Geometry and Stability of Bubbles with Gravity

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ABSTRACT. We study the variational theory of surfaces whose mean curvature is prescribed to be a linear function of their height above a horizontal plane (PMC surfaces). We develop a flux formula and use it to prove nonexistence results for closed PMC surfaces. The perturbation theory for PMC surfaces is studied. We obtain necessary conditions for the stability of PMC surfaces with planar boundaries. A height estimate is obtained for stable PMC graphs.

1. INTRODUCTION

It is often said that surfaces with constant mean curvature serve as geometric models for bubbles; they minimize the surface tension of a homogeneous membrane, subject to the constraint that the surface contains a fixed three dimensional volume. Of course, this is only true when other forces besides the surface tension are neglected. In particular, if the force of gravity is considered as acting on the material inside the bubble, then the mean curvature is no longer constant. Near the surface of the earth, the Euler-Lagrange equation gives that the mean curvature is a linear function of the vertical coordinate, that is

$$H = \kappa x_3 + H_0, \tag{1.1}$$

where κ and H_0 are constants. For convenience, we will refer to immersed surfaces satisfying (1.1) as prescribed mean curvature (PMC) surfaces. The equation (1.1) can be derived from the variational principle

$$\delta(A + G_\kappa) = 0, \tag{1.2}$$

where A is the area of the surface and G_κ is the gravitational potential energy:

$$G_\kappa := 2\kappa \int_{\mathcal{V}} x_3 \, dV,$$

where $\mathcal{V}$ denotes the volume inside the surface. In (1.2) we consider only variations which preserve the enclosed three dimensional volume.

The equation (1.1) has a long history beginning with the investigations of Young and Laplace early in the 19th century concerning the height of a liquid in a capillary tube. We refer the reader to the book of Finn [4] for background and historical material. However, the equation (1.1) is usually considered with a free boundary condition and/or for embedded surfaces. Here, we will be concerned with PMC surfaces mostly with fixed boundary.

The paper is organized as follows.

In the second section, we will develop the basic ideas concerning the first and the second variations for the functional $A + G_\kappa$, and we will give basic notations.

In the third section, we study a version of the flux and/or balancing formulas for PMC surfaces and apply them to derive some geometric properties of PMC surfaces: We will prove that, except for CMC surfaces, there exist no closed PMC surfaces with non zero volume (Corollary 3.5), and that the round spheres are the only compact stable PMC surfaces without boundary (Theorem 3.7). It will be proved that under certain assumptions on H and κ, for any PMC surface bounded by a circle with radius r in a horizontal plane, the absolute value of the mean curvature at any boundary point is not greater than $1/r$ (Corollary 3.9), which was known only for disc-type surfaces with constant mean curvature (Heinz [5]). Moreover, we will obtain a condition for an embedded PMC surface with planar boundary to be contained in a halfspace determined by its boundary, and a condition for an embedded PMC surface to have the same symmetry as its boundary.

In the fourth section, we apply the implicit function theorem to show that under certain conditions on the spectrum of the Jacobi operator, we can deform a given PMC surface to obtain a new one whose values of κ and H_0 are close to those of the original surface. In particular, beginning with a CMC (constant mean curvature) surface, we can produce many nearby PMC surfaces, for $\kappa \approx 0$, having the same boundary values.

Following this, in the fifth section we study the stability of PMC surfaces. The stability of embedded PMC surfaces of revolution was studied intensively in Wente [9]. We will investigate primarily embedded PMC surfaces with boundary in a horizontal plane, and derive some information on each part of the considered surface above or below that plane. Moreover, we will obtain a criterion of the stability for PMC graphs.

In the sixth section we derive a height estimate for a strongly stable PMC graph in terms of its "initial data."

Finally, in the seventh section, we give numerical results which indicate the necessity of some assumptions in the results of Section 3 and Section 6.

Much of what is carried out in this paper goes through without difficulty in higher dimensions. Because of the connections with problems in mechanics, we have chosen to concentrate on the two dimensional case.

In closing, we would like to thank the referee for the statement and proof of Proposition 5.1.

2. Preliminaries

Let Σ be a two dimensional orientable compact connected C^∞ manifold (with or without boundary). Consider a smooth immersion

$$X = (x_1, x_2, x_3) : \Sigma \to \mathbb{R}^3,$$

with Gauss map $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2$. We assign to X the following three functionals:

$$A(X) := \int_\Sigma d\Sigma,$$
$$V(X) := \int_\Sigma x_3\nu_3 \, d\Sigma,$$
$$G_\kappa(X) := \kappa \int_\Sigma x_3^2 \nu_3 \, d\Sigma,$$

where $d\Sigma$ is the area element of Σ induced by X. Then $V(X)$ represents the algebraic volume between the surface and the plane $\{x_3 = 0\}$. $G_\kappa(X)$ represents the (gravitational) potential energy, where κ is a constant which is determined from the density change across the surface, the (constant) surface tension, and the gravitational constant. When the orientation of the surface is chosen suitably, the sign $\kappa > 0$ (resp. $\kappa < 0$) is determined if the material inside the surface is denser (resp. less dense) than the surrounding air. It may be natural that we only consider surfaces for which $x_3 \geq 0$ holds, otherwise this expression for G_κ is not realistic. For generality, however, we do not make any a priori restrictions on the region containing the surface. It is also possible to consider more general potentials like

$$\tilde{G}_\kappa(X) := \kappa \int_\Sigma f(x_3) \, dV,$$

where f is a smooth function.

Consider a smooth variation $X_\varepsilon : \Sigma \to \mathbb{R}^3$ of X, with $X_\varepsilon - X \in C_0^\infty(\Sigma)$. Under such a deformation, the first variations of the above quantities are given by

$$A'(0) = -2 \int_\Sigma uH \, d\Sigma,$$
$$V'(0) = \int_\Sigma u \, d\Sigma,$$
$$G_\kappa'(0) = 2\kappa \int_\Sigma x_3 u \, d\Sigma,$$

where u is the normal component of the variation vector field of X_ε, that is,

$$u = \left\langle \frac{\partial X_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}, \nu \right\rangle,$$

where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product. If we look for critical points for the functional $A + G_\kappa$ when only volume-preserving variations are allowed, then we arrive at the necessary and sufficient condition for a critical point:

$$H = \kappa x_3 + H_0, \tag{2.1}$$

where H_0 is a constant.

Assume that a critical point has been found. The second variation of the functional $A + G_\kappa$ for volume-preserving variations is derived by a standard way (cf. Wente [9]) and we see that

$$(A + G_\kappa)''(0) = -\int_\Sigma u(\Delta u + (|d\nu|^2 - 2\kappa\nu_3)u)\, d\Sigma, \tag{2.2}$$

where Δ is the Laplacian in the metric induced by X.

The formula (2.2) is understood also in the following way: Let us postulate that the second variation is of the form:

$$(A + G_\kappa)''(0) = -\int_\Sigma u \cdot L[u]\, d\Sigma$$

and look for the form of the operator L.

Denote by $\{E_1, E_2, E_3\}$ the canonical orthonormal base in $\mathbb{R}^3$:

$$E_1 = (1,0,0), \quad E_2 = (0,1,0), \quad E_3 = (0,0,1).$$

Observe that translations in the E_1 and E_2 directions (but not the E_3 direction) are symmetries of the functional $A + G_\kappa$. Moreover, observe that rotation about a vertical axis is also a symmetry. We can infer from this that:

$$L[\nu_j] = 0, \quad j = 1, 2. \tag{2.3}$$

$$L[\psi] = 0, \quad \text{where } \psi := \langle E_3 \times X, \nu\rangle. \tag{2.4}$$

Recall that the tension field of the Gauss map can be related to the mean curvature by

$$\Delta\nu + |d\nu|^2\nu = -2\nabla H,$$

where ∇ denotes the covariant differentiation. Hence, in the present case we have

$$\Delta\nu + |d\nu|^2\nu = -2\kappa\nabla x_3 = -2\kappa(E_3 - \nu_3\nu). \tag{2.5}$$

Therefore, if L is defined by $L[u] = \Delta u + (|d\nu|^2 - 2\kappa\nu_3)u$, then (2.3) holds.

A computation also shows that, in general

$$\Delta\psi + |d\nu|^2\psi = -2\langle\nabla H, E_3 \times X\rangle.$$

In the present case, this gives

$$\Delta\psi + |d\nu|^2\psi = -2\kappa\langle E_3 - \nu_3\nu, E_3 \times X\rangle = 2\kappa\nu_3\psi,$$

so that (2.4) holds.

As for ν_3, from (2.5) we obtain the following formula:

$$L[\nu_3] = -2\kappa. \tag{2.6}$$

Definition. Let $X : \Sigma \to \mathbb{R}^3$ be a smooth immersion which satisfies the Euler-Lagrange equation (2.1). The immersion will be called *stable* if,

$$-\int_\Sigma u(\Delta u + (|d\nu|^2 - 2\kappa\nu_3)u)\, d\Sigma \geq 0 \tag{2.7}$$

holds for all $u \in C_0^\infty(\Sigma)$ such that

$$\int_\Sigma u\, d\Sigma = 0.$$

The immersion will be called *strongly stable* if (2.7) holds for all $u \in C_0^\infty(\Sigma)$.

It follows by the standard estimate of the Rayleigh quotient that X is strongly stable if and only if the first eigenvalue $\lambda_1(L, \Sigma)$ of the eigenvalue problem

$$L[u] = -\lambda u, \quad u|_{\partial\Sigma} = 0, \quad u \in H_0^1(\Sigma) - \{0\}$$

(the definition of the function space $H_0^1(\Sigma)$ will be given in Section 4) is nonnegative.

Basic Notation.

- $(\mathbb{R}^3)_c^+$: closed half space $= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq c\}$.
- $(\mathbb{R}^3)_c^- = \{x \in \mathbb{R}^3 \mid x_3 \leq c\}$.
- Π_c: horizontal plane $= \{x \in \mathbb{R}^3 \mid x_3 = c\}$.
- $\overline{W}$ = closure of the point set $W = W \cup \partial W$.
- $Du = (\partial u/\partial x_1, \partial u/\partial x_2)$ = gradient of the function $u = u(x_1, x_2)$ in a domain in $\mathbb{R}^2$.

For a PMC surface $X : \Sigma \to \mathbb{R}^3$, we will adopt the following notation:

- ∇u = the gradient of the function $u : \Sigma \to \mathbb{R}$ with respect to the Riemannian metric in Σ induced by X.

- $\nabla u \nabla v$ = the inner product of ∇u and ∇v with respect to the metric in Σ induced by X.
- $|\nabla u|^2 = \nabla u \nabla u$.
- $\lambda_i(\tilde{L}, \Sigma)$: the i-th eigenvalue of the eigenvalue problem

$$\tilde{L}[u] = -\lambda u, \quad u|_{\partial\Sigma} = 0, \quad u \in H_0^1(\Sigma) - \{0\}$$

  for a linear operator $\tilde{L} : H_0^1(\Sigma) \to L^2(\Sigma)$.
- A *region* will mean the union of a finite number of closed domains.

