1,$$

where 1 means the identity endomorphism field. In view of (2), the *anisotropic mean curvature* Λ of X is defined as (cf. [10], [9])

$$\Lambda := -\operatorname{div}_{\Sigma} DF + 2HF = -\operatorname{trace}_{\Sigma} A d\nu.$$

Note that the operator A is symmetric on S^2 so there exists a locally defined frame $\{e_1, e_2\}$ on S^2 such that $Ae_i = (1/\mu_i)e_i$. If $(-h_{ij})$ is the matrix of $d\nu$ with respect to this frame, we obtain the expression

$$\Lambda = h_{11}/\mu_1 + h_{22}/\mu_2. \quad (3)$$

Note that the anisotropic mean curvature of the Wulff shape χ is -2 with respect to the outward pointing unit normal. This follows since $d\chi = D^2F + F1$, $d\nu = (d\chi)^{-1}$. Also, $1/\mu_j$ are negatives of the reciprocals of the principal curvatures of χ with respect to the outward unit normal.

In the special case we will be concerned with where $F = F(\nu_3)$, the endomorphism field $D^2F + F1$ has eigendirections corresponding to

$$E_1 := (0, 0, 1) - \nu_3\nu, \quad E_2 := \nu \times E_1.$$

The eigenvalues $1/\mu_j$ corresponding to these directions are given by

$$1/\mu_1 = (1 - \nu_3^2)F'' + 1/\mu_2, \quad 1/\mu_2 = F - \nu_3F'. \quad (4)$$

Note that since $D^2F + F1$ is assumed to be positive definite, μ_1 and μ_2 are positive functions on S^2 which depend only on ν_3 .

3 Capillary problems

Let $\Omega \subset \mathbf{R}^3$ be a closed domain with nonempty boundary $\partial\Omega$ which we assume to be a union of smooth, embedded surfaces. The region Ω is not assumed to be bounded. For a fixed two dimensional, orientable compact connected C^∞ manifold with boundary $(\Sigma, \partial\Sigma)$, we let \mathcal{I} denote the ‘‘manifold’’ of all C^∞ immersions $(\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \partial\Omega)$ which enclose a fixed volume V_0 , here the volume is defined locally as follows. Let $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \partial\Omega)$ be an immersion. For a variation $X_\epsilon : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \partial\Omega)$ of X , we define the volume of X_ϵ as follows (cf. [2]):

$$V[X_\epsilon] = \int_{[0, \epsilon] \times \Sigma} X_\epsilon^* dV,$$

where dV is the canonical volume element of \mathbf{R}^3 , and $X_\epsilon^* dV$ is the pull back of dV by X_ϵ . $V[X_\epsilon]$ represents the (algebraic) volume enclosed between the surfaces X and X_ϵ .

We seek a condition that X be a critical point of $\mathcal{F}|_{\mathcal{I}}$. To do this, we consider smooth curves $(-\epsilon, \epsilon) \rightarrow \mathcal{I}$ of the form $X_\epsilon = X + \epsilon\dot{X} + \mathcal{O}(\epsilon^2)$. We will refer to such a curve as an *admissible variation* and we refer to its tangent field \dot{X} as an *admissible variation field*. Let N denote a unit normal field to $\partial\Omega$. Then if $\dot{X} = \xi + \varphi\nu$ is the expression of a variation field in terms of tangential and normal components, admissibility is equivalent to the two conditions

$$\int_{\Sigma} \varphi d\Sigma = 0, \quad (5)$$

$$\langle \dot{X}, N \rangle|_{\partial\Sigma} \equiv 0. \quad (6)$$

The condition (5) is the infinitesimal condition that the enclosed volume is preserved ([2]), while (6) is the infinitesimal condition that $X_\epsilon(\partial\Sigma) \subset \partial\Omega$ for all ϵ . In fact, a standard argument using the implicit function theorem shows that (5) and (6) are also sufficient to embed X in an admissible variation with the given variation field \dot{X} .

Proposition 3.1 *An immersion $X \in \mathcal{I}$ is a critical point of $\mathcal{F}|_{\mathcal{I}}$ if and only if there holds:*

$$\Lambda \equiv \Lambda_0, \quad \text{in } \Sigma, \quad (7)$$

for some constant Λ_0 and

$$\langle \chi, N \rangle \equiv 0, \quad \text{on } \partial\Sigma. \quad (8)$$

Proof. For an admissible variation field \dot{X} as above, we have

$$\begin{aligned}
\delta\mathcal{F} &= \int_{\Sigma} \langle DF, -\nabla\varphi + d\nu(\xi) \rangle + F(\operatorname{div}\xi - 2H\varphi)d\Sigma \\
&= \int_{\Sigma} \varphi(\operatorname{div}_{\Sigma}DF - 2HF)d\Sigma + \oint_{\partial\Sigma} -\varphi\langle DF, n \rangle + F\langle \xi, n \rangle d\tilde{s} \\
&= - \int_{\Sigma} \varphi\Lambda d\Sigma + \oint_{\partial\Sigma} \langle \chi \times \dot{X}, dX \rangle,
\end{aligned} \tag{9}$$

where n is the outward pointing unit normal along $\partial\Sigma$, $d\tilde{s}$ is the line element on $\partial\Sigma$, and $dX := (\nu \times n)d\tilde{s}$. Assume first that (7) and (8) hold. Because of (5), the surface integral in (9) vanishes for any admissible variation. Define a unit tangent t to $\partial\Sigma$ as $t := \nu \times n$. Note that (8) implies that $t \times \chi$ is proportional to N along $\partial\Sigma$ and so for any admissible variation field the boundary integrals in (9) vanish.

Conversely assume that $\delta\mathcal{F} = 0$ for all admissible variations. By considering admissible, compactly supported normal variations of X , one sees that (7) holds.

Suppose that (8) does not hold on some boundary arc $\alpha \subset \partial\Sigma$. We may assume that $\langle \chi, N \rangle > 0$ holds on α and let u be a non-constant smooth function on $\partial\Sigma$ which is non-negative and has compact support in α . Let \dot{X} be any smooth, admissible variation field on Σ with $\dot{X}|_{\partial\Sigma} = ut \times N$. Then we obtain a contradiction since

$$0 = \delta\mathcal{F} = - \oint_{\partial\Sigma} u\langle \chi, t \times (t \times N) \rangle d\tilde{s} = \oint_{\partial\Sigma} u\langle \chi, N \rangle d\tilde{s} > 0.$$

q.e.d.

Definition 3.1 We call an immersion $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \partial\Omega)$ with $\partial\Sigma \neq \emptyset$ a *capillary surface* (for the energy \mathcal{F}) if X satisfies (7) for some constant Λ_0 and (8) holds.

We will concentrate on the case where $F(\nu)$ is a function of one variable, say $F = F(\nu_3)$. Let Π_1, Π_2 be two horizontal planes $\{x_3 = \text{constant}\}$, and let Ω be the closed domain bounded by $\Pi_1 \cup \Pi_2$.

