Area minimization among marginally trapped surfaces

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The most important property of zero mean curvature surfaces in Euclidean space is that they locally minimize area with respect to their boundary curves.

The following theorem is proved using the method of calibrations.

Theorem

(H. A. Schwarz, $\approx$ 1860)
Let $\Sigma \subset \mathbf{R}^3$ be a minimal surface which can be represented as a graph over a convex domain. Then

$$\mathrm{Area}[\Sigma] \leq \mathrm{Area}[S] \; ,$$

for any sufficiently smooth surface with $\partial\Sigma = \partial S$.

Space-like immersion:

$$X : \Sigma^n \to M_1^4 = \text{Lorentzian Manifold}\,.$$

Variation:

$$X_\epsilon = X + \epsilon(\delta X) + \ldots$$

The first variation formula defines the mean curvature vector $\vec{H} \in \Gamma(\perp X)$:

$$\delta A[X] = -n \int_\Sigma \vec{H} \cdot \delta X \, d\Sigma + \text{boundary terms}$$

Any space-like hypersurface $M^3 \to M_1^4$ with $\vec{H} \equiv 0$ in M_1^4 locally maximizes area with respect to its boundary.

We now consider a space-like surface with $\vec{H} \equiv 0$

$$\Sigma^2 \to \mathbf{R}^4_1 := (\mathbf{R}^4, dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2)$$

. Simplest example

$$D_r = \{(x_1, x_2, 0, 0) |\, x_1^2 + x_2^2 \leq r^2\} \,.$$

For $f \in C_c^\infty(D)$ with $|\nabla f| < 1$ in D, define

$$\Sigma_1 = \{(x_1, x_2, f(x_1, x_2), 0) |\, x_1^2 + x_2^2 \leq r^2\} \subset \mathbf{R}^3 \subset \mathbf{R}^4_1 \,,$$

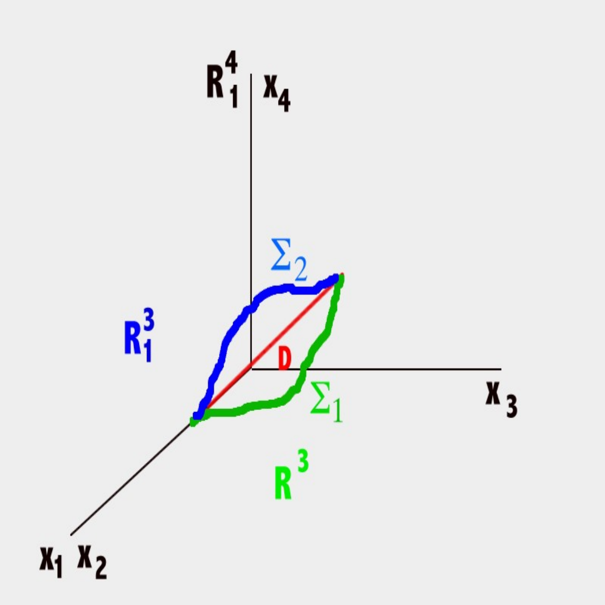
$$\Sigma_2 = \{(x_1, x_2, 0, f(x_1, x_2)) |\, x_1^2 + x_2^2 \leq r^2\} \subset \mathbf{R}^3_1 \subset \mathbf{R}^4_1 \,.$$

We have

$$\int_{D_r} \sqrt{1 - |\nabla f|^2}\, d^2x \leq \pi r^2 \leq \int_{D_r} \sqrt{1 + |\nabla f|^2}\, d^2x \,,$$

and so

$$\mathrm{Area}(\Sigma_2) \leq \mathrm{Area}(D_r) \leq \mathrm{Area}(\Sigma_1) \,,$$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Since D is a space-like zero mean curvature surface in $\mathbf{R}^4_1$, this shows that such surfaces fail to either minimize or maximize area in a rather dramatic way. This behavior is typical of all space-like zero mean curvature surface in any Lorentzian four-manifold.