3. Flux Formulas and Their Applications

One of the most important tools for dealing with surfaces of constant mean curvature is the flux or balancing formula, which was first introduced for CMC surfaces in [7]. Here we will develop and apply analogous formulas in the case of prescribed mean curvature. We will then apply them to obtain some geometric results for PMC surfaces: nonexistence of compact PMC surfaces without boundary, volume estimates, estimate for the mean curvature, conditions for an embedded PMC surface with planar boundary to be contained in a halfspace determined by its boundary, and the condition for an embedded PMC surface to have the same symmetry as its boundary.

Throughout this section, we assume that $X : \Sigma \to \mathbb{R}^3$ is a compact PMC surface (with or without boundary) with mean curvature $H = \kappa x_3 + H_0$. We will denote by n the exterior normal of X along $\partial\Sigma$.

Let us represent $\partial\Sigma$ as the union of a finite number of topological circles as follows:

$$\partial\Sigma = \Gamma_1 \cup \cdots \cup \Gamma_m.$$

We attach topological discs Ω_i to Σ along Γ_i and obtain a two dimensional compact connected topological manifold

$$\hat{\Sigma} = \Sigma \cup \Big(\bigcup_{i=1}^{m} \Omega_i\Big)$$

without boundary. We set $\Omega = \bigcup_{i=1}^{m} \Omega_i$.

When we consider the special case where $X(\partial\Sigma)$ lies in the horizontal plane $\Pi_c = \{x_3 = c\}$, we always assume the following condition:

Condition P. Each restriction $X|_{\Gamma_i} : \Gamma_i \to \mathbb{R}^3$ of X to Γ_i is an embedding.

Denote by S_i the closed domain of Π_c bounded by $X(\Gamma_i)$. We can extend $X : \Sigma \to \mathbb{R}^3$ continuously to $\hat{X} : \hat{\Sigma} \to \mathbb{R}^3$ such that each $\hat{X}|_{\bar{\Omega}_i}$ is a diffeomorphism of $\bar{\Omega}_i$ onto S_i. We extend the unit normal vector field $\nu : \Sigma \to S^2$ to $\hat{\nu} : \hat{\Sigma} \to S^2$

such that the orientation of $\hat{\nu}$ is that which orients the cycle $\hat{\Sigma} = \Sigma \cup \Omega$. Then, $\hat{\nu}$ is a constant vector $(0,0,1)$ or $(0,0,-1)$ on each Ω_i. For $i \in \{1,\ldots,m\}$, set

$$\operatorname{sgn}(\nu, i) = \begin{cases} +1, & \text{if } \hat{\nu} = (0,0,-1) \text{ on } \Omega_i, \\ -1, & \text{if } \hat{\nu} = (0,0,1) \text{ on } \Omega_i. \end{cases}$$

We denote by $|S_i|$ the area of S_i. When $\partial\Sigma$ has only one component (i.e., $\partial\Sigma = \Gamma_1$), we set

$$\operatorname{sgn}(\nu) = \operatorname{sgn}(\nu, 1), \quad S = S_1.$$

First we state a flux formula for general PMC surfaces.

Theorem 3.1. *Let $X : \Sigma \to \mathbb{R}^3$ be a compact PMC surface with mean curvature $H = \kappa x_3 + H_0$. Then the following vector identity holds:*

$$\oint_{\partial\Sigma} n\, ds = \frac{1}{3}\oint_{\partial\Sigma} (2H + H_0)(X \times dX) + 2\kappa\tilde{V}(X)E_3,$$

where $\tilde{V}(X) := \frac{1}{3}\int_\Sigma \langle X, \nu\rangle\, d\Sigma$ is the algebraic volume of the cone-like region constructed by X and the origin of $\mathbb{R}^3$. In the special case where $X(\partial\Sigma) \subset \Pi_c$ and Condition P is satisfied, we see that

$$\oint_{\partial\Sigma} n\, ds = \Big(2H_0 \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| + 2\kappa V(X)\Big)E_3. \tag{3.1}$$

Proof. Let (u, v) be local coordinates in Σ. Set

$$P = \frac{1}{3}(2H + H_0)X \times X_u, \quad Q = \frac{1}{3}(2H + H_0)X \times X_v,$$

and define a vector valued 1-form ω as

$$\omega = P\, du + Q\, dv = \frac{1}{3}(2H + H_0)(X \times dX).$$

By using $H = \kappa x_3 + H_0$, we see

$$\begin{aligned} &3(Q_u - P_v) \\ &\quad = 2H_u X \times X_v + (2H + H_0)(X \times X_v)_u - 2H_v X \times X_v - (2H + H_0)(X \times X_u)_v \\ &\quad = 2\kappa(x_3)_u X \times X_v - 2\kappa(x_3)_v X \times X_u + 2(2H + H_0)X_u \times X_v, \end{aligned}$$

from which we obtain

$$d\omega = (Q_u - P_v)\, du \wedge dv = \left(2H\nu - \frac{2\kappa}{3}\langle X, \nu\rangle E_3\right) d\Sigma.$$

Denote by $\partial/\partial n$ the partial derivative with respect to n. Then, we see that

$$\begin{aligned}\oint_{\partial\Sigma} n\, ds &= \oint_{\partial\Sigma}\left(\frac{\partial X}{\partial n}\right) ds = \int_\Sigma (\Delta X)\, d\Sigma = 2\int_\Sigma H\nu\, d\Sigma \\ &= \int_\Sigma \left(d\omega + \frac{2\kappa}{3}\langle X,\nu\rangle E_3\, d\Sigma\right) \\ &= \frac{1}{3}\oint_{\partial\Sigma}(2H + H_0)(X\times dX) + 2\kappa\tilde{V}(X)E_3.\end{aligned}$$

When $X(\partial\Sigma)\subset\Pi_c$, by using the Green's formula, we see that

$$\oint_{\partial\Sigma}(X\times dX) = \Big(2\sum_{i=1}^m(\operatorname{sgn}(\nu,i))|S_i|\Big)E_3.$$

Therefore,

$$\begin{aligned}\oint_{\partial\Sigma} n\, ds &= \frac{2\kappa c + 3H_0}{3}\oint_{\partial\Sigma}(X\times dX) + 2\kappa\tilde{V}(X)E_3 \\ &= \Big(2H_0\sum_{i=1}^m(\operatorname{sgn}(\nu,i))|S_i| + 2\kappa V(X)\Big)E_3.\end{aligned}$$ □

Next, we pose a condition which is weaker than Condition P.

Condition E. $X:\Sigma\to\mathbb{R}^3$ can be extended to a continuous mapping $\tilde{X}:\hat{\Sigma}\to\mathbb{R}^3$ such that each restriction $\tilde{X}|_{\bar{\Omega}_i}:\bar{\Omega}_i\to\mathbb{R}^3$ is an immersion.

Under Condition E, we extend the unit normal vector field $\nu:\Sigma\to S^2$ to $\tilde{\nu}:\hat{\Sigma}\to S^2$ such that the orientation of $\tilde{\nu}$ orients the cycle $\hat{\Sigma}=\Sigma\cup\Omega$.

The following is a generalization of the *balancing formula* in [7].

Theorem 3.2. *Let $X:\Sigma\to\mathbb{R}^3$ be a compact PMC surface satisfying Condition E, with mean curvature $H=\kappa x_3+H_0$. Assume that H does not vanish at any point in Σ. Then, for any fixed vector v in $\mathbb{R}^3$, the following equality holds.*

$$\sum_{i=1}^m\int_{\Omega_i}\langle v,\tilde{\nu}\rangle\, d\Omega_i = \int_\Sigma \nabla\left(\frac{1}{2H}\right)\nabla\langle v,X\rangle\, d\Sigma - \oint_{\partial\Sigma}\frac{1}{2H}\langle v,n\rangle\, ds. \tag{3.2}$$

Proof. Let X_t, Y_t be the variations of X, $\tilde{X}|_\Omega$, respectively, induced by the parallel transform in the direction v. Then

$$0 = \frac{d}{dt}\Big|_{t=0}(V(X_t)+V(Y_t)) = \int_\Sigma\langle v,\nu\rangle\, d\Sigma + \int_\Omega\langle v,\tilde{\nu}\rangle\, d\Omega.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{m}\int_{\Omega_i}\langle v,\bar{\nu}\rangle\, d\Omega_i &= \int_{\Omega}\langle v,\bar{\nu}\rangle\, d\Omega = -\int_{\Sigma}\langle v,\nu\rangle\, d\Sigma\\ &= -\int_{\Sigma}\frac{1}{2H}\langle v,\Delta X\rangle\, d\Sigma = -\int_{\Sigma}\frac{1}{2H}\Delta\langle v,X\rangle\, d\Sigma\\ &= \int_{\Sigma}\nabla\left(\frac{1}{2H}\right)\nabla\langle v,X\rangle\, d\Sigma - \oint_{\partial\Sigma}\frac{1}{2H}\frac{\partial\langle v,X\rangle}{\partial n}\, ds\\ &= \int_{\Sigma}\nabla\left(\frac{1}{2H}\right)\nabla\langle v,X\rangle\, d\Sigma - \oint_{\partial\Sigma}\frac{1}{2H}\langle v,n\rangle\, ds. \qquad \square\end{aligned}$$

Corollary 3.3. *Let $X : (\Sigma,\partial\Sigma)\to(\mathbb{R}^3,\Pi_c)$ be a compact PMC surface satisfying Condition P, with mean curvature $H=\kappa x_3+H_0$. Assume that H does not vanish at any point in Σ. Then, the following formula holds.*

$$-\sum_{i=1}^{m}(\mathrm{sgn}(\nu,i))|S_i| = \frac{-1}{2\kappa}\int_{\Sigma}\frac{1}{H^2}|\nabla H|^2 d\Sigma - \frac{1}{2H_c}\oint_{\partial\Sigma}\langle E_3,n\rangle\, ds, \tag{3.3}$$

where $H_c := \kappa c + H_0$, and the first term of the right hand side does not appear if $\kappa = 0$.

Proof. By putting $v = E_3$ in (3.2), we see

$$\begin{aligned}&-\sum_{i=1}^{m}(\mathrm{sgn}(\nu,i))|S_i|\\ &= \sum_{i=1}^{m}\int_{S_i}\langle E_3,\hat{\nu}\rangle dS_i = \int_{\Sigma}\nabla\left(\frac{1}{2H}\right)\nabla x_3\, d\Sigma - \oint_{\partial\Sigma}\frac{1}{2H}\langle E_3,n\rangle\, ds\\ &= \frac{1}{2}\int_{\Sigma}\nabla\left(\frac{1}{\kappa x_3+H_0}\right)\nabla x_3\, d\Sigma - \frac{1}{2H_c}\oint_{\partial\Sigma}\langle E_3,n\rangle\, ds\\ &= \frac{1}{2\kappa}\int_{\Sigma}\nabla\left(\frac{1}{\kappa x_3+H_0}\right)\nabla(\kappa x_3+H_0)\, d\Sigma - \frac{1}{2H_c}\oint_{\partial\Sigma}\langle E_3,n\rangle\, ds\\ &= \frac{-1}{2\kappa}\int_{\Sigma}\frac{1}{(\kappa x_3+H_0)^2}|\nabla(\kappa x_3+H_0)|^2\, d\Sigma - \frac{1}{2H_c}\oint_{\partial\Sigma}\langle E_3,n\rangle\, ds. \qquad\square\end{aligned}$$

Remark 3.4. Later we will give a sufficient condition for H to have a definite sign (Corollary 5.3, Theorem 5.7).