Theorem 3.1 *Let $F = F(\nu_3)$. If X is either a closed embedded surface of constant anisotropic mean curvature or an embedded capillary surface $(\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_1 \cup \Pi_2)$, then $X(\Sigma)$ is rotationally symmetric with respect to a vertical line and its genus is zero. In particular, if X is closed, then $X(\Sigma)$ is, up to translation and homothety, the Wulff shape.*

Proof. Since $D^2F + F1$ is positive definite, the equation “ $\Lambda = \text{constant}$ ” is absolutely elliptic (cf. [7, Chapter 16]). Hence, by the Hopf maximum principle [8, Corollary 4.6]), if two surfaces with $\Lambda \equiv \Lambda_0$ are in oriented contact at a point from one side, then they are identical near that point. Therefore, if X is either a closed embedded surface of constant anisotropic mean curvature or an embedded capillary surface $(\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_1 \cup \Pi_2)$, then, for any vertical plane P_0 , there exists a plane P which is parallel to P_0 such that $X(\Sigma)$ is the union of a graph over a domain in P and its reflection with respect to P . Therefore, $X(\Sigma)$ is rotationally symmetric with respect to a

vertical line and its genus is zero. In view of the classification of rotationally symmetric surfaces of constant anisotropic mean curvature (cf. Section 4), up to translation and homothety, the Wulff shape is the only closed surface of revolution with constant anisotropic mean curvature. **q.e.d.**

Proposition 3.2 *Let $F = F(\nu_3)$. Then, the angle between a capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ and the bounding planes $\Pi_1 \cup \Pi_2$ must be locally constant along the boundary in the sense that there exists a unique value η which depends only on F such that $\nu_3 = \eta$ holds on $\partial\Sigma$. Moreover, a capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ meets the planes $\Pi_1 \cup \Pi_2$ in a right angle if and only if $F'(0) = 0$ holds.*

Proof. Set $E_3 := (0, 0, 1)$. The boundary condition (8) becomes

$$\langle F'(\nu_3)E_3 + (F(\nu_3) - F'(\nu_3)\nu_3)\nu, N \rangle = 0,$$

which can be expressed as

$$g(\nu_3) := (1 - \nu_3^2)F' + \nu_3F = 0.$$

This and (4) gives

$$g'(\nu_3) = (1 - \nu_3^2)F'' - \nu_3F' + F = 1/\mu_1 > 0.$$

Hence, g is monotonically increasing. Since

$$g(1) = F(1) > 0, \quad g(-1) = -F(-1) < 0$$

hold, there exists a unique value η which depends only on F such that $g(\eta) = 0$ holds, which implies the first assertion of the proposition. The second assertion follows from $g(0) = F'(0)$. **q.e.d.**

Proposition 3.3 *Let $F = F(\nu_3)$ satisfy $F'(0) = 0$. And let $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ be a capillary surface. Then for admissible variations with variation vector field $\xi + \varphi\nu$, the second variation of energy is given by*

$$\delta^2 \mathcal{F} = - \int_{\Sigma} \varphi L[\varphi] d\Sigma + \oint_{\partial\Sigma} \varphi \langle A\nabla\varphi, n \rangle d\tilde{s}, \quad (10)$$

where L is the self-adjoint Jacobi operator

$$L[\varphi] := \operatorname{div}(A\nabla\varphi) + \langle Adv, d\nu \rangle \varphi.$$

Proof. We will denote by “ \cdot ” the derivative with respect to the variation parameter. The first variation formula above gives,

$$\delta \mathcal{F} = - \int_{\Sigma} \varphi \dot{\Lambda} d\Sigma + \oint_{\partial\Sigma} \langle \chi \times \dot{X}, dX \rangle,$$

using $\dot{X} = \xi + \varphi\nu$. This is valid for any surface, so we can take the variation of the previous formula at an equilibrium surface $\Lambda \equiv \Lambda_0$, to obtain:

$$\delta^2 \mathcal{F} = - \int_{\Sigma} \varphi \dot{\Lambda} d\Sigma - \Lambda_0 \delta \int_{\Sigma} \varphi d\Sigma + \delta \oint_{\partial\Sigma} \langle \chi \times \dot{X}, dX \rangle.$$

The second integral vanishes since the variation is volume-preserving to all orders. As was shown in [9],

$$\dot{\Lambda} = L[\varphi] + \langle \nabla \Lambda, \xi \rangle \quad (11)$$

holds. Now since $\Lambda \equiv \Lambda_0$,

$$\dot{\Lambda} = L[\varphi]$$

holds. Expanding out the last term gives,

$$\delta^2 \mathcal{F} = - \int_{\Sigma} \varphi L[\varphi] d\Sigma + \oint_{\partial\Sigma} \langle \dot{\chi} \times \dot{X}, dX \rangle + \oint_{\partial\Sigma} \langle \chi \times \ddot{X}, dX \rangle + \oint_{\partial\Sigma} \langle \chi \times \dot{X}, d\dot{X} \rangle. \quad (12)$$

We now use the boundary values to compute the last three terms. Set $E_3 := (0, 0, 1)$. Since $F'(0) = 0$, $DF = F'(\nu_3)E_3^T$ vanishes on $\partial\Sigma$, i.e. $\chi|_{\partial\Sigma} = F \cdot \nu$. Since $\langle \dot{X}, E_3 \rangle \equiv 0 \equiv \nu_3$ on $\partial\Sigma$, we have that $\xi = \langle \xi, t \rangle t$ where $t := \nu \times n$.

Note that

$$\dot{\chi} = d\chi(\dot{\nu}) = A(-\nabla\varphi + d\nu(\xi)).$$

Therefore

$$\begin{aligned} \langle \dot{\chi} \times \dot{X}, dX \rangle &= \langle A(-\nabla\varphi + d\nu(\xi)) \times (\varphi\nu + \xi), t \rangle d\tilde{s} \\ &= -\langle t \times (\varphi\nu + \xi), A(-\nabla\varphi + d\nu(\xi)) \rangle d\tilde{s} \\ &= \varphi \langle A\nabla\varphi, n \rangle d\tilde{s}, \end{aligned}$$

using Joachimsthal's theorem and the fact that ξ is parallel to t .

Also,

$$\langle \chi \times \ddot{X}, dX \rangle = F \langle \nu \times \ddot{X}, t \rangle d\tilde{s} = F \langle n, \ddot{X} \rangle d\tilde{s} = 0$$

since $n = \pm E_3$ and the variation is tangent to the boundary to all orders.

Finally,

$$\begin{aligned} \langle \chi \times \dot{X}, d\dot{X} \rangle &= F \langle \nu \times (\varphi\nu + \xi), \frac{d}{d\tilde{s}}(\varphi\nu + \xi) \rangle d\tilde{s} \\ &= -F \langle \xi, t \rangle \langle n, \frac{d}{d\tilde{s}}(\varphi\nu + \xi) \rangle d\tilde{s}. \end{aligned}$$

Since $\langle E_3, \varphi\nu + \xi \rangle \equiv 0$ on the boundary, we have $\langle E_3, \frac{d}{d\tilde{s}}(\varphi\nu + \xi) \rangle \equiv 0$ on the boundary and so, by the previous proposition, $\langle n, \frac{d}{d\tilde{s}}(\varphi\nu + \xi) \rangle \equiv 0$.