Space-like zero mean curvature surface in Lorentzian manifolds have important applications in General Relativity where they are used to study singularities of space-time. Since their variational definition, (first variation of area equals zero), is important in these application, one would expect that there should be a good variational approach to studying these surfaces.

Space-like zero mean curvature surfaces in $\mathbf{R}^4_1$ can all be described by a Weierstrass representation using holomorphic data (recent work of Changping Wang, Peng Wang and Xiang Ma):

$$X = \Re \int (\phi + \psi, -i(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi)\, dh\,,$$

where ϕ, ψ are holomorphic functions and dh is a holomorphic 1-form.

Let $\Sigma \to \mathbf{R}^4_1$ be a spacelike immersion. At each point $p \in \Sigma$, the induced metric on the normal bundle $\perp_p \Sigma$ is Lorentzian.

Definition: Let $\mathcal{L}^4$ be a Lorentzian manifold. A space-like surface $\Sigma \to \mathcal{L}$ is called *marginally trapped* if its mean curvature vector $\vec{H}$ is light-like, $\vec{H} \cdot \vec{H} \equiv 0$.

Theorem

(Alías, P. 1997) Let $X : \Sigma \to \mathcal{L}$ *be a space-like zero mean curvature surface in a Lorentzian manifold which satisfies the Null Convergence Condition (Ricci*$(N, N) \geq 0$ *for all null vectors N). Then each point p in* Σ *has a neighborhood U on which the second variation of area is non negative when restricted to variations through marginally trapped surface which have the same boundary values as* $X|_U$ *to first order.*

An arbitrary relatively compact neighborhood of a point in Σ has finite Morse index when restricted to variations through marginally trapped surface which have the same boundary values as $X|_U$ to first order.

We will call a space-like surface $Y : \Sigma \to \mathbf{R}^4_1$ a spherical graph if there exists a null section of the normal bundle $\perp Y$ of the form $\xi = (\nu, 1)$ such that $\nu : \Sigma \to S^2$ is injective.

If Y is a spherical graph and if $\nu(\Sigma) =: \Omega$, we will say that Y is a spherical graph over Ω.

If a surface is a space-like spherical graph over Ω, then the surface can be parameterized by $\nu^{-1} =: Y$ and we write the surface as $Y : \Omega \to \mathbf{R}^4_1$.

If $Y : \Omega \to \mathbf{R}^4_1$ is a spherical graph over Ω and the mean curvature $\vec{H}_Y$ of Y satisfies $\vec{H}_Y \cdot (\nu, 1) \equiv 0$, then we will call Y a *marginally trapped spherical graph over* Ω.

Theorem

Consider a space-like zero mean curvature surface which can be represented as a spherical graph $X : \Omega \to \mathbf{R}^4_1$*. Let* $Y : \Omega \to \mathbf{R}^4_1$ *be any marginally trapped spherical graph over* Ω *with* $Y|_{\partial\Omega} \equiv X|_{\partial\Omega}$*. Then,*

$$\mathrm{Area}[X] \leq \mathrm{Area}[Y] \,, \tag{1}$$

holds.

Remark: If $\Sigma \to \mathbf{R}^4_1$ is a space-like surface and $K(p) \neq 0$, then near p the surface can be represented as a spherical graph.

Lemma

Let $Y : \Sigma \to \mathbf{R}^4_1$ be a space-like surface. Assume that its mean curvature vector satisfies $\vec{H} \cdot (\nu, 1) \equiv 0$, with $|\nu| \equiv 1$ and $(\nu, 1) \in \Gamma(\perp Y)$, Then the map $\nu : \Sigma \to S^2$ is weakly conformal, i.e.

$$d\nu \cdot d\nu = h\, ds^2_Y \,,$$

for some function $h \geq 0$.