From the flux formulas (3.1) and (3.3), we immediately get the following nonexistence theorem for compact PMC surfaces without boundary.

Corollary 3.5. *Let $X : \Sigma \to \mathbb{R}^3$ be a compact PMC surface with mean curvature $H = \kappa x_3 + H_0$, $\kappa \neq 0$. If $\partial\Sigma = \emptyset$, then both of the following* (i) *and* (ii) *are satisfied.*

(i) $V(X) = 0$.

(ii) *H changes its sign in Σ.*

Hence, in particular, there exists no compact embedded PMC surface with $\kappa \neq 0$ without boundary in $\mathbb{R}^3$.

Remark 3.6. As for the last statement of Corollary 3.5, it is obtained easily by using the Divergence Theorem.

Moreover, by using Corollary 3.5, we obtain the following result.

Theorem 3.7. *The only compact stable PMC surfaces without boundary in $\mathbb{R}^3$ are round spheres.*

Proof. Let $X : \Sigma \to \mathbb{R}^3$ be a compact stable PMC surface with mean curvature $H = \kappa x_3 + H_0$. Assume that $\partial\Sigma = \emptyset$. Since the only compact stable CMC surfaces without boundary in $\mathbb{R}^3$ are round spheres (Barbosa-do Carmo [2]; see also Wente [10]), it is sufficient for us to prove that $\kappa = 0$.

Suppose that $\kappa \neq 0$. Then, by Corollary 3.5, we see that

$$V(X) = 0.$$

Set $\varphi := \langle X, \nu \rangle$. Then, from $V(X) = 0$ and $\partial\Sigma = \emptyset$, we have

$$\int_\Sigma \varphi \, d\Sigma = 3V(X) = 0. \tag{3.4}$$

Moreover, we claim the following equality.

$$L[\varphi] := \Delta\varphi + (|d\nu|^2 - 2\kappa\nu_3)\varphi = -2H - 2\kappa x_3. \tag{3.5}$$

In fact, by using (2.5), we have

$$\begin{aligned}\Delta\varphi &= \langle \Delta X, \nu \rangle + \langle X, \Delta\nu \rangle + 2\sum_i \nabla x_i \nabla \nu_i \\ &= \langle 2H\nu, \nu \rangle + \langle X, -|d\nu|^2\nu - 2\kappa(E_3 - \nu_3\nu) \rangle - 4H \\ &= -2H - 2\kappa x_3 - |d\nu|^2\varphi + 2\kappa\nu_3\varphi.\end{aligned}$$

On the other hand, the following formula is known as a Minkovski integral formula.

$$\int_\Sigma (H\varphi + 1) \, d\Sigma = 0. \tag{3.6}$$

By the stability of X, (3.4), (3.5), and (3.6), we observe that

$$\begin{aligned}0 &\le -\int_\Sigma \varphi L[\varphi]\, d\Sigma \\ &= \int_\Sigma \varphi(2H + 2\kappa x_3)\, d\Sigma = 2\int_\Sigma \varphi(2H - H_0)\, d\Sigma \\ &= 4\int_\Sigma H\varphi\, d\Sigma - 2H_0 \int_\Sigma \varphi\, d\Sigma = -4\int_\Sigma d\Sigma < 0,\end{aligned}$$

which is a contradiction. ◻

The following volume formula is derived from flux formulas (3.1) and (3.3) by a simple calculation.

Corollary 3.8. *Suppose the same assumption as in Corollary 3.3 holds. If $\kappa \neq 0$, then the following equality holds:*

$$2V_c = -\frac{H_c}{\kappa^2}\int_\Sigma \frac{1}{H^2}|\nabla H|^2\, d\Sigma = -H_c \int_\Sigma \frac{|\nabla x_3|^2}{(\kappa x_3 + H_0)^2}\, d\Sigma,$$

where $V_c := V(X) - c\sum_{i=1}^m (\mathrm{sgn}(\nu, i))|S_i|$ is the algebraic volume of $\hat{X} : \hat{\Sigma} \to \mathbb{R}^3$.

In the rest of this section, we will investigate the case where

$$\kappa \sum_{i=1}^{m} (\mathrm{sgn}(\nu, i))|S_i| \le 0 \tag{3.7}$$

holds. In the final section, we will give numerical results which indicate that our results need not hold without the assumption (3.7).

By using the balancing formula (3.3), we observe the following estimate for the mean curvature which was known for disc-type surfaces with constant mean curvature ([5]).

Corollary 3.9. *Suppose the same assumption as in Corollary 3.3 holds. Assume further that $m = 1$, and $X(\partial\Sigma)$ is a circle with radius r. If $(\mathrm{sgn}\,\nu)\kappa \le 0$, then*

$$|H_c| \le \frac{1}{r}, \qquad H_c := \kappa c + H_0.$$

Proof. When $\kappa \neq 0$, from (3.3), we observe that

$$|H_c| = \left| \frac{-\kappa \oint_{\partial\Sigma} \langle E_3, n\rangle\, ds}{-2(\mathrm{sgn}\,\nu)\kappa|S| + \int_\Sigma \frac{1}{H^2}|\nabla H|^2\, d\Sigma} \right| \le

$$\leq \frac{\oint_{\partial\Sigma} |\langle E_3, n\rangle|\, ds}{2|S|} \leq \frac{\oint_{\partial\Sigma} 1 \cdot ds}{2\pi r^2} = \frac{1}{r}.$$

When $\kappa = 0$, again from (3.3), we see that

$$|H_c| = \left| \frac{\oint_{\partial\Sigma} \langle E_3, n\rangle\, ds}{2(\operatorname{sgn}\nu)|S|} \right| \leq \frac{1}{r}. \qquad \square$$

Next we will consider conditions for a compact PMC surface $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ to be contained in a halfspace determined by Π_c. First we state a lemma.

Lemma 3.10. *Suppose the same assumption as in Corollary 3.3 holds and $\partial\Sigma \neq \emptyset$. Assume further that $\langle E_3, n\rangle$ has a definite sign on $\partial\Sigma$, and*

$$\kappa \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| \leq 0.$$

Then, if $\kappa \neq 0$,

$$\kappa H_c \langle E_3, n\rangle < 0 \tag{3.8}$$

holds. If $\kappa = 0$, then

$$H_c \langle E_3, n\rangle \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| > 0 \tag{3.9}$$

holds.

Proof. When $\kappa \neq 0$, from (3.3), we see that

$$\oint_{\partial\Sigma} \langle E_3, n\rangle\, ds = \frac{-2H_c}{\kappa}\left(-\kappa \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| + \frac{1}{2}\int_\Sigma \frac{1}{H^2}|\nabla H|^2\, d\Sigma \right).$$

Therefore, we have

$$\begin{aligned} \frac{\kappa}{2H_c}\oint_{\partial\Sigma} \langle E_3, n\rangle\, ds &= \kappa \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| - \frac{1}{2}\int_\Sigma \frac{1}{H^2}|\nabla H|^2\, d\Sigma \\ &< \kappa \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i| < 0, \end{aligned}$$

which implies (3.8).

When $\kappa = 0$, again from (3.3),

$$\oint_{\partial\Sigma} \langle E_3, n \rangle \, ds = 2H_c \sum_{i=1}^{m} (\operatorname{sgn}(\nu, i))|S_i|,$$

which implies (3.9). ◻

The following is a generalization of Theorem 1 in [3].

Proposition 3.11. *Let $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ be a compact embedded PMC surface with mean curvature $H = \kappa x_3 + H_0$. Assume that $X(\partial\Sigma)$ is a convex curve. Assume further that H has a definite sign in Σ, $\langle E_3, n \rangle$ has a definite sign on $\partial\Sigma$, and $(\operatorname{sgn} \nu)\kappa \le 0$ holds. Then $X(\Sigma)$ is contained in either $(\mathbb{R}^3)^+_c$ or $(\mathbb{R}^3)^-_c$.*

Proof. We may assume, without loss of generality, that $H < 0$ holds on Σ. We will prove the result under the assumption that $\kappa \le 0$ holds. A similar proof works also in the case that $\kappa > 0$.

When $\kappa < 0$, from the assumption, $\operatorname{sgn} \nu = 1$ holds. When $\kappa = 0$, by taking the symmetry of the surface with respect to the plane Π_c if necessary, we may assume that $\operatorname{sgn} \nu = 1$ holds. Therefore, by Lemma 3.10,

$$\langle E_3, n \rangle < 0 \quad \text{on } \partial\Sigma,$$

and hence $X(\Sigma - \partial\Sigma)$ is contained in $\{x \in \mathbb{R}^3 \mid x_3 > c\}$ near the boundary.

It is proved that the whole $X(\Sigma - \partial\Sigma)$ is contained in $\{x \in \mathbb{R}^3 \mid x_3 > c\}$ by using the balancing formula (3.3) and the Alexandrov reflection methods. Since the proof is similar to the proof of Theorem 1 in [3], we will give only its outline.

Set $M = X(\Sigma)$ and $C = X(\partial\Sigma)$. Denote by M_1 the closure of the connected component of $M - (M \cap \Pi_c)$ which contains C. Denote by $\operatorname{Ext}(S)$ the exterior of S in Π_c, and by $\operatorname{Int}(S)$ the interior of S in Π_c; here S is the closed domain of Π_c bounded by C.

First assume that $M \cap \operatorname{Ext}(S) = \emptyset$ and $M \cap \operatorname{Int}(S) \ne \emptyset$. Then it is clear that $M \cap \{x \in \mathbb{R}^3 \mid x_3 < c\} \ne \emptyset$. For, otherwise, $H > 0$ at each point in $M \cap \operatorname{Int}(S)$, which is a contradiction. Now, denote by S_1 the subregion of S with $\partial S_1 = \partial M_1$. By applying the balancing formula (3.3) to M and M_1, in case $\kappa \ne 0$, we observe that

$$\begin{aligned} &|S_1| + \frac{-1}{2\kappa} \int_{M_1} \frac{1}{H^2} |\nabla H|^2 \, d\Sigma \\ &= \frac{1}{2H_c} \oint_{\partial M_1} \langle E_3, n \rangle \, ds > \frac{1}{2H_c} \oint_{\partial M} \langle E_3, n \rangle \, ds \\ &= |S| + \frac{-1}{2\kappa} \int_M \frac{1}{H^2} |\nabla H|^2 \, d\Sigma > |S_1| + \frac{-1}{2\kappa} \int_{M_1} \frac{1}{H^2} |\nabla H|^2 \, d\Sigma, \end{aligned}$$

which is a contradiction. Also in case $\kappa = 0$, a similar argument yields a contradiction. Therefore, if $M \cap \operatorname{Ext}(S) = \emptyset$, then $M \cap \operatorname{Int}(S) = \emptyset$.