Combining these formulas with (12) shows that (10) holds. **q.e.d.**

Definition 3.2 A capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ is said to be *stable* if the second variation $\delta^2 \mathcal{F}$ of \mathcal{F} is nonnegative for all admissible variations of X , otherwise it is said to be *unstable*.

Remark 3.1 Second variation formulas for anisotropic surface energies with fixed boundary conditions appear elsewhere (cf. [4, 9].)

4 Anisotropic Delaunay Surfaces

In [9], we studied the surfaces of revolution with constant anisotropic mean curvature which we called *anisotropic Delaunay surfaces*. It turns out that for a general, convex $F = F(\nu_3)$, the classification of such surfaces is remarkably similar to the classical CMC case. We summarize some of the results from [9] which we need in Sections 3 and 5.

Let Σ be a surface of revolution with vertical rotation axis which has constant anisotropic mean curvature Λ for rotationally symmetric energy

$$\mathcal{F}[X] = \int_{\Sigma} F(\nu_3) d\Sigma.$$

The surface Σ will be parametrized by

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad x \geq 0,$$

where s is the arc length of the generating curve $(x(s), z(s))$. We choose the orientation of X so that its Gauss map is given by

$$\nu(s, \theta) = (z'(s) \cos \theta, z'(s) \sin \theta, -x'(s)).$$

Then, by using a flux formula for the functional \mathcal{F} , we have shown ([9]) that the generating curve $(x(s), z(s))$ satisfies the conservation law:

$$2(1/\mu_2)z'(s)x(s) + \Lambda(x(s))^2 \equiv \text{constant} =: c. \quad (13)$$

The constant c arises as the flux with respect to vertical translation which is a symmetry of the variational problem. Conversely, a solution $(x(s), z(s))$ of (13) gives a surface of revolution with vertical rotation axis and constant anisotropic mean curvature Λ for $\mathcal{F}[X]$.

By changing the orientation of the generating curve if it is necessary, we may assume that $\Lambda \leq 0$ holds. The solution curves of (13) can be periodically continued to complete surfaces which are classified as follows:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane*.
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid*.
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to vertical translation and homothety)*.
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *cylinder*.
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid*.
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid*.

Some justification for the terminology is given in Lemma 4.1 below.

Lemma 4.1 (i) *The generating curve C of an anisotropic catenoid is a graph over an interval of the z -axis. C is perpendicular to the horizontal line at a unique point.*

(ii) Let $(x(s), z(s))$, $(x \geq 0)$, be the generating curve of an anisotropic unduloid or an anisotropic nodoid. Then, there is a unique local maximum B and a unique local minimum $N > 0$ of x , which we will call a bulge and a neck respectively.

(iii) The generating curve C of an anisotropic unduloid is a graph over the z -axis. C is a periodic curve with respect to the vertical translation, and from a neck to the next neck (and/or a bulge to the next bulge) gives one period. Therefore, C has a unique inflection point (x, z) between each neck and the next bulge, which satisfies $x = \sqrt{c/(-\Lambda)}$.

(iv) The curvature of the generating curve C of an anisotropic nodoid has a definite sign. C is a non-embedding periodic curve with respect to the vertical translation. From a neck to the next neck (and/or a bulge to the next bulge) gives one period.

Proof. (ii) \sim (iv) were given in [9]. We will prove (i). From the equation $2\mu_2^{-1}z'x = c$, z' does not change sign, and so $(x(s), z(s))$ is a graph over an interval of the z -axis. $z' = \pm 1$ if and only if $x = |c|(\mu_2|_{\nu_3=0})/2$. Hence, C is orthogonal to $\{z = \text{constant}\}$ at a unique point. **q.e.d.**

For our stability analysis, we will need a representation formula for the profile curves which is summarized in the following result from [9].

Proposition 4.1 ([9]) *Let \mathcal{W} be the Wulff shape of a rotationally symmetric anisotropic surface energy \mathcal{F} . Let*

$$\sigma \mapsto (u(\sigma), v(\sigma)), \quad \sigma \in (-\infty, \infty),$$

be the profile curve of \mathcal{W} , where σ is the arc length. Then

$$\mu_2^{-1}v_\sigma - u = 0$$

holds. Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be a surface with constant anisotropic mean curvature $\Lambda \leq 0$, and let the Gauss map of X coincide with that of \mathcal{W} at $s = s(\sigma)$. Then X is given as follows.

(i) *When X is an anisotropic catenoid,*

$$x = c/(2u)$$

for some nonzero constant c .

(ii) *When X is an anisotropic unduloid,*

$$x = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c > 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{\sigma | u \geq \sqrt{-\Lambda c}\}$.

(iii) *When X is an anisotropic nodoid,*

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c < 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{-\infty < \sigma < \infty\}$.

In all cases above, z is given by

$$z = \int^u v_u x_u du. \tag{14}$$

Conversely, for a Wulff shape \mathcal{W} defined as above, define x and z as in (i) \sim (iii) and (14). Then $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic Delaunay surface which satisfies

$$2\mu_2^{-1}z_s x + \Lambda x^2 = c,$$

where s is the arc length of (x, z) , and Λ is supposed to be zero for Case (i). Moreover, X has the same regularity as that of \mathcal{W} .

5 Stability of critica

We study the stability of anisotropic Delaunay surfaces with free boundary on the union of two horizontal planes $\Pi_1 \cup \Pi_2$. As in §3, we denote by Ω the closed domain bounded by $\Pi_1 \cup \Pi_2$. We will assume that the energy functional is rotationally symmetric, i.e. $F = F(\nu_3)$ and $F(\nu_3)$ is an even function.

Let

$$(u(\sigma), v(\sigma)), \quad -2L \leq \sigma \leq 2L$$

be the generating curve C of the Wulff shape \mathcal{W} , where σ is the arc-length parameter and the length of C is $4L$ such that

$$u(0) = \max u(\sigma), \quad \begin{cases} v(\sigma) > 0, & 0 < \sigma < 2L, \\ v(\sigma) < 0, & -2L < \sigma < 0 \end{cases}$$

holds. The curvature κ of C with respect to the inward pointing normal is given by

$$\kappa = -u''v' + u'v'' \geq 0.$$

We pose the following condition for C :

$$\begin{cases} \kappa'(\sigma) \geq 0, & 0 < \sigma < L, \\ \kappa'(\sigma) \leq 0, & -L < \sigma < 0. \end{cases} \quad (15)$$

Theorem 5.1 and Corollary 5.1 below are precise statements of the main results of this paper.

Theorem 5.1 *Assume that $F = F(\nu_3)$ is an even function. Let $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ be a rotationally symmetric capillary surface, then, X is either the upper or the lower half of the Wulff shape \mathcal{W} , a part of a cylinder, an anisotropic unduloid, or an anisotropic nodoid.*

(i) *If X is either the upper or the lower half of \mathcal{W} , then X is stable.*

(ii) *If X is a part of a cylinder, then X is stable if and only if*

$$\frac{\mu_1(0)|\Lambda|}{R} \leq (\pi/h)^2$$

holds, where R and h denote the radius and height of X .