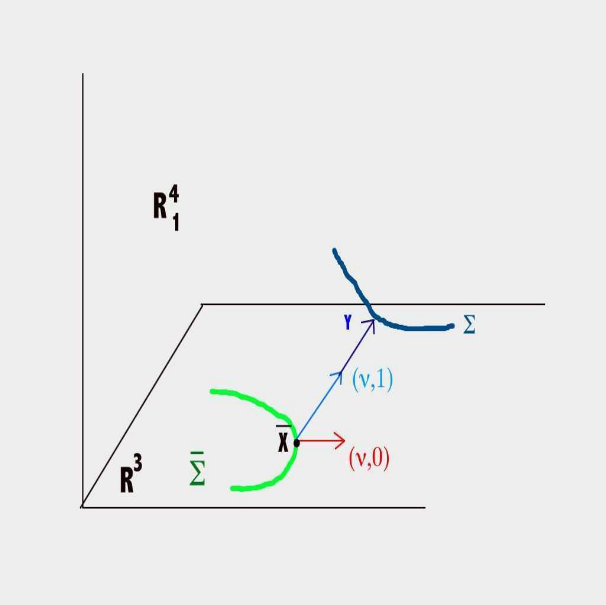
We denote the Laplacian on S^2 by $\hat{\Delta}$. The gradient of a function u on S^2 will be denoted Du.

Proposition

Let Y be a marginally trapped spherical graph over $\Omega \subset S^2$. Let $\xi = (\nu, 1)$ and define $f := Y \cdot \xi$. Then,

$$Y = (Df - (\frac{1}{2}\hat{\Delta} f)\nu \, , \, -\frac{1}{2}[\hat{\Delta} f + 2f]) \, . \tag{2}$$

Conversely, let f be any smooth function on a domain $\Omega \subset S^2$ and define $Y = Y_f : \Omega \to \mathbf{R}^4_1$ by (2). Then, Y is marginally trapped. Consequently, if g is any smooth function on Ω such that f and g agree to order $k+2$ on $\partial\Omega$, then Y_g is a marginally trapped spherical graph over Ω which agrees with Y_f to order k on $\partial\Omega$.



$$Y = \vec{X} + \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)(\nu, 1)\,.$$

Corollary

Y has zero mean curvature only if

$$\hat{\Delta}(\hat{\Delta} f + 2f) = 0 \qquad (3)$$

Lemma

Let Y be a marginally trapped spherical graph. Then the surface area is given by

$$\mathrm{Area}[Y] = \frac{1}{4}\int_{\Omega} \hat{\Delta}[f](\hat{\Delta}+2)[f] d\omega + \frac{1}{2}\oint_{\partial\Omega} \frac{1}{2}\partial_n |Df|^2 - (\hat{\Delta}+2)[f]\partial_n f d\sigma. \qquad (4)$$

Observe that the area integral in (4) is exactly the energy integral for equation (3).

The lemma follows from a compuation of the induced metric using (2) ,

$$dY \cdot dY = \frac{1}{4}\big((\hat{\Delta}[f])^2 - 4M[f]\big)\, d\mathcal{S}^2 \,,$$

where $M[f]$ is the Monge-Ampere operator and $d\mathcal{S}^2$ is the metric on S^2. We now recall a version of the Lichnerowicz formula. If Σ is any surface with boundary and u is any sufficiently smooth function

$$\begin{aligned}\oint_{\partial\Sigma} \frac{1}{2}\partial_n|\nabla u|^2 - (\Delta u)\partial_n u \, ds &= \int_\Sigma |\nabla^2 u|^2 - (\Delta u)^2 + K|\nabla u|^2 \, d\Sigma \\ &= \int_\Sigma -2M[u] + K|\nabla u|^2 \, d\Sigma.\end{aligned}$$

We apply this to f using our current notation and using $K \equiv 1$ on S^2.

Recall the classical Dirichlet Principle:

If $h \in C^2(\bar{\Omega})$ solves $\Delta h = 0$ and $g \equiv h$ on $\partial\Omega$, then

$$\int_\Omega |\nabla h|^2 \, d^n x \leq \int_\Omega |\nabla g|^2 d^n x \, .$$

Lemma

(Dirichlet's Principle). If q solves

$$\hat{\Delta}(\hat{\Delta}q + 2q) = 0 \tag{5}$$

in $\Omega \subset\subset S^2$ *and* f *is any smooth function with* $f - q \in W_0^{2,2}(\Omega)$*, then*

$$\int_\Omega \hat{\Delta}[q]\,(\hat{\Delta} + 2)[q]\,d\omega \le \int_\Omega \hat{\Delta}[f]\,(\hat{\Delta} + 2)[f]\,d\omega\,, \tag{6}$$

holds.