Next we will prove that $M_1 \cap \mathrm{Ext}(S) = \emptyset$. Assume that $M_1 \cap \mathrm{Ext}(S) \neq \emptyset$. We will derive a contradiction by using the Alexandrov reflection methods. For simplicity, we assume that the straight line $\{(a, x_2, c) \mid x_2 \in \mathbb{R}\} \subset \Pi_c$ is tangent to C and

$$M_1 \cap \mathrm{Ext}(S) \cap \{(x_1, x_2, c) \mid x_1 > a\} \neq \emptyset. \tag{3.10}$$

Consider a family $P(t) = \{x \in \mathbb{R}^3 \mid x_1 = t\}$ of parallel vertical planes. Assume that

$$P(t) \cap M_1 = \emptyset \quad \text{for all } t > t_0, \tag{3.11}$$

and

$$P(t_0) \cap M_1 \neq \emptyset. \tag{3.12}$$

For $t < t_0$, we denote by $\widetilde{M}_1(t)$ the reflection of $M_1(t) := \{x \in M_1 \mid x_1 \geq t\}$ with respect to $P(t)$. Denote by $D_1(t)$ the bounded domain bounded by $M_1(t) \cup \widetilde{M}_1(t) \cup \mathrm{Ext}(S)$. Set

$$t_1 = \inf\{t \mid D_1(t) \cap M_1 = \emptyset\}.$$

We have the following two cases:

(I) $\widetilde{M}_1(t_1)$ touches M_1.
(II) $\widetilde{M}_1(t_1) \cap C \neq \emptyset$.

In the case (I), by using the Alexandrov reflection methods, we see that $P(t_1)$ is a plane of symmetry of M_1, which is a contradiction. So we assume the case (II). Let $Q \in M_1(t_1)$ be a point such that its reflection $\tilde{Q}(t_1)$ with respect to $P(t_1)$ is contained in C. Denote by ρ the connected component of $M_1 \cap \Pi_c$ which contains Q. Since the mean curvature H does not vanish at any point in M, the mean curvature vector along $C \cup \rho$ points into $\overline{D_1(t_1)} \cap \Pi_c$, in particular, along C. Therefore, from $\hat{\nu} = (0, 0, -1)$ on S, we see that $H_c > 0$, which contradicts the assumption.

Now, let us prove that $M \cap \mathrm{Ext}(S) = \emptyset$ by using a similar method to the above. Assume that $M \cap \mathrm{Ext}(S) \neq \emptyset$. We may assume that the straight line $\{(a, x_2, c) \mid x_2 \in \mathbb{R}\} \subset \Pi_c$ is tangent to C and (3.10), (3.11), (3.12) hold for M instead of M_1. Denote by $\widetilde{M}(t)$ the reflection of $M(t) := \{x \in M \mid x_1 \geq t\}$ with respect to $P(t)$, and by $D(t)$ the bounded domain bounded by $M(t) \cup \widetilde{M}(t)$. Set

$$t_2 = \inf\{t \mid D(t) \cap M = \emptyset\}.$$

Then, $\widetilde{M}(t_2)$ touches M, and, again by using the Alexandrov reflection methods, we see that $P(t_2)$ is a plane of symmetry of M, which is a contradiction.

Consequently, $M - C \subset \{x \in \mathbb{R}^3 \mid x_3 > c\}$ holds. $\square$

By using Proposition 3.11 and the Alexandrov reflection methods, we observe the following result.

Proposition 3.12. *Suppose the same assumption as in Proposition 3.11 holds. Assume further that $X(\partial\Sigma)$ is symmetric with respect to some plane P which is perpendicular to Π_c. Then, $X(\Sigma)$ is contained in either $(\mathbb{R}^3)_c^+$ or $(\mathbb{R}^3)_c^-$, and it is symmetric with respect to the plane P. In the special case where $X(\partial\Sigma)$ is a round circle, $X(\Sigma)$ is a surface of revolution contained in either $(\mathbb{R}^3)_c^+$ or $(\mathbb{R}^3)_c^-$.*

4. Deformations

In this section, we give sufficient conditions under which a PMC surface has a uniquely determined PMC deformation fixing the boundary. Our results also give a geometric example of bifurcation with two bifurcation parameters. Theorems 4.1, 4.2 below are generalizations of Theorems 1.1, 1.2 in [6], respectively.

Let α be a constant with $0 < \alpha < 1$, and let $X : \Sigma \to \mathbb{R}^3$ be a compact $C^{3+\alpha}$ immersion whose mean curvature H satisfies

$$H = \kappa_0 x_3 + h_0$$

for some constants κ_0, h_0.

Denote by $L^2(\Sigma)$ the usual Hilbert space completion of $C^\infty(\Sigma)$ with respect to the norm defined by the inner product

$$(u, v)_{L^2} = \int_\Sigma uv \, d\Sigma,$$

and denote by $H_0^1(\Sigma)$ the completion of $C_0^\infty(\Sigma)$ with respect to the norm defined by the inner product

$$(u, v)_{H^1} = \int_\Sigma (uv + \nabla u \nabla v) \, d\Sigma.$$

We define a linear operator $L : H_0^1(\Sigma) \to L^2(\Sigma)$ as

$$L[u] = \Delta u + (|d\nu|^2 - 2\kappa_0 \nu_3)u,$$

and consider the following eigenvalue problem:

$$L[u] = -\lambda u, \quad u|_{\partial\Sigma} = 0, \quad u \in H_0^1(\Sigma) - \{0\}. \tag{4.1}$$

It will be proved that the following deformation theorem holds for the case where (4.1) does not have zero as an eigenvalue.

Theorem 4.1. *Let $X \in C^{3+\alpha}(\Sigma, \mathbb{R}^3)$ be an immersion whose mean curvature H satisfies*

$$H = \kappa_0 x_3 + h_0,$$

for some constants κ_0, h_0, and $\nu : \Sigma \to S^2$ be its Gauss map. Assume that the eigenvalue problem (4.1) does not have zero as an eigenvalue. Then, there exist a neighborhood $W = W_1 \times W_2$ of (κ_0, h_0) in $\mathbb{R}^2$ and a unique C^1 mapping $\varphi : W \to C_0^{2+\alpha}(\Sigma)$ such that $\varphi(\kappa_0, h_0) = 0$ and each

$$Y := X + \varphi(\kappa, h)\nu, \quad (\kappa, h) \in W,$$

is a $C^{2+\alpha}$ immersion of Σ into $\mathbb{R}^3$ with mean curvature $\kappa y_3 + h$. Moreover, in a small neighborhood of X in $C^{2+\alpha}(\Sigma, \mathbb{R}^3)$, there exist no other PMC surfaces (modulo $C^{2+\alpha}$ diffeomorphisms of Σ) with the same boundary values as X. Furthermore, $\varphi(\kappa_0, *) : W_2 \to C_0^{2+\alpha}(\Sigma)$ is injective for any fixed κ_0, while, for each h_0, $\varphi(*, h_0) : W_1 \to C_0^{2+\alpha}(\Sigma)$ is injective if and only if x_3 is not identically zero. φ itself is injective if and only if x_3 is not constant.

When the problem (4.1) has zero eigenvalues, denote by E the eigenspace of zero eigenvalues, and by $E^\perp$ its orthogonal complement in $L^2(\Sigma)$. The following deformation theorems will be proven.

Theorem 4.2. *Let $X \in C^{3+\alpha}(\Sigma, \mathbb{R}^3)$ be an immersion whose mean curvature H satisfies*

$$H = \kappa_0 x_3 + h_0,$$

for some constants κ_0, h_0, and $\nu : \Sigma \to S^2$ be its Gauss map. Assume that the dimension of E is one, and that $\int_\Sigma e \, d\Sigma$ does not vanish for any eigenfunction e in E. Then, there exist a neighborhood $\Omega = \Omega_1 \times \Omega_2$ of $(0, \kappa_0)$ in $E \times \mathbb{R}$ and a unique C^1 mapping $\psi = (v, h) : \Omega \to (C_0^{2+\alpha} \cap E^\perp) \times \mathbb{R}$ such that $\psi(0, \kappa_0) = (0, h_0)$ and each

$$Y := X + (u + v(u, \kappa))\nu, \quad (u, \kappa) \in \Omega,$$

is a $C^{2+\alpha}$ immersion of Σ into $\mathbb{R}^3$ with mean curvature $\kappa y_3 + h(u, \kappa)$. Moreover, in a small neighborhood of X in $C^{2+\alpha}(\Sigma, \mathbb{R}^3)$, there exist no other PMC surfaces (modulo $C^{2+\alpha}$ diffeomorphisms of Σ) with the same boundary values as X.

Remark 4.3. When 0 is the first eigenvalue of (4.1), the assumption in Theorem 4.2 is satisfied.

Remark 4.4. The function $h(*, \kappa_0) : \Omega_1 \to \mathbb{R}$ in Theorem 4.2 is not necessarily injective. In fact, each hemisphere gives an example such that $h(*, \kappa_0)$ is not injective. This is seen from Corollary 3.9.

Theorem 4.5. *Let $X \in C^{3+\alpha}(\Sigma, \mathbb{R}^3)$ be an immersion whose mean curvature H satisfies*

$$H = \kappa_0 x_3 + h_0,$$

for some constants κ_0, h_0, *and* $\nu : \Sigma \to S^2$ *be its Gauss map. Assume that the dimension of* E *is one, and that* $\int_\Sigma x_3 e \, d\Sigma$ *does not vanish for any eigenfunction* e *in* E. *Then, there exist a neighborhood* $U = U_1 \times U_2$ *of* $(0, h_0)$ *in* $E \times \mathbb{R}$ *and a unique* C^1 *mapping* $\psi = (v, \kappa) : U \to (C_0^{2+\alpha} \cap E^\perp) \times \mathbb{R}$ *such that* $\psi(0, h_0) = (0, \kappa_0)$ *and each*

$$Y := X + (u + v(u,h))\nu, \quad (u,h) \in U,$$

is a $C^{2+\alpha}$ *immersion of* Σ *into* $\mathbb{R}^3$ *with mean curvature* $\kappa(u,h)y_3 + h$. *Moreover, in a small neighborhood of* X *in* $C^{2+\alpha}(\Sigma, \mathbb{R}^3)$, *there exist no other PMC surfaces (modulo* $C^{2+\alpha}$ *diffeomorphisms of* Σ) *with the same boundary values as* X.

Remark 4.6. When $x_3 \geq 0$, x_3 is not identically zero, and 0 is the first eigenvalue, the assumption in Theorem 4.5 is satisfied.

Remark 4.7. If $(A + G_\kappa)'' > 0$ for all nontrivial volume-preserving variation (i.e., $-\int_\Sigma uL[u] \, d\Sigma > 0$ for all $u \in C_0^\infty(\Sigma) - \{0\}$ such that $\int_\Sigma u \, d\Sigma = 0$), then, by a standard method, we see that the second eigenvalue of (4.1) is positive. Therefore, in this case, by virtue of Theorems 4.1, 4.2, and 4.5, X has a PMC deformation fixing the boundary.

Before proving Theorems 4.1, 4.2 and 4.5, we state the following lemma, which is verified by using the Riesz-Schauder alternative theorem and the regularity theorem for solutions of strictly elliptic partial differential equations.

Lemma 4.8. *Let* λ *be a real number.*

(a) *Assume that* λ *is not an eigenvalue of* (4.1). *Then, for any function* $f \in L^2(\Sigma)$, *the equation*

$$\lambda u - L[u] = f$$

has a uniquely determined solution $u \in H_0^1(\Sigma)$. *Moreover, if* $f \in C^\alpha(\Sigma)$, *then the solution* u *is in* $C_0^{2+\alpha}(\Sigma)$.