(iii) *If X is a part of an anisotropic nodoid, then X is unstable.*

(iv) *Suppose X is a part of an anisotropic unduloid. If the generating curve of \mathcal{W} satisfies the condition (15) on the part of \mathcal{W} satisfying $u^2 + \Lambda c \geq 0$, then X is unstable.*

From Theorems 3.1 and 5.1, we obtain

Corollary 5.1 *Assume that $F = F(\nu_3)$ is an even function. Assume also that the generating curve of the Wulff shape \mathcal{W} satisfies the condition (15). Then,*

(i) *Only the upper and the lower half of \mathcal{W} and sufficiently short cylinders are stable embedded capillary surfaces $(\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_1 \cup \Pi_2)$.*

(ii) *Only the upper and the lower half of \mathcal{W} and sufficiently short cylinders are stable rotationally-symmetric capillary surfaces $(\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$.*

Remark 5.1 In the CMC case, both F and κ are constant. Therefore, Corollary 5.1 (i) supplies a new proof of the uniqueness result for stable embedded CMC surfaces with free boundary on $\Pi_1 \cup \Pi_2$ proved by [1] and [13] which we mentioned in Section 1.

Remark 5.2 Although a half-period of an anisotropic nodoid from a bulge to a neck is not contained between the supporting planes, such a surface might arise physically if the supporting surfaces consists of two horizontal discs instead of planes. Theorem 5.1 (iii) implies that such a surface is unstable. We remark that in the CMC case, the stability of the nodoid for the free boundary problem was not considered in the references [1] and [13].

Remark 5.3 The assumption that $F(\nu_3)$ is an even function can be weakened to an assumption that $F'(0) = 0$ holds. Here, for simplicity, we assume the stronger condition.

In the rest of this section, we will prove Theorem 5.1. The first assertion follows from the list of anisotropic Delaunay surfaces and Lemma 4.1 (i) which were given in Section 4.

Proof of Theorem 5.1 (i). Wulff's theorem states that the Wulff shape \mathcal{W} is the absolute minimizer of \mathcal{F} among all embedded surfaces enclosing the same (usual) volume ([12]). If X admitted a volume-preserving variation which diminishes \mathcal{F} and keeps the boundary on the horizontal plane, then reflecting the deformation across the plane would give an energy decreasing deformation of \mathcal{W} which contradicts its minimizing property.

Now let $X : (\Sigma, \partial\Sigma) \rightarrow (\mathbf{R}^3, \Pi_1 \cup \Pi_2)$ be a rotationally symmetric capillary surface with vertical rotation axis which satisfies both of $X(\partial\Sigma) \cap \Pi_1 \neq \emptyset$ and $X(\partial\Sigma) \cap \Pi_2 \neq \emptyset$. Then, X is parametrized as

$$\begin{aligned} X(s, \theta) &= (x(s) \cos \theta, x(s) \sin \theta, z(s)), \quad 0 \leq s \leq l, \\ (x')^2 + (z')^2 &= 1, \quad 0 \leq s \leq l. \end{aligned}$$

We may assume that

$$\Pi_1 = \{z = z(0)\}, \quad \Pi_2 = \{z = z(l)\}, \quad z(0) < z(l)$$

holds.

Since X is perpendicular to $\Pi_1 \cup \Pi_2$ on the boundary, it is sufficient to consider only admissible normal variations

$$X_\epsilon = X + \epsilon \varphi \nu + \mathcal{O}(\epsilon^2)$$

of X . By some simple computation, we get

$$L[\varphi] = x^{-1}[(\mu_1^{-1} x \varphi_s)_s + (\mu_2^{-1} x^{-1} \varphi_\theta)_\theta] + Q_1 \varphi,$$

where

$$Q_1 := \langle Adv, d\nu \rangle = \frac{(h_{11})^2}{\mu_1} + \frac{(h_{22})^2}{\mu_2}. \quad (16)$$

Therefore, by Proposition 3.3, the second variation of \mathcal{F} is given by

$$\begin{aligned} \mathcal{I}[\varphi] &:= \delta^2 \mathcal{F} = - \int_0^l \int_0^{2\pi} [(\mu_1^{-1} x \varphi_s)_s + (\mu_2^{-1} x^{-1} \varphi_\theta)_\theta + Q_1 x \varphi] \varphi d\theta ds + \oint_{\partial \Sigma} \varphi \langle \mu_1^{-1} \varphi_s \frac{\partial}{\partial s} + \mu_2^{-1} x^{-2} \varphi_\theta \frac{\partial}{\partial \theta}, n \rangle d\tilde{s} \\ &= - \int_0^l \int_0^{2\pi} [(\mu_1^{-1} x \varphi_s)_s + (\mu_2^{-1} x^{-1} \varphi_\theta)_\theta + Q_1 x \varphi] \varphi d\theta ds + \int_0^{2\pi} [\mu_1^{-1} x \varphi \varphi_s]_0^l d\theta. \end{aligned} \quad (17)$$

If X_ϵ is a rotationally symmetric admissible variation of X , then X_ϵ is represented as a variation C_ϵ of the generating curve

$$C(s) = (x(s), z(s))$$

of X . Set

$$C_\epsilon = C + \epsilon \psi \tilde{\nu} + \mathcal{O}(\epsilon^2),$$

where

$$\tilde{\nu} = (z', -x').$$

From (17), the second variation of the energy is given by,

$$\mathcal{I}[\psi] = 2\pi \left(- \int_0^l \psi \tilde{L}[\psi] ds + [\psi B_1[\psi]]_0^l \right), \quad (18)$$

where

$$\tilde{L}[\psi] := (\mu_1^{-1} x \psi')' + Q_1 \cdot x \psi, \quad B_1[\psi] := \mu_1^{-1} x \psi'.$$

Proof of Theorem 5.1 (ii). Since we take the outward pointing unit normal, we have

$$h_{11} \equiv 0, \quad h_{22} \equiv -R^{-1}. \quad (19)$$

First we prove that, in order to determine the stability of X , it is sufficient to consider only rotationally symmetric variations. For a general admissible normal variation $X_\epsilon = X + \epsilon \varphi \nu + \mathcal{O}(\epsilon^2)$, we have from (16), (17), and (19),

$$\mathcal{I}[\varphi] = \int_0^h \int_0^{2\pi} \{ \mu_1^{-1} (\varphi_s)^2 + R^{-2} \mu_2^{-1} (\varphi_\theta)^2 - R^{-2} \mu_2^{-1} \varphi^2 \} R d\theta ds. \quad (20)$$

Define functions ψ and ζ as

$$\psi := (2\pi)^{-1} \int_0^{2\pi} \varphi d\theta, \quad \zeta := \varphi - \psi.$$

Note that μ_1, μ_2 and ψ are independent of θ , and ζ has a period 2π with respect to variable θ and satisfies the following equalities:

$$\int_0^{2\pi} \zeta d\theta = \int_0^{2\pi} \zeta_s d\theta = 0.$$

Using this, the second variation formula (20) gives

$$\begin{aligned} \mathcal{I}[\varphi] &= 2\pi R \int_0^h (\mu_1^{-1} \psi_s^2 - R^{-2} \mu_2^{-1} \psi^2) ds \\ &\quad + R(\mu_1(0))^{-1} \int_0^h \int_0^{2\pi} \zeta_s^2 d\theta ds + R^{-1}(\mu_2(0))^{-1} \int_0^h \int_0^{2\pi} (\zeta_\theta^2 - \zeta^2) d\theta ds. \end{aligned} \quad (21)$$