To prove the theorem, we represent the zero mean curvature surface using the representation theorem (2), i.e.

$$X = (Dq - (\frac{1}{2}\hat{\Delta}q)\nu\,,\, -\frac{1}{2}[\hat{\Delta}q + 2q])\,,$$

where q solves $\hat{\Delta}(\hat{\Delta}q + 2q) = 0$.
If Y is any marginally trapped spherical graph which agrees with X on the boundary then we can use (2) to represent Y using a function f. We have

$$Y = (Df - (\frac{1}{2}\hat{\Delta}f)\nu\,,\, -\frac{1}{2}[\hat{\Delta}f + 2f])\,.$$

$f \equiv q$ on $\partial\Omega$ since $Y \cdot (\nu, 1) \equiv X \cdot (\nu, 1)$ on $\partial\Omega$.
By projecting both X and Y to $\mathbf{R}^3$ and using that ν is orthogonal to both Df and Dq, we have that $Df \equiv Dq$ on $\partial\Omega$ also.

Since $f - q \in W_0^{2,2}(\Omega)$, we have by Dirichlet's Principle:

$$\int_\Omega \hat{\Delta}[q]\,(\hat{\Delta}+2)[q]\,d\omega \le \int_\Omega \hat{\Delta}[f]\,(\hat{\Delta}+2)[f]\,d\omega\,,$$

Recall:

$$\mathrm{Area}[Y] = \frac{1}{4}\int_\Omega \hat{\Delta}[f](\hat{\Delta}+2)[f]d\omega + \frac{1}{2}\oint_{\partial\Omega} \frac{1}{2}\partial_n|Df|^2 - (\hat{\Delta}+2)[f]\partial_n f d\sigma.$$

We will show that the boundary integral in the area formula only depends on f and Df restricted to the boundary.

$$\begin{aligned}
\frac{1}{2}\partial_n|Df|^2 - (\hat{\Delta}+2)[f]\partial_n f &= [D_n(Df)\cdot Df] - (D_tDf\cdot t + D_nDf\cdot n)f_n - \\
&= [(D_n(Df)\cdot t)f_t + (D_n(Df)\cdot n)f_n] - (D_tDf \\
&\quad -2ff_n \\
&= (D_n(Df)\cdot t)f_t - (D_tDf\cdot t)f_t - 2ff_n \\
&= (D_t(Df)\cdot n)f_t - (D_tDf\cdot t)f_t - 2ff_n,
\end{aligned}$$

where we have used that the Hessian is symmetric. By assumption $f\equiv q$, $Df\equiv Dq$ on $\partial\Omega$, and thus $f_n\equiv q_n$ and $D_tDf\equiv D_tDq$ on $\partial\Omega$. This gives the result.

Remark:

The inequality (1) can be improved to

$$4A[X] + (\beta_1 - 2) \int_\Sigma ||\vec{X}^T - \vec{Y}^T||^2 \, d\Sigma \leq 4A[Y] \,,$$

where

$$\beta_1(\Omega) = \inf_{C_c^\infty(\Omega)} \frac{\int_\Omega (\Delta f)^2}{\int_\Omega |Df|^2} \,.$$

is the buckling eigenvalue of $\Omega \subset S^2$.

Questions

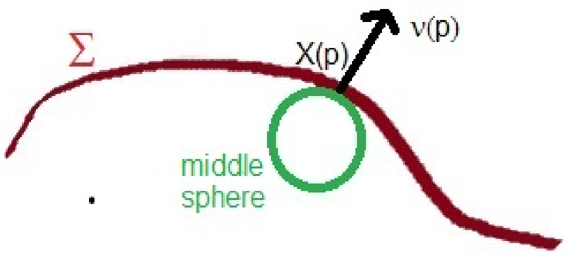