(b) *Assume that* λ *is an eigenvalue of* (4.1). *Then, for each function* $f \in L^2(\Sigma)$, *the equation*

$$\lambda u - L[u] = f$$

has a solution $u \in H_0^1(\Sigma)$ *if and only if*

$$\int_\Sigma f e \, d\Sigma = 0,$$

for all eigenfunctions e *belonging to* λ. *Moreover, if* $f \in C^\alpha(\Sigma)$, *then the solution* u *is in* $C_0^{2+\alpha}(\Sigma)$.

Now let us begin to prove Theorems 4.1, 4.2, and 4.5.

Choose a neighborhood V of 0 in $C_0^{2+\alpha}(\Sigma)$ so that, for any $u \in V$, $X + u\nu : \Sigma \to \mathbb{R}^3$ is an immersion. For each $u \in V$, denote by H_u and ν_u the mean curvature and the Gauss map of $X + u\nu$, respectively.

Define a mapping $\Phi : V \times \mathbb{R}^2 \to C^\alpha(\Sigma)$ by

$$\Phi(u, c, \mu) = 2((c + \kappa_0)(x_3 + u\nu_3) + \mu + h_0 - H_u).$$

Then $\Phi(0, 0, 0) = 0$. An immersion $Y := X + u\nu$ is a PMC surface (i.e., $H_u = \kappa y_3 + h$ for some constants κ, h) if and only if

$$\Phi(u, c, \mu) = 0$$

for some c, $\mu \in \mathbb{R}$.

$\Phi(u, c, \mu)$ is Fréchet differentiable, and

$$D_{(u,c,\mu)}\Phi(0, 0, 0)(u, c, \mu) = 2(cx_3 + \mu) - L[u]$$

for $(u, c, \mu) \in C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2$. Set

$$F := D_{(u,c,\mu)}\Phi(0, 0, 0) : C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2 \to C^\alpha(\Sigma).$$

Then,

$$\operatorname{Ker} F = \{(u, c, \mu) \in C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2 \mid L[u] = 2(cx_3 + \mu)\}.$$

Proof of Theorem 4.1. We will apply the implicit mapping theorem to Φ. By virtue of Lemma 4.8 (a), for each $f \in C^\alpha(\Sigma)$, there exists a unique function $u_f \in C_0^{2+\alpha}(\Sigma)$ such that $L[u_f] = -f$. Define a map $P : C^\alpha(\Sigma) \to C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2$ as

$$P(f) = (u_f, 0, 0).$$

Then, P is linear and

$$F \circ P = id_{C^\alpha(\Sigma)}.$$

Let us consider $\operatorname{Ker} F$ and $P(C^\alpha(\Sigma))$. Again by Lemma 4.8 (a), for each $(c, \mu) \in \mathbb{R}^2$, there exists a unique function $u(c, \mu) \in C_0^{2+\alpha}(\Sigma)$ such that

$$L[u(c, \mu)] = 2(cx_3 + \mu).$$

Therefore,

$$\operatorname{Ker} F = \{(u(c, \mu), c, \mu) \mid (c, \mu) \in \mathbb{R}^2\},$$

which is diffeomorphic to $\mathbb{R}^2$ by the map: $(u(c, \mu), c, \mu) \mapsto (c, \mu)$. On the other hand, by definition,

$$P(C^\alpha(\Sigma)) = C_0^{2+\alpha}(\Sigma) \times \{0\} \times \{0\}.$$

Now let us apply the implicit mapping theorem to Φ. First we observe that

$$C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2 = \operatorname{Ker} F \oplus P(C^\alpha(\Sigma)),$$

where $\oplus$ denotes the direct sum. By regarding Φ as a mapping

$$\Phi : \operatorname{Ker} F \oplus P(C^\alpha(\Sigma)) \to C^\alpha(\Sigma),$$

the equation $\Phi(\xi_1, \xi_2) = 0$ ($\xi_1 \in \operatorname{Ker} F$, $\xi_2 \in P(C^\alpha(\Sigma))$) is solvable with respect to ξ_2 in a neighborhood of $(\xi_1, \xi_2) = (0, 0)$, which implies the existence of the mapping φ stated in Theorem 4.1.

The statement on the nonexistence follows from the fact that any immersion Y in a small neighborhood of X in $C^{2+\alpha}(\Sigma, \mathbb{R}^3)$ with $Y|_{\partial\Sigma} = X|_{\partial\Sigma}$ is represented as $Y \circ \tau = X + u\nu$, where $\tau : \Sigma \to \Sigma$ is a $C^{2+\alpha}$ diffeomorphism of Σ and $u \in C_0^{2+\alpha}(\Sigma)$.

In order to get the condition for injectivity of φ, let us assume that

$$\varphi(\kappa_1, h_1) = \varphi(\kappa_2, h_2).$$

Set

$$Y := X + \varphi(\kappa_1, h_1)\nu = X + \varphi(\kappa_2, h_2)\nu.$$

Then,

$$\kappa_1 y_3 + h_1 = H_{\varphi(\kappa_1, h_1)} = H_{\varphi(\kappa_2, h_2)} = \kappa_2 y_3 + h_2.$$

Therefore,

$$(\kappa_1 - \kappa_2) y_3 + (h_1 - h_2) = 0. \tag{4.2}$$

Let us divide the situation into two cases:

(I) x_3 is not identically constant.

(II) x_3 is identically constant.

In case (I), by taking a small neighborhood of X, we may assume that y_3 is not identically constant. Therefore, from (4.2), we see that $\kappa_1 = \kappa_2$ and $h_1 = h_2$, which implies the injectivity of φ. In case (II), let $x_3 \equiv a$. Then we see that $\varphi(\kappa, -a\kappa) = 0$ for all $\kappa \in \mathbb{R}$.

As for the injectivity of $\varphi(\kappa_0, *)$ and $\varphi(*, h_0)$, we can prove the desired results by similar ways to the above. $\square$

Proof of Theorem 4.2. Let us choose an eigenfunction $e_0 \in E$ such as

$$\int_\Sigma {e_0}^2 \, d\Sigma = 1.$$

Since $\dim E = 1$, E is represented as

$$E = \{a e_0 \mid a \in \mathbb{R}\}.$$

For any $c \in \mathbb{R}$, set

$$\mu(c) = -\frac{c \displaystyle\int_\Sigma x_3 e_0 \, d\Sigma}{\displaystyle\int_\Sigma e_0 \, d\Sigma}.$$

Then, for each $c \in \mathbb{R}$, there exists a unique function $v_c \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ such that

$$L[v_c] = 2(cx_3 + \mu(c)).$$

In fact, in view of Lemma 4.8 (b), there exists a function $u \in C_0^{2+\alpha}(\Sigma)$ such that

$$L[u] = 2(cx_3 + \mu(c)).$$

Set

$$v = u - \Big(\int_\Sigma u e_0 \, d\Sigma\Big) e_0.$$

Then, $v \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ and

$$L[v] = 2(cx_3 + \mu(c)). \tag{4.3}$$

The uniqueness of $v \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ satisfying (4.3) is clear.

So we get the following expression of $\operatorname{Ker} F$.

$$\operatorname{Ker} F = \{(ae_0 + v_c, c, \mu(c)) \mid (a, c) \in \mathbb{R}^2\}.$$

Let us define a mapping $P : C^\alpha(\Sigma) \to C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2$. For each $f \in C^\alpha(\Sigma)$, set

$$\mu_f = \frac{\displaystyle\int_\Sigma f e_0 \, d\Sigma}{2\displaystyle\int_\Sigma e_0 \, d\Sigma}.$$

Then, by a similar way to the above, we see that there exists a unique function $u_f \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ such that

$$L[u_f] = 2\mu_f - f.$$

Set

$$P(f) = (u_f, 0, \mu_f).$$

Then, $P : C^\alpha(\Sigma) \to C_0^{2+\alpha}(\Sigma) \times \mathbb{R}^2$ is linear, and

$$F \circ P = id_{C^\alpha(\Sigma)}.$$

We claim

$$P(C^\alpha(\Sigma)) = (E^\perp \cap C_0^{2+\alpha}(\Sigma)) \times \{0\} \times \mathbb{R}.$$

In fact, for any $u \in E^\perp \cap C_0^{2+\alpha}(\Sigma)$ and $\mu \in \mathbb{R}$, by setting $f = 2\mu - L[u]$, we get $P(f) = (u, 0, \mu)$.

Therefore, by applying the implicit mapping theorem to Φ, the equation

$$\Phi(\xi_1, \xi_2) = 0, \quad \xi_1 \in \operatorname{Ker} F, \ \xi_2 \in P(C^\alpha(\Sigma)),$$

is solvable with respect to ξ_2 near $(0, 0)$. Hence, we obtain the desired result by a similar way to the proof of Theorem 4.1. □

Proof of Theorem 4.5. The proof can be achieved by a similar way to the proof of Theorem 4.2. □

5. Stability

We will investigate to what extent a stable PMC surface with boundary in a plane must lie on one side of the plane. Consider a cylinder of height c and radius R. If we consider the top of the cylinder as a PMC surface with $\nu_3\kappa < 0$ as in Example 5.2, then we see that if R is sufficiently large, the disc $S(R)$ is unstable. If we assume that a minimizer for the functional $A + G_\kappa$ exists with the volume fixed at $\pi R^2 c$, then clearly this surface cannot lie on one side of Π_c. On the other hand, if R is sufficiently small, then $\lambda_1(L, S(R)) > 0$. In this case, $S(R)$ has a uniquely determined PMC deformation fixing the boundary (Theorems 4.1, 4.2, and 4.5), which is constructed by a two-parameter family of PMC surfaces (one-parameter family if κ is fixed). From Theorem 5.7 and Proposition 5.9 below, we see that each of them must lie on one side of Π_c.

We have already mentioned this subject in Section 3, where we treated only the case where the PMC surface is assumed to be embedded and intersects with the plane containing its boundary transversally. In this section, we turn our attention to the stability of the surface and consider mainly embedded stable PMC surfaces. Since we will treat only embedded surfaces, we will adopt an intuitively clearer notation on the orientation of the considered surface than Section 3 when it is more convenient. However, we should always note that our variational problem is divided essentially into three cases. For embedded surfaces bounded by a Jordan curve in a horizontal plane, these three cases are represented as follows: $(\operatorname{sgn}\nu)\kappa < 0$, $(\operatorname{sgn}\nu)\kappa > 0$, and $(\operatorname{sgn}\nu)\kappa = 0$ in the notation defined in Section 3.

First we will investigate the stability of PMC graphs. After that, we will consider general compact embedded PMC surfaces with boundary in a horizontal plane. Let Π_c denote the horizontal plane $\{x_3 = c\}$ as in Section 3.

Define the energy of an immersed surface by

$$E(X) = A(X) + G_\kappa(X) + 2H_0V(X).$$

The following proposition and its proof were supplied by the referee. It is based on a calibration type argument of Schwarz, which can be used to show that a minimal graph over a convex domain is absolutely area minimizing with respect to its boundary.