By considering the function $a \cos \theta + b \sin \theta$, one sees that

$$\inf_{f \in \mathcal{B}_0 \setminus \{0\}} \frac{\int_0^{2\pi} (f')^2 d\theta}{\int_0^{2\pi} f^2 d\theta} = 1,$$

where \mathcal{B}_0 is the space of piecewise smooth functions with period 2π and with mean value zero. From this,

$$\mathcal{I}[\varphi] \geq 2\pi R \int_0^h (\mu_1^{-1} \psi_s^2 - R^{-2} \mu_2^{-1} \psi^2) ds$$

holds, where the equality holds if and only if $\zeta(s, \theta) = a \cos \theta + b \sin \theta$ for some constants a, b . Therefore, the cylinder X is stable if and only if the second variation of the energy is nonnegative for all rotationally symmetric admissible variations.

The formula (21) with (19) and (3) gives for $\psi = \psi(s)$,

$$\mathcal{I}[\psi] = \frac{2\pi R}{\mu_1(0)} \int_0^h (\psi')^2 + \left(\frac{\Lambda \mu_1(0)}{R}\right) \psi^2 ds.$$

By considering the function $\cos(\pi s/h)$, one sees that

$$\inf_{f \in \mathcal{A}_0 \setminus \{0\}} \frac{\int_0^h (f')^2 ds}{\int_0^h f^2 ds} = (\pi/h)^2,$$

where \mathcal{A}_0 is the space of piecewise smooth functions on $[0, h]$ with mean value zero. From this the result follows.

Now we assume that X is a part of an anisotropic unduloid or an anisotropic nodoid. We will estimate the second variation of the energy for rotationally symmetric admissible variations. In view of (18), we consider two eigenvalue problems as follows: For $j = 0, 1$, set

$$\tilde{L}[\psi] = -\lambda \psi \text{ in } (0, l), \quad B_j[\psi] = 0 \text{ on } \partial[0, l], \quad (\text{EP}_j)$$

where

$$B_0[\psi] := \psi, \quad B_1[\psi] := \mu_1^{-1} x \psi'.$$

(EP₁) is the eigenvalue problem associated with the second variation of the energy for our free boundary problem, while (EP₀) is the eigenvalue problem associated with the second variation of the energy for the fixed boundary problem. Denote by $\lambda_1^j < \lambda_2^j < \dots$ the eigenvalues of (EP_j).

Note that if $\lambda_2^1 < 0$, then X is unstable. In fact, in this case, one can construct a linear combination of eigenfunctions belonging to λ_1^1, λ_2^1 which gives an admissible variation with negative second variation of the energy.

Also, note that, since (EP_j) is a Sturm-Liouville eigenvalue problem, the multiplicity of each eigenvalue is one, and the number of nodal domains of an eigenfunction belonging to λ_k^j is equal to k .

Since a translation does not change the anisotropic mean curvature, by considering a variation

$$X_\epsilon = X + \epsilon E_3, \quad E_3 = (0, 0, 1)$$

and using (11), we have

$$0 = \delta\Lambda = L[\nu_3].$$

Since ν_3 is independent of θ ,

$$\tilde{L}[x'] = \tilde{L}[-\nu_3] = -xL[\nu_3] = 0 \tag{22}$$

holds. Also note that, from Lemma 4.1 (ii), x' vanishes only at the necks and the bulges of X . We get

Lemma 5.1 *Let X be a part of an anisotropic unduloid or an anisotropic nodoid that is in equilibrium for the free boundary problem. If X includes one period, then X is unstable.*

Proof. Since X is orthogonal to horizontal planes along the boundary, each boundary component of X must be a neck or a bulge. Therefore, from the above remarks, $-x' = \nu_3$ gives an eigenfunction of the problem (EP_0) which belongs to λ_k^0 , where $k-1$ is the number of extrema of x inside of X . Therefore, if X includes one period, then λ_2^0 is non-positive. Therefore, by Lemma 6.3 (i), λ_2^1 is negative and hence X is unstable. **q.e.d.**

Next, we assume that X is an embedding onto a half-period of an anisotropic unduloid or nodoid from a bulge to a neck. We may assume that $s = 0, l$ correspond to the bulge and the neck of X , respectively.

In the case where X is an embedding onto a half-period of an anisotropic unduloid, let $s = s_1$ ($0 < s_1 < l$) correspond to the unique inflection point (cf. Lemma 4.1 (iii)) of the generating curve $(x(s), z(s))$. In the case where X is an embedding onto a half-period of an anisotropic nodoid, let $s = s_1$ ($0 < s_1 < l$) correspond to the top of the generating curve $(x(s), z(s))$, that is, $s = s_1$ is the unique point where $z'(s) = 0$ ($0 < s_1 < l$) holds.

Set

$$f(s) = x'(s) \int_{s_1}^s \frac{\mu_1}{x(x')^2} ds. \tag{23}$$

Lemma 5.2

$$\tilde{L}[f] = 0. \tag{24}$$

Proof. From (22), x' is a solution of $\tilde{L} = 0$. By general theory there exists a fundamental set of solutions $\{x', f\}$ for the second order ODE $\tilde{L} = 0$ satisfying $\mu_1^{-1}x(x'(f') - f(x')') \equiv 1$. Considering this as a first order equation for f , integration yields (23). **q.e.d.**

Lemma 5.3 (i) $f(s) > 0$ on $0 \leq s < s_1$, and $f(s) < 0$ on $s_1 < s \leq l$.

(ii) *In the case where X is an embedding onto a half-period of an anisotropic unduloid, assume that (15) holds on the part of C satisfying $u^2 + \Lambda c \geq 0$. Then*

$$f'(0) > 0, \quad f'(l) > 0 \tag{25}$$

holds.

(iii) In the case where X is an embedding of a half-period of an anisotropic nodoid, (25) holds without any extra assumption.

We will prove Lemma 5.3 at the end of this section.

Proof of Theorem 5.1 (iii). In view of Lemma 5.1, we need to prove only the instability of a half period of an anisotropic nodoid from bulge to neck. Consider the following eigenvalue problem

$$\tilde{L}[\psi] = -\lambda\psi \text{ in } (0, l), \quad B_2[\psi] = 0 \text{ on } \partial[0, l], \quad (26)$$

where

$$B_2[\psi] := \rho\psi' - (\rho f'/f)\psi, \quad \rho := \mu_1^{-1}x.$$

Denote by λ_k^2 the k th eigenvalue of (26). From (24) and Lemma 5.3 (i), we see that f is an eigenfunction of (26) belonging to the second eigenvalue λ_2^2 , and $\lambda_2^2 = 0$.