Proposition 5.1. *Let $X : \Omega \to \mathbb{R}^3$, $(x, y) \to (x, y, u(x, y))$ be a compact PMC graph with $H = \kappa x_3 + H_0$ defined on a horizontal region $\Omega \subset \mathbb{R}^2$. Assume that $\nu_3\kappa \geq 0$ holds. Let Ω_1 be an abstract oriented surface such that there exists a smooth homeomorphism $\varphi : \partial\Omega_1 \to \partial\Omega$, and let $Y : \Omega_1 \to \mathbb{R}^3$ be any other immersion such that*

- *$X \circ \varphi \equiv Y$ on $\partial\Omega_1$,*
- *$Y(\Omega_1) \subset \Omega \times \mathbb{R}$,*

- *the immersion obtained by glueing X to Y along the boundaries of their respective domains via φ gives the oriented boundary of an oriented 3-chain U.*

Then,

$$E(X) \le E(Y), \tag{5.1}$$

and consequently X is strongly stable.

Proof. If the surface is given as a graph $X(x,y) = (x,y,u(x,y))$, $(x,y) \in \Omega$, then the upward pointing normal is $(1+|Du|^2)^{-1/2}(-Du,1)$. We may assume that $\nu(x,y) = (1+|Du|^2)^{-1/2}(-Du,1)$, without loss of generality. In this case $\nu_3 \ge 0$ holds, and so $\kappa \ge 0$ holds. We define a field $\tilde{\nu}(x,y,z) = \nu(x,y)$ on the cylinder $\Omega \times \mathbb{R}$. Note that on this cylinder,

$$\mathrm{Div}_{\mathbb{R}^3} \tilde{\nu} = -\mathrm{Div}_{\mathbb{R}^2} Du = -2H_\Sigma = -2(\kappa u + H_0).$$

Therefore, the field

$$\vec{F} := \tilde{\nu} + (\kappa x_3^2 + 2H_0 x_3)E_3$$

satisfies

$$\mathrm{Div}_{\mathbb{R}^3} \vec{F} = 2\kappa(x_3 - u),$$

on the cylinder.

Let Y be another immersion as in the statement of the proposition. We let N be the unit normal field along Y consistent with its orientation. We have by the Divergence Theorem,

$$\begin{aligned}\int_U 2\kappa(x_3-u)\,dU &= \int_\Omega \langle \vec{F},\nu\rangle\,d\Sigma_X - \int_{\Omega_1} \langle \vec{F},N\rangle\,d\Sigma_Y \\ &= E(X) - \left(\int_{\Omega_1} \langle \tilde{\nu},N\rangle\,d\Sigma_Y + G\kappa(Y) + 2H_0V(Y)\right) \\ &\ge E(X) - E(Y),\end{aligned}$$

by the Cauchy-Schwarz inequality. Here $dU = +dx_1\,dx_2\,dx_3$ on each positively oriented component of the chain U, i.e., on each component for which ν points out of the component, while $dU = -dx_1\,dx_2\,dx_3$ on each negatively oriented component. Because $Y(\Omega_1)$ is contained in the cylinder, we have that on the positively oriented components $u(x,y) \ge x_3$ holds, and for the negatively oriented components, we have $u(x,y) \le x_3$. Therefore, the integral on the left above is nonpositive for $\kappa \ge 0$ and (5.1) holds.

For $w \in C_0^\infty(\Omega)$, consider the variation $X_\varepsilon = X + \varepsilon w \nu$. It can be checked that

$$\partial^2_{\varepsilon\varepsilon} E(X_\varepsilon)_{\varepsilon=0} = -\int_\Omega wL[w]\,d\Sigma,$$

and so the energy minimizing property implies strong stability. ∎

Example 5.2. When $\nu_3\kappa < 0$ holds, the result is no longer true. In fact, a graph with $\nu_3\kappa < 0$ may not even be stable in the usual sense that the second variation is non negative for volume preserving variations which vanish on the boundary. We give the following simple but important example.

Let S be a relatively compact domain in the plane $x_3 = c$. We can consider S as a PMC surface with $H = \kappa c + H_0$, where κ is an arbitrary constant and $H_0 := -\kappa c$. Choose the unit normal $\nu = (0, 0, 1)$. It is then clear that S will be unstable if for some function $\zeta \in C_0^\infty(S)$ with

$$\int_S \zeta \, d^2x = 0,$$

we have

$$\int_S |D\zeta|^2 + 2\kappa\zeta^2 \, d^2x < 0,$$

where $d^2x := dx_1 \wedge dx_2$. This will hold, for example, if $\kappa < 0$ and $\lambda_2(\Delta, S) < -2\kappa$ hold. In particular, if S contains a disc of radius $R > \sqrt{14.682/(-2\kappa)}$, a well known monotonicity principle for eigenvalues implies that S is unstable (cf. [1]).

Corollary 5.3. *Let $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ be a compact PMC graph with mean curvature $H = \kappa x_3 + H_0$. Assume that $X(\Sigma)$ is not a planar region. Then the following* (I) *and* (II) *hold, where we write Σ instead of $X(\Sigma)$.*

I. *Suppose that $\nu_3\kappa \geq 0$ holds. Then,*

(i) *X is strongly stable,*
(ii) *H has a definite sign on Σ, and Σ is contained in either $(\mathbb{R}^3)_c^+$ or $(\mathbb{R}^3)_c^-$.*

II. *Suppose that $\nu_3\kappa < 0$ holds. Then, the following* (i) *and* (ii) *hold.*

(i) *If X is strongly stable, then H has a definite sign on Σ, and Σ is contained in either $(\mathbb{R}^3)_c^+$ or $(\mathbb{R}^3)_c^-$.*
(ii) *Suppose that X is stable and not strongly stable. If neither of $\Sigma \cap (\mathbb{R}^3)_c^\pm$ is empty, then either*
   (a) *$\lambda_1(L, \Sigma \cap (\mathbb{R}^3)_c^+) \geq 0$, $\lambda_1(L, \Sigma \cap (\mathbb{R}^3)_c^-) < 0$, $H_c < 0$, and $0 \neq |V_c^+| < |V_c^-|$ hold, or*
   (b) *$\lambda_1(L, \Sigma \cap (\mathbb{R}^3)_c^-) \geq 0$, $\lambda_1(L, \Sigma \cap (\mathbb{R}^3)_c^+) < 0$, $H_c > 0$, and $0 \neq |V_c^-| < |V_c^+|$ hold,*

   where $|V_c^\pm|$ is the usual volume of the compact region bounded by $(\Sigma \cap (\mathbb{R}^3)_c^\pm) \cup \Pi_c$.

Remark 5.4. Cases (I), (II) of Corollary 5.3 are exactly the cases $(\operatorname{sgn}\nu)\kappa \geq 0$, $(\operatorname{sgn}\nu)\kappa < 0$, respectively.

Corollary 5.3 I(i) follows from Proposition 5.1 above. The other parts of Corollary 5.3 follow from Theorem 5.7 below, which is a result for general compact embedded PMC surfaces with boundary in a horizontal plane.

Moreover, on the stability in case II of Corollary 5.3, we will obtain the following:

Theorem 5.5. *Let* $X : \Sigma \to \mathbb{R}^3$ *be a PMC graph defined on a compact region* $S \subset \Pi_c$*. Suppose that the mean curvature* H_Σ *of* X *satisfies*

$$H_\Sigma = \kappa x_3 + H_0$$

on Σ *and consider* S *as a PMC surface with mean curvature*

$$0 = H_S = \kappa c + (H_S)_0 := \kappa c + (-\kappa c).$$

Then, if S *is unstable,* X *is also unstable. Similarly, if* S *is not strongly stable, neither is* X*.*

Remark 5.6. In Theorem 5.5 we do not assume that Σ lies on one side of Π_c or that the graph has its boundary in Π_c. In fact, the parameter c plays no part in the result, the instability of S is characterized by the inequality (5.2) below.

Proof of Theorem 5.5. Let us choose the unit normal ν of any graph such that $\nu_3 > 0$. Assume that S is not stable. Then, κ must be negative by Proposition 5.1, and we can find $f \in C_0^\infty(S)$ with

$$\int_S f\, d^2x = 0$$

and

$$\int_S |Df|^2 + 2\kappa f^2\, d^2x < 0. \tag{5.2}$$

Let Σ be the graph of a function w. Then $\nu_3 = (1 + |Dw|^2)^{-1/2}$, $d\Sigma = \nu_3^{-1} d^2x$, and so

$$0 = \int_S f\, d^2x = \int_S f\nu_3(\nu_3)^{-1} d^2x = \int_\Sigma f\nu_3\, d\Sigma.$$

However,

$$\begin{aligned} -\int_\Sigma f\nu_3 L[f\nu_3]\, d\Sigma &= \int_\Sigma \nu_3^2 |\nabla f|^2\, d\Sigma - \int_\Sigma f^2\nu_3 L[\nu_3]\, d\Sigma \\ &= \int_\Sigma \nu_3^2 |\nabla f|^2\, d\Sigma + \int_\Sigma 2\kappa f^2 \nu_3\, d\Sigma \\ &= \int_S \nu_3 |\nabla f|^2 + 2\kappa f^2\, d^2x \\ &\le \int_S |Df|^2 + 2\kappa f^2\, d^2x < 0 \end{aligned}$$

since $|Df|^2 = |\nabla f|^2 + \langle Df, (\nu_1, \nu_2)\rangle^2 \ge |\nabla f|^2 \ge \nu_3 |\nabla f|^2$. ◻

Now let us consider a more general case. We will consider a compact smooth embedded PMC surface $X : \Sigma \to \mathbb{R}^3$, with mean curvature $H = \kappa x_3 + H_0$. We will denote $X(\Sigma)$ also by Σ.

When $\partial\Sigma \subset \Pi_c$ but Σ is not a planar region, we adopt the following notation: Set

$$\Sigma - \Pi_c = \Sigma_1 \cup \cdots \cup \Sigma_k,$$

where each Σ_i is a connected component of $\Sigma - \Pi_c$. Let us denote by S_i the (not necessarily connected) region in Π_c such that $\partial S_i = \partial\Sigma_i$ and that $\Sigma_i \cup S_i$ forms a two dimensional compact connected topological manifold without boundary. Denote by U_i the bounded domain in $\mathbb{R}^3$ bounded by $\Sigma_i \cup S_i$. Set

$$\operatorname{sgn}\Sigma_i = \begin{cases} +1, & \text{if } \Sigma_i \subset (\mathbb{R}^3)^+_c, \\ -1, & \text{if } \Sigma_i \subset (\mathbb{R}^3)^-_c. \end{cases}$$

Extend the unit normal vector field $\nu|_{\Sigma_i} : \Sigma_i \to S^2$ to $\nu_i : \Sigma_i \cup S_i \to S^2$ in the natural manner, and set

$$\operatorname{sgn}\nu_i = \begin{cases} +1, & \text{if } \nu_i|_{S_i} = (0,0,-1), \\ -1, & \text{if } \nu_i|_{S_i} = (0,0,1). \end{cases}$$

By renumbering Σ_i if necessary, we can set the following:

$$\Sigma^+_c = \bigcup\{\overline{\Sigma_i} \mid \operatorname{sgn}\Sigma_i \cdot \operatorname{sgn}\nu_i = 1\} = \bigcup_{i=1}^{\ell} \overline{\Sigma_i},$$

$$\Sigma^-_c = \bigcup\{\overline{\Sigma_i} \mid \operatorname{sgn}\Sigma_i \cdot \operatorname{sgn}\nu_i = -1\} = \bigcup_{i=\ell+1}^{k} \overline{\Sigma_i}.$$

See Figure 5.1.

It is possible that one of $\Sigma^\pm_c$ is empty. Set

$$U^+_c = \bigcup_{i=1}^{\ell} U_i, \quad U^-_c = \bigcup_{i=\ell+1}^{k} U_i.$$

We will denote by $|U^\pm_c|$ the volume of $U^\pm_c$ in the usual Lebesgue measure.

Theorem 5.7. *Let $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ be a compact embedded PMC surface with mean curvature $H = \kappa x_3 + H_0$. Assume that $X(\Sigma)$ is not a planar region. Then the following* (i) *and* (ii) *hold.*

(i) *If X is strongly stable, then H has a definite sign on Σ, and one of $\Sigma^\pm_c$ is empty.*

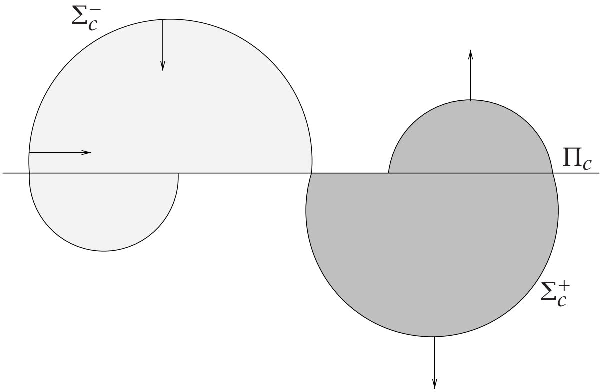

FIGURE 5.1.

(ii) *Suppose that X is stable and not strongly stable. If neither of $\Sigma_c^\pm$ is empty, then either*
 (a) *$\lambda_1(L, \Sigma_c^+) \geq 0$, $\lambda_1(L, \Sigma_c^-) < 0$, $H_c < 0$, Σ_c^- is connected and contained in one of $(\mathbb{R}^3)_c^\pm$, and $0 \neq |U_c^+| < |U_c^-|$ hold, or*
 (b) *$\lambda_1(L, \Sigma_c^-) \geq 0$, $\lambda_1(L, \Sigma_c^+) < 0$, $H_c > 0$, Σ_c^+ is connected and contained in one of $(\mathbb{R}^3)_c^\pm$, and $0 \neq |U_c^-| < |U_c^+|$ hold.*

Before we prove Theorem 5.7, we make a few observations.

Lemma 5.8. *Let Σ be a compact embedded PMC surface with mean curvature $H = \kappa x_3 + H_0$. Assume that $\partial\Sigma \subset \Pi_c$ and Σ is not a planar region. Then, the following hold.*
 (i) *If $\pm x_3$ attains a local maximum at $p = (p_1, p_2, p_3) \in \{\pm(x_3 - c) > 0\} \cap \Sigma_c^\pm$, then $\pm H(p) \leq 0$ holds, that is, $\pm(\kappa p_3 + H_0) \leq 0$ holds.*
 (ii) *If $\pm x_3$ attains a local minimum at $q = (q_1, q_2, q_3) \in \{\pm(x_3 - c) < 0\} \cap \Sigma_c^\pm$, then $\pm H(q) \leq 0$ holds, that is, $\pm(\kappa q_3 + H_0) \leq 0$.*

Proof. By assumption, $\nu_3(p) = 1$ and $\nu_3(q) = -1$. By using $\Delta x_3 \leq 0$ (resp. $\Delta x_3 \geq 0$) at the local interior maximum (resp. minimum) value of x_3 and $\Delta x_3 = 2H\nu_3$, we arrive at the desired results. □

Proposition 5.9. *Let Σ be a compact embedded PMC surface with mean curvature $H = \kappa x_3 + H_0$. Assume that $\partial\Sigma \subset \Pi_c$ and Σ is not a planar region. Then, the following* (i)–(iii) *hold.*
 (i) *If $\kappa > 0$, then at least one of $\Sigma_c^+ \cap \{x_3 > c\}$ and $\Sigma_c^- \cap \{x_3 < c\}$ is empty. Moreover, if neither of $\Sigma_c^\pm$ is empty, then the following inequalities hold.*

$$\max\{x_3 \mid x \in \Sigma_c^+\} \leq \max\{x_3 \mid x \in \Sigma_c^-\},$$
$$\min\{x_3 \mid x \in \Sigma_c^+\} \leq \min\{x_3 \mid x \in \Sigma_c^-\}.$$

(ii) *If $\kappa < 0$, then at least one of $\Sigma_c^+ \cap \{x_3 < c\}$ and $\Sigma_c^- \cap \{x_3 > c\}$ is empty. Moreover, if neither of $\Sigma_c^\pm$ is empty, then the following inequalities hold.*

$$\max\{x_3 \mid x \in \Sigma_c^+\} \geq \max\{x_3 \mid x \in \Sigma_c^-\},$$
$$\min\{x_3 \mid x \in \Sigma_c^+\} \geq \min\{x_3 \mid x \in \Sigma_c^-\}.$$

(iii) *If $\kappa = 0$, then $\Sigma_c^+ = \varnothing$ or $\Sigma_c^- = \varnothing$.*

Proof. Assume that neither of $\Sigma_c^+ \cap \{x_3 > c\}$ and $\Sigma_c^- \cap \{x_3 < c\}$ is empty. Then, by Lemma 5.8, we see that $\kappa p_3^1 + H_0 \leq 0 \leq \kappa p_3^4 + H_0$. Therefore, if $\kappa > 0$, then we get $p_3^1 \leq p_3^4$. On the other hand, $p_3^1 > c$ and $p_3^4 < c$, which is a contradiction. So we have proved the first half of (i). The second half of (i) is also observed easily by using Lemma 5.8.

The proof of (ii) is similar to the proof of (i).

Note that H is not identically zero by assumption. In the case where $\kappa = 0$, by using Lemma 5.8, the condition that neither of $\Sigma_c^\pm$ is empty leads to $H = H_0 = 0$, which is a contradiction. □

Proof of Theorem 5.7. Let $u := x_3 - c$. Using that

$$\int_{\Sigma_c^\pm} 2Hu\nu_3\, d\Sigma = \int_{\Sigma_c^\pm} u\Delta u\, d\Sigma = -\int_{\Sigma_c^\pm} |\nabla u|^2\, d\Sigma$$
$$= -\int_{\Sigma_c^\pm} \nu_3^2 |\nabla u|^2\, d\Sigma - \int_{\Sigma_c^\pm} (1-\nu_3^2)|\nabla u|^2\, d\Sigma,$$

we have

$$-\int_{\Sigma_c^\pm} u\nu_3 L[u\nu_3]\, d\Sigma = \int_{\Sigma_c^\pm} \nu_3^2|\nabla u|^2 + 2\kappa\nu_3 u^2\, d\Sigma \tag{5.3a}$$
$$= -2H_c \int_{\Sigma_c^\pm} \nu_3 u\, d\Sigma - \int_{\Sigma_c^\pm} (1-\nu_3^2)|\nabla u|^2\, d\Sigma \tag{5.3b}$$
$$= \mp 2H_c |U_c^\pm| - \int_{\Sigma_c^\pm} (1-\nu_3^2)|\nabla u|^2\, d\Sigma. \tag{5.3c}$$

We see from (5.3c), that

$$\lambda_1(L, \Sigma_c^+) \geq 0 \implies H_c < 0, \tag{5.4}$$
$$\lambda_1(L, \Sigma_c^-) \geq 0 \implies H_c > 0. \tag{5.5}$$

Next, set

$$\bar{u} = \begin{cases} u, & \text{in } \Sigma_c^+, \\ 0, & \text{in } \Sigma_c^-, \end{cases} \qquad \underline{u} = \begin{cases} 0, & \text{in } \Sigma_c^+, \\ u, & \text{in } \Sigma_c^-. \end{cases}$$

Assume that neither of $\Sigma_c^{\pm}$ is empty. Define a constant σ by

$$0 = \sigma \int_\Sigma \bar{u} \nu_3 \, d\Sigma + \int_\Sigma \underline{u} \nu_3 \, d\Sigma = \sigma |U_c^+| - |U_c^-|.$$

Note that

$$\sigma = |U_c^-| / |U_c^+|. \tag{5.6}$$

For $f \in C_0^\infty(\Sigma)$, we have, with $f = \sigma \bar{u} + \underline{u}$, by using (5.3c) and (5.6),

$$\begin{aligned} & -\int_\Sigma f \nu_3 L[f \nu_3] \, d\Sigma \\ &= -\sigma^2 \int_{\Sigma_c^+} u \nu_3 L[u \nu_3] \, d\Sigma - \int_{\Sigma_c^-} u \nu_3 L[u \nu_3] \, d\Sigma \\ &= 2H_c(1-\sigma)|U_c^-| - \sigma^2 \int_{\Sigma_c^+} (1-\nu_3^2)|\nabla u|^2 \, d\Sigma - \int_{\Sigma_c^-} (1-\nu_3^2)|\nabla u|^2 \, d\Sigma. \end{aligned} \tag{5.7}$$

If $\lambda_1(L, \Sigma_c^{\pm}) < 0$ holds, then by a standard argument, the surface is unstable. From (5.4) and (5.5), both of $\lambda_1(L, \Sigma_c^{\pm}) \geq 0$ cannot hold. Returning to (5.3c), we see that if $H_c > 0$ (resp. $H_c < 0$) holds, then multiple components of Σ_c^+ (resp. Σ_c^-) would yield disjoint subdomains on which the first eigenvalue of L is negative. These observations combined with (5.7) and Lemma 5.8 yield the desired result. □

Theorem 5.7, combined with flux formula (3.1), gives an upper bound for the volume above (or below) Π_c of a certain class of strongly stable PMC surfaces, depending only on the length of the boundary curve and the constant κ.

Corollary 5.10. *Let $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ be a compact embedded PMC surface with $H = \kappa x_3 + H_0$. Assume that Σ is strongly stable and contained in one of $(\mathbb{R}^3)_c^{\pm}$. Assume further that $(\operatorname{sgn} \nu)\kappa < 0$ holds. Then, the following inequality holds:*

$$\frac{|\partial\Sigma|}{2|\kappa|} > |V_c|,$$

where $|\partial\Sigma|$ is the length of $X(\partial\Sigma)$, and $V_c := V(X) - c\sum_i |S_i|$, that is, $|V_c|$ is the usual volume of the compact region bounded by $\Sigma \cup \Pi_c$.

Proof. It is sufficient to prove the result for the case where Σ is not a planar region. We will only prove it on the case $\Sigma \subset (\mathbb{R}^3)_c^+$. The proof for the case $\Sigma \subset (\mathbb{R}^3)_c^-$ is similar.