On the other hand, by Lemma 5.3,

$$(\rho f'/f)|_{s=0} > 0 \quad \text{and} \quad (\rho f'/f)|_{s=l} < 0.$$

Therefore, in view of Lemma 6.3 (ii), we see that

$$0 = \lambda_2^2 > \lambda_2^1$$

holds. This implies that X is unstable.

Proof of Theorem 5.1 (iv). The proof is achieved by a similar way to the proof of (iii). **q.e.d.**

Now we will prove Lemma 5.3.

Proof of Lemma 5.3. (i) clearly holds because $x' < 0$ ($0 < s < l$) holds.

We will prove (ii) and (iii). Note that, from (4) and $\nu_3 = -x'$, μ_i can be regarded as functions of x' . Let the Gauss map of X at s coincide with that of \mathcal{W} at $\sigma = \sigma(s)$. Then,

$$x_s = u_\sigma$$

holds. Since $-1/\mu_j$ are the reciprocals of the principal curvatures of the Wulff shape with respect to the outward unit normal, we have

$$\mu_1 = -h_{11} = -u_{\sigma\sigma}v_\sigma + u_\sigma v_{\sigma\sigma}. \quad (27)$$

First we will prove (ii). Set $\sigma_1 := \sigma(s_1)$. Then, $\sigma_1 > 0$. Because u attains its maximum value at $\sigma = 0$,

$$\sigma(0) = \sigma(l) = 0 \quad (28)$$

holds. Since, from Proposition 4.1,

$$x = \begin{cases} \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}, & 0 < s < s_1, \\ \frac{u - \sqrt{u^2 + \Lambda c}}{-\Lambda}, & s_1 < s < l \end{cases}$$

holds, we have

$$ds = \frac{ds}{d\sigma} d\sigma = x_u d\sigma = \begin{cases} \frac{x}{\sqrt{u^2 + \Lambda c}} d\sigma, & 0 < s < s_1, \\ -\frac{x}{\sqrt{u^2 + \Lambda c}} d\sigma, & s_1 < s < l. \end{cases}$$

Hence, by using (27), we obtain

$$f(s) = \begin{cases} u_\sigma \int_{\sigma_1}^{\sigma} \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma, & 0 < s < s_1, \\ -u_\sigma \int_{\sigma_1}^{\sigma} \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma, & s_1 < s < l, \end{cases}$$

$$f'(s) = f_\sigma \sigma_s = \frac{\sqrt{u^2 + \Lambda c}}{x} \frac{d}{d\sigma} \left[u_\sigma \int_{\sigma_1}^{\sigma} \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma \right]. \quad (29)$$

Set

$$h(\sigma) := u_\sigma \int_{\sigma_1}^{\sigma} \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma.$$

From now on, we will denote by “ ’ ” the derivative with respect to σ . In order to prove (25), from (28) and (29) with $\sigma_1 > 0$, it is sufficient to prove $h'(0) > 0$.

Since

$$v' = \sqrt{1 - (u')^2},$$

we have

$$h(\sigma) = u' \int_{\sigma_1}^{\sigma} \frac{-u''}{(u')^2 \sqrt{(1 - (u')^2)(u^2 + \Lambda c)}} d\sigma.$$

Set

$$a := \sqrt{-\Lambda c}.$$

Then

$$u(\sigma_1) = a$$

holds.

For $0 < \sigma < \sigma_1$, we have

$$\begin{aligned} h'(\sigma) &= u'' \int_{\sigma_1}^{\sigma} \frac{-u''}{(u')^2 \sqrt{(1 - (u')^2)(u^2 - a^2)}} d\sigma - \frac{u''}{u' \sqrt{(1 - (u')^2)(u^2 - a^2)}} \\ &= u'' \int_{\sigma_1}^{\sigma} \left(\frac{\sqrt{u^2 - a^2}}{u'} \right)' \frac{-u''}{\{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}} d\sigma - \frac{u''}{u' \sqrt{(1 - (u')^2)(u^2 - a^2)}} \\ &= u'' \left[\frac{-u'' \sqrt{u^2 - a^2}}{u' \{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}} \right]_{\sigma_1}^{\sigma} \\ &\quad + u'' \int_{\sigma_1}^{\sigma} \frac{\sqrt{u^2 - a^2}}{u'} \left(\frac{u''}{\{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}} \right)' d\sigma - \frac{u''}{u' \sqrt{(1 - (u')^2)(u^2 - a^2)}} \\ &=: P + Q + R. \end{aligned}$$

$$\begin{aligned}
P + R &= \frac{-(u'')^2 \sqrt{u^2 - a^2}}{u' \{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}} - \frac{u''}{u' \sqrt{(1 - (u')^2)(u^2 - a^2)}} \\
&= \frac{u'' \{-u''(u^2 - a^2) - (u(u')^2 - u''(u^2 - a^2))\}}{u' \{u(u')^2 - u''(u^2 - a^2)\} \sqrt{(1 - (u')^2)(u^2 - a^2)}} \\
&= \frac{-uu'u''}{\{u(u')^2 - u''(u^2 - a^2)\} \sqrt{(1 - (u')^2)(u^2 - a^2)}} \rightarrow 0 \quad (\text{as } \sigma \rightarrow +0). \quad (30)
\end{aligned}$$

Therefore,

$$h'(0) = u''(0) \int_{\sigma_1}^0 \frac{\sqrt{u^2 - a^2}}{u'} \left(\frac{u''}{\{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}} \right)' d\sigma. \quad (31)$$

Set

$$\varphi(\sigma) := \frac{u''}{\{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2}}.$$

Then,

$$\begin{aligned}
&\{u(u')^2 - u''(u^2 - a^2)\}^2 (1 - (u')^2) \varphi'(\sigma) \\
&= u''' \{u(u')^2 - u''(u^2 - a^2)\} \sqrt{1 - (u')^2} \\
&\quad - u'' \left[\{(u')^3 + 2uu'u'' - u'''(u^2 - a^2) - 2uu'u''\} \sqrt{1 - (u')^2} \right. \\
&\quad \left. + \{u(u')^2 - u''(u^2 - a^2)\} \frac{-u'u''}{\sqrt{1 - (u')^2}} \right] \\
&= \frac{1}{\sqrt{1 - (u')^2}} \left[(1 - (u')^2)(u(u')^2 u''' - (u')^3 u'') + u'(u'')^2 \{u(u')^2 - u''(u^2 - a^2)\} \right] \\
&= \frac{u'}{\sqrt{1 - (u')^2}} \left[u \{ (1 - (u')^2) u' u''' + (u')^2 (u'')^2 \} \right. \\
&\quad \left. - (1 - (u')^2)(u')^2 u'' - (u'')^3 (u^2 - a^2) \right]. \quad (32)
\end{aligned}$$

Set

$$(u'(\sigma), v'(\sigma)) = (\cos \vartheta(\sigma), \sin \vartheta(\sigma)).$$

Then,

$$\vartheta' = \kappa, \quad \vartheta(0) = \pi/2, \quad \vartheta(L) = \pi, \quad 0 > u'' = -\vartheta' \sin \vartheta, \quad \vartheta' \geq 0, \quad u''' = -\vartheta'' \sin \vartheta - (\vartheta')^2 \cos \vartheta$$

holds. Therefore,

$$\begin{aligned}
&\{u(u')^2 - u''(u^2 - a^2)\}^2 (1 - (u')^2)^{3/2} \varphi'(\sigma) \\
&= u' \left[-(\sin^3 \vartheta \cos \vartheta) \vartheta'' u + (\sin^3 \vartheta \cos^2 \vartheta) \vartheta' + (\sin^3 \vartheta) (\vartheta')^3 (u^2 - a^2) \right] \\
&= u' (\sin^3 \vartheta) \left\{ -\vartheta'' (\cos \vartheta) u + \vartheta' \cos^2 \vartheta + (\vartheta')^3 (u^2 - a^2) \right\}. \quad (33)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
h'(0) &= u''(0) \int_{\sigma_1}^0 \frac{\sqrt{u^2 - a^2}}{u'} \varphi'(\sigma) d\sigma \\
&= u''(0) \int_{\sigma_1}^0 \frac{\sqrt{u^2 - a^2} \{-\vartheta''(\cos \vartheta)u + \vartheta' \cos^2 \vartheta + (\vartheta')^3(u^2 - a^2)\}}{\{u(u')^2 - u''(u^2 - a^2)\}^2} d\sigma. \tag{34}
\end{aligned}$$

Recall from Proposition 4.1 that the map $\chi = DF + F\nu$ maps the unduloid onto the part of the Wulff shape satisfying $u^2 + \Lambda c \geq 0$. Hence, by assumption,

$$\vartheta'' = \kappa' \geq 0 \quad \text{on} \quad 0 < \sigma < \sigma_1 \tag{35}$$

holds. Since

$$\sigma_1 > 0, \quad \cos \vartheta \leq 0, \quad \vartheta' \geq 0, \quad u^2 - a^2 \geq 0, \quad \sin \vartheta > 0 \quad \text{on} \quad 0 < \sigma < \sigma_1,$$

from (34) and (35), we see that

$$h'(0) > 0$$

holds, which implies (25).

Next, we will prove (iii).

$$\sigma(0) = 0, \quad \sigma(l) = 2L, \tag{36}$$

$$L = \sigma(s_1),$$

$$v' = \begin{cases} \sqrt{1 - (u')^2}, & 0 < \sigma < L, \\ -\sqrt{1 - (u')^2}, & L < \sigma < 2L \end{cases}$$

holds.

From Proposition 4.1, we have

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}.$$

By a similar way to the proof of (25) for an anisotropic unduloid, we obtain

$$\begin{aligned}
f(s) &= u_\sigma \int_L^\sigma \frac{-u_{\sigma\sigma}v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma, \\
f'(s) = f_\sigma \sigma_s &= \frac{\sqrt{u^2 + \Lambda c}}{x} \frac{d}{d\sigma} \left[u_\sigma \int_L^\sigma \frac{-u_{\sigma\sigma}v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma \right]. \tag{37}
\end{aligned}$$

Set

$$h(\sigma) := u_\sigma \int_L^\sigma \frac{-u_{\sigma\sigma}v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma.$$

In order to prove (iii), from (36) and (37), it is sufficient to prove $h'(0) > 0$ and $h'(2L) > 0$.

For $0 < \sigma < L$, we have

$$h(\sigma) = u' \int_L^\sigma \frac{-u''}{(u')^2 \sqrt{(1 - (u')^2)(u^2 + \Lambda c)}} d\sigma.$$

Set

$$a := \sqrt{\Lambda c}.$$

Then,

$$\begin{aligned}
h'(\sigma) &= u'' \int_L^\sigma \frac{-u''}{(u')^2 \sqrt{(1-(u')^2)(u^2+a^2)}} d\sigma - \frac{u''}{u' \sqrt{(1-(u')^2)(u^2+a^2)}} \\
&= u'' \int_L^\sigma \left(\frac{\sqrt{1-(u')^2}}{u'} \right)' \frac{1}{\sqrt{u^2+a^2}} d\sigma - \frac{u''}{u' \sqrt{(1-(u')^2)(u^2+a^2)}} \\
&= u'' \left[\frac{\sqrt{1-(u')^2}}{u' \sqrt{u^2+a^2}} \right]_L^\sigma - u'' \int_L^\sigma \frac{\sqrt{1-(u')^2}}{u'} \left(\frac{1}{\sqrt{u^2+a^2}} \right)' d\sigma \\
&\quad - \frac{u''}{u' \sqrt{(1-(u')^2)(u^2+a^2)}} \\
&= \frac{-u'u''}{\sqrt{(1-(u')^2)(u^2+a^2)}} + u'' \int_L^\sigma \frac{u\sqrt{1-(u')^2}}{(u^2+a^2)^{3/2}} d\sigma \\
&\rightarrow u''(0) \int_L^0 \frac{u\sqrt{1-(u')^2}}{(u^2+a^2)^{3/2}} d\sigma > 0 \quad (\text{as } \sigma \rightarrow +0),
\end{aligned}$$

here we used $u''(0) < 0$.

As for $L < \sigma < 2L$, since $v' = -\sqrt{1-(u')^2}$, we have

$$h(\sigma) = -u' \int_L^\sigma \frac{-u''}{(u')^2 \sqrt{(1-(u')^2)(u^2+a^2)}} d\sigma.$$

Therefore, by a similar way to the above, we obtain

$$h'(2L) = -u''(2L) \int_L^{2L} \frac{u\sqrt{1-(u')^2}}{(u^2+a^2)^{3/2}} d\sigma > 0,$$

here we used $u''(2L) > 0$ and $u(\sigma) < 0$ for $L < \sigma \leq 2L$. **q.e.d.**

6 Appendix: Minimizing properties of eigenvalues

The theory in this section is probably well-known. It is included for the reader's convenience.

Let $\rho > 0$, Q be sufficiently smooth functions on the interval $[0, l]$, and let R be a function on $\partial[0, l]$. Set

$$J[\psi] = (\rho\psi')' + Q\psi \quad \text{in } [0, l].$$

And set, on $\partial[0, l]$,

$$B_0[\psi] := \psi, \quad B_1[\psi] := \rho\psi', \quad B_2[\psi] := \rho\psi' + R\psi,$$

$$\begin{aligned}
\mathcal{S} &:= C^2([0, l]), \\
\mathcal{S}_0 &:= \{\psi \in \mathcal{S}; \psi(0) = \psi(l) = 0\}, \quad \mathcal{S}_1 = \mathcal{S}_2 := \mathcal{S}.
\end{aligned}$$

For $\phi, \psi \in \mathcal{S}_j$ ($j = 0, 1, 2$), set

$$I_j[\psi] := - \int_0^l \psi J[\psi] ds + [\psi B_j[\psi]]_0^l,$$

$$H[\psi] := \int_0^l \psi^2 ds, \quad H[\phi, \psi] := \int_0^l \phi \psi ds.$$

For $\psi \in \mathcal{S}_j$, $I_j[\psi]$ satisfies the following:

$$I_0[\psi] = \int_0^l (\rho(\psi')^2 - Q\psi^2) ds, \quad (38)$$

$$I_1[\psi] = \int_0^l (\rho(\psi')^2 - Q\psi^2) ds, \quad (39)$$

$$I_2[\psi] = \int_0^l (\rho(\psi')^2 - Q\psi^2) ds + R(l)(\psi(l))^2 - R(0)(\psi(0))^2. \quad (40)$$

We consider eigenvalue problems

$$\begin{cases} J[\psi] = -\lambda\psi & \text{in } [0, l], \\ B_j[\psi] = 0 & \text{on } \partial[0, l]. \end{cases} \quad (41)$$

Denote by $\lambda_1^j < \lambda_2^j < \dots$ the eigenvalues of (41). Let e_i^j be an eigenfunction of (41) belonging to λ_i^j which satisfies

$$H[e_i^j] = 1, \quad H[e_i^j, e_k^j] = 0 \quad (i \neq k).$$

Then, the following Lemma holds (c.f. [3]).

Lemma 6.1

$$\lambda_i^j = I_j[e_i^j] = \min\{I_j[\psi]; \psi \in \mathcal{S}_j, H[\psi] = 1, H[\psi, e_k^j] = 0 \ (k = 1, \dots, i-1)\}$$

holds. Moreover, $I_j[\psi] = \lambda_i^j$ holds if and only if $\psi \in C^2([0, l])$ satisfies (41) for $\lambda = \lambda_i^j$.

Also, the following so-called min-max principle holds: Denote by \mathcal{V}_i^j the set of all i -dimensional subspaces of \mathcal{S}_j . Then,

Lemma 6.2

$$\lambda_i^j = \min_{V \in \mathcal{V}_i^j} \max_{\psi \in V - \{0\}} (I_j[\psi]/H[\psi]).$$

By using the above lemmas, we get the following

Lemma 6.3 For each $i \in \mathbf{N}$, the following inequalities hold.

- (i) $\lambda_i^0 > \lambda_i^1$.
- (ii) If $R(0) \leq 0$ and $R(l) \geq 0$ hold, then $\lambda_i^2 \geq \lambda_i^1$ holds, here the equality holds if and only if $R(0) = R(l) = 0$ holds.

Proof. Note that $\mathcal{S}_0 \subset \mathcal{S}_1 = \mathcal{S}_2$ holds.

- (i) From (38), (39), and Lemma 6.1, the inequality $\lambda_i^0 > \lambda_i^1$ holds.
- (ii) From (39) and (40),

$$I_1[\psi] \leq I_2[\psi]$$

holds. Hence, by Lemma 6.2, $\lambda_i^2 \geq \lambda_i^1$ holds. Moreover, by Lemma 6.1, the equality holds only for the case where $R \equiv 0$. **q.e.d.**

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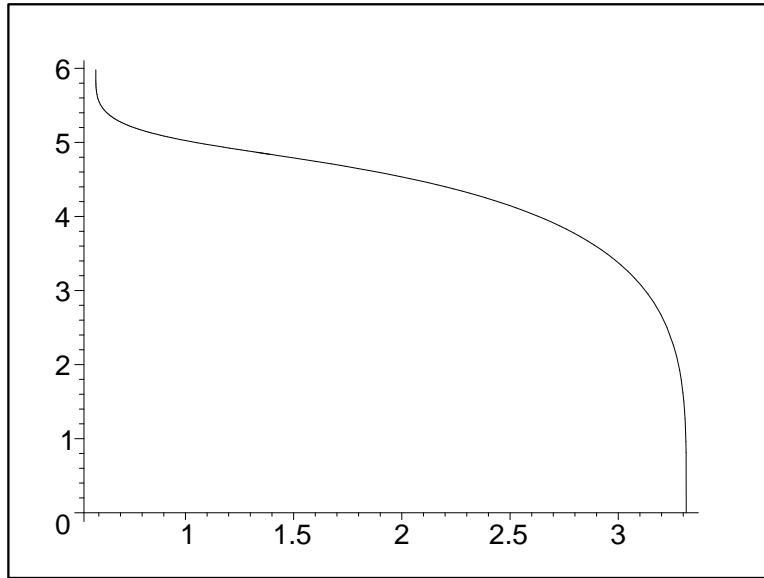


Figure 1: The profile curve of an equilibrium anisotropic unduloid The Wulff shape is $u^4 + v^4 = 1$, $\Lambda = -1/2$ and $c = 1$.

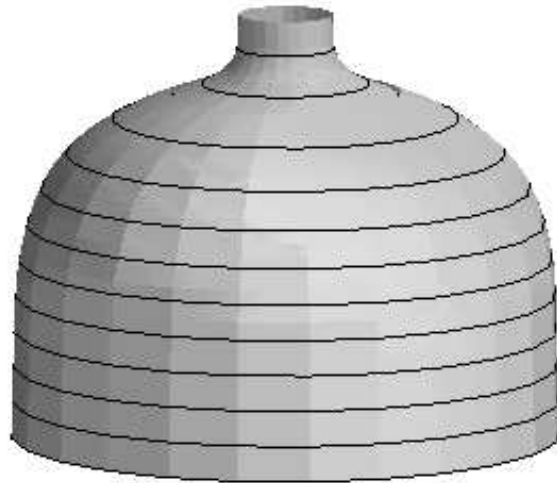


Figure 2: The corresponding anisotropic unduloid

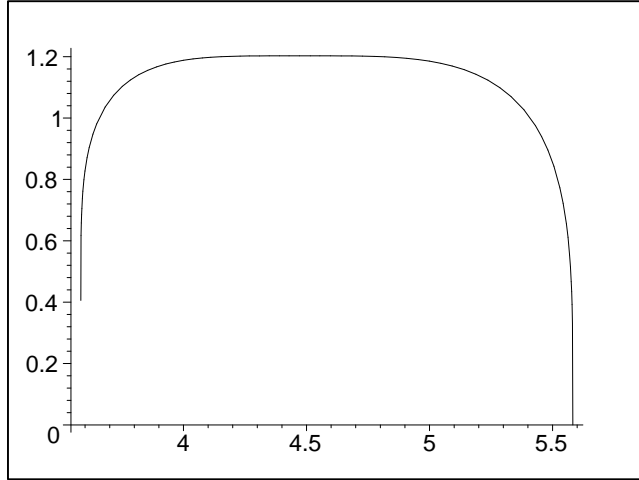


Figure 3: The profile curve of an equilibrium anisotropic nodoid. The Wulff shape is $u^4 + v^4 = 1$, $\Lambda = -1$ and $c = -20$.

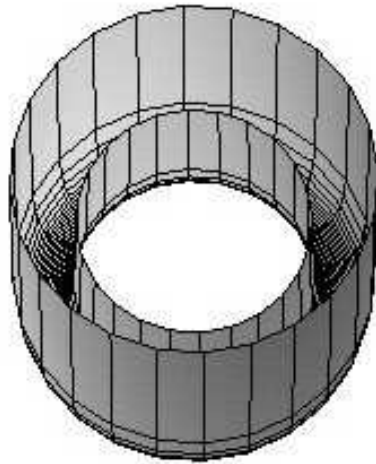


Figure 4: The corresponding anisotropic nodoid