We will write $V = V(X)$. We may assume that $\operatorname{sgn} \nu_i = 1$ holds for all i, without loss of generality. Then, $\kappa < 0$ holds. By using the flux formula (3.1), we observe that

$$\oint_{\partial\Sigma} \langle E_3, n \rangle \, ds = 2H_0 \sum_i |S_i| + 2\kappa V = 2(c\kappa + H_0) \sum_i |S_i| + 2\kappa V_c.$$

Note that $c\kappa + H_0 < 0$ by virtue of Theorem 5.7 (i). Since $\langle E_3, n\rangle \le 0$ on $\partial\Sigma$ and $V_c > 0$ in the present case, we see that

$$\oint_{\partial\Sigma} |\langle E_3, n\rangle|\, ds = -\oint_{\partial\Sigma} \langle E_3, n\rangle\, ds = 2|c\kappa + H_0| \sum_i |S_i| + 2|\kappa|\, |V_c|.$$

Consequently, we have

$$0 < |V_c| < \frac{\oint_{\partial\Sigma} |\langle E_3, n\rangle|\, ds}{2|\kappa|} \le \frac{|\partial\Sigma|}{2|\kappa|}. \qquad \square$$

6. Height Estimate

We will consider a smooth, embedded PMC surface

$$X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c),$$

with mean curvature $H = \kappa x_3 + H_0$. For any graph, we will choose the unit normal ν such that $\nu_3 > 0$. We assume that $\kappa < 0$ holds.

Theorem 6.1. (i) *Let $X : (\Sigma, \partial\Sigma) \to (\mathbb{R}^3, \Pi_c)$ be a compact embedded PMC surface. Assume that Σ_c^+ is a strongly stable graph above the plane Π_c. Then the maximum height u_M over the plane Π_c satisfies*

$$\frac{|\partial\Sigma_c^+| - 2|H_c|\,|S_c^+|}{-2\kappa} > \frac{2\pi}{3} \cdot \frac{|H_c| u_M^3}{-2\kappa u_M + |H_c|}, \tag{6.1}$$

where $H_c := \kappa c + H_0$.

(ii) *If Σ_c^- is a strongly stable graph below the plane Π_c, then,* (6.1) *holds, where now u_M is interpreted as the maximum depth below the plane Π_c, and $\partial\Sigma_c^+$, S_c^+ are replaced by $\partial\Sigma_c^-$, S_c^-, respectively.*

Remark 6.2. The proof is similar to the derivation of the height estimate for CMC surfaces (cf. Sperb [8, Chap.11]).

We have seen above, that a PMC graph with $\kappa < 0$ need not be stable (Example 5.2).

In the following section, we will give numerical results which indicate that the height estimate needs not hold without the hypothesis of strong stability.

Since $|\partial S_c^+| = |\partial\Sigma_c^+|$, the height estimate depends only on the "initial data," i.e., on S_c^+ and the parameters κ, H_0 and c. For CMC surfaces, one can drop the stronger of the two hypotheses that the surface is a graph and obtain, from (6.6) below, that if Σ_c^+ is strongly stable, then

$$|V_c^+| \ge \frac{2\pi}{3} u_M^3.$$

By considering a hemisphere, one can see that this is sharp.

Proof of Theorem 6.1. We will only prove (i). The proof of (ii) is similar.

If $\lambda_1(L, \Sigma_c^+) \geq 0$, we can use $u := x_3 - c$ in the second variation formula to obtain

$$0 \leq -\int_{\Sigma_c^+} uL[u]\, d\Sigma = -2(\kappa c + H_0)|V_c^+| - \int_{\Sigma_c^+} |d\nu|^2 u^2\, d\Sigma. \tag{6.2}$$

Note that

$$|d\nu|^2 \geq 2H^2 \geq 2|\kappa c + H_0|^2$$

holds, and that the first equality can only hold if the surface is totally umbilic, in which case $\kappa = 0$ holds. The second inequality above uses that $\kappa c + H_0 < 0$ holds when $\lambda_1(L, \Sigma_c^+) \geq 0$ holds (Theorem 5.7 or Corollary 5.3).

We estimate, using the coarea formula,

$$\int_{\Sigma_c^+} |d\nu|^2 u^2\, d\Sigma \geq 2(\kappa c + H_0)^2 \int_{\Sigma_c^+} u^2\, d\Sigma \tag{6.3a}$$

$$= 2(\kappa c + H_0)^2 \int_0^{u_M} t^2 \int_{u=t} \frac{1}{|\nabla u|}\, ds_t\, dt, \tag{6.3b}$$

where ds_t is the line element on the level set $u = t$. By Hölder's inequality and the isoperimetric inequality, there holds

$$4\pi S_t \leq \left(\int_{u=t} ds_t\right)^2 \leq \int_{u=t} \frac{1}{|\nabla u|}\, ds_t \cdot \int_{u=t} |\nabla u|\, ds_t, \tag{6.4}$$

where S_t is the area inside the curve $u = t$ in the plane Π_{c+t}. By the flux formula (3.1),

$$\begin{aligned} \int_{u=t} |\nabla u|\, ds_t &= -\int_{u=t} \nabla u \cdot n\, ds_t \\ &= -\int_{u=t} \langle E_3, n\rangle\, ds_t = -2\kappa V_t - 2(\kappa(t+c) + H_0)S_t, \end{aligned} \tag{6.5}$$

where V_t is the volume over the plane Π_{c+t}. Under the condition that Σ_c^+ is strongly stable, we recall that $\kappa c + H_0 < 0$ holds. Under the assumption that Σ_c^+ is a graph, we have

$$V_t := \int_{u\geq t} u\nu_3\, d\Sigma \leq u_M \int_{u\geq t} \nu_3\, d\Sigma = u_M S_t.$$

We therefore obtain from (6.4) and (6.5),

$$\frac{2\pi}{-2\kappa u_M - H_c} \leq \int_{u=t} \frac{1}{|\nabla u|}\, ds_t.$$

Using this and integrating in (6.3b) yields from (6.2)

$$|V_c^+| \geq \frac{2\pi}{3} \cdot \frac{|H_c| u_M^3}{-2\kappa u_M - H_c}. \tag{6.6}$$

Finally from flux formula (3.1), we see that

$$-\oint_{\partial\Sigma_c^+} \langle E_3, n\rangle \, ds = -2\kappa |V_c^+| - 2(\kappa c + H_0)|S_c^+|,$$

so that we obtain $|\partial\Sigma_c^+| \geq -2\kappa|V_c^+| - 2H_c|S_c^+|$, which combined with (6.6) yields the result. □

7. Concluding Remarks and Examples

We will consider an example of a certain PMC surface of revolution which will indicate the necessity of several assumptions of Corollary 3.9, Lemma 3.10, Propositions 3.11, 3.12, and Theorem 6.1. We wish to emphasize that the results of this section are numerical in nature.

Figure 7.1 shows the generating curve C_0 of a rotationally symmetric PMC surface whose mean curvature satisfies:

$$H = \kappa x_3 + H_0, \quad \kappa = -1, \; H_0 = 0.4808.$$

Figure 7.2 gives a part of C_0.

We denote by C the curve in Figure 7.2, and by Σ the surface generated by C. The curve C has a vertical tangent at

$$P = (p_1, p_3) \approx (0.62, 2.45).$$

Also C appears to have a singularity at

$$Q = (q_1, q_3) \approx (0, 1.9).$$

We remark that the unit normal ν along the surface is chosen in such a way that it points out from the rotation axis in Figure 7.2, that is,

$$\nu = (1, 0)$$

at the vertical tangent P.

Now let us divide Σ into two parts

$$\Sigma_1 := \Sigma \cap \{x_3 \geq q_3\}, \quad \Sigma_2 := \Sigma \cap \{x_3 \leq q_3\}.$$

First we consider Σ_1 and observe the necessity of the assumption

$$(\operatorname{sgn} \nu)\kappa \leq 0 \tag{7.1}$$

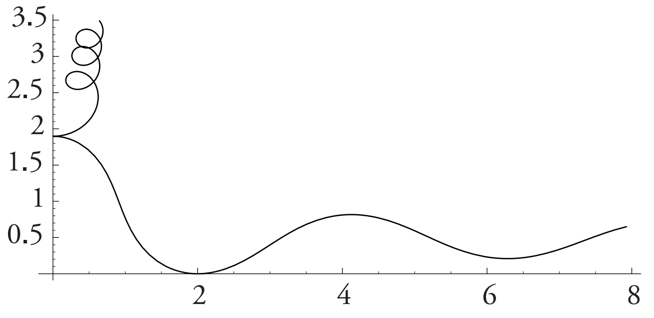

FIGURE 7.1.

in Corollary 3.9, Lemma 3.10, and Propositions 3.11, 3.12.

Note that $H < 0$ holds at any point in Σ_1.

In order to examine Corollary 3.9, we consider the part

$$\Sigma_1' := \Sigma_1 \cap \{x_3 \leq p_3\}$$

of Σ_1. Then,

$$\operatorname{sgn} \nu = -1, \quad (\operatorname{sgn} \nu)\kappa = 1 > 0$$

holds. We observe

$$c = p_3 \approx 2.45, \quad H_c \approx -2.45 + 0.48 = -1.97.$$

On the other hand,

$$r \approx 0.62, \quad r^{-1} \approx 1.61 < 1.97 \approx |H_c|.$$

Therefore, the conclusion of Corollary 3.9 does not hold.

Next, we consider the whole of Σ_1. In this case also we have

$$\operatorname{sgn} \nu = -1, \quad (\operatorname{sgn} \nu)\kappa = 1 > 0.$$

On the other hand,

$$\langle E_3, n \rangle < 0, \quad H(\operatorname{sgn} \nu) > 0$$

holds on the boundary, which implies that the conclusion of Lemma 3.10 does not hold.

Moreover, Σ_1 is not contained in either of the half spaces determined by its boundary. Therefore, the conclusions of Propositions 3.11, 3.12 do not hold.

On the other hand, consider the surface generated by $C_0 \cap \{x_3 \leq q_3\}$ and its subregions bounded by a horizontal circle. Then, the inequality (7.1) is satisfied.

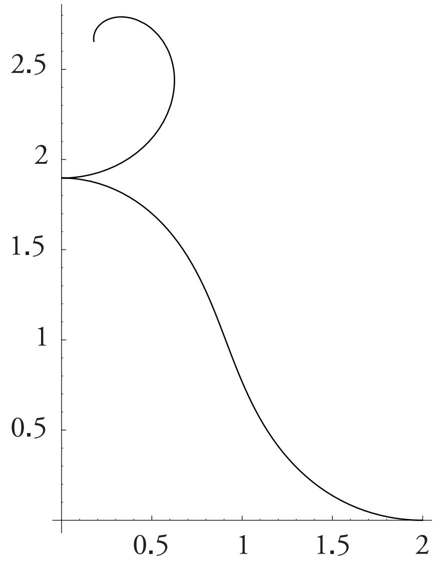

FIGURE 7.2.

By examining appropriate subregions, we see the necessity of the condition that H has a definite sign in Lemma 3.10, and Propositions 3.11, 3.12.

Next we consider Σ_2 and see the necessity of the condition that the considered surface is strongly stable in Theorem 6.1.

Note that $\lambda_1 < 0$ holds. In fact, since the normal is vertical on the boundary, the functions v_j, $j = 1, 2$ are Jacobi fields which vanish on the boundary and change sign in the interior.

We calculate the left and right hand sides of the height estimate (6.1) in Theorem 6.1. Here we use

$$c = 0,\ |\partial\Sigma_c^+| = 4\pi,\ |H_c| = 0.4808,\ |S_c^+| = 4\pi,\ \kappa = -1,\ u_M \approx 1.9.$$

Then, the value of the left hand side of (6.1) is approximately 0.2413. On the other hand, the value of the right hand side of (6.1) is approximately 1.6135. This indicates that the height estimate (6.1) is not necessarily satisfied by a PMC graph which is not strongly stable.

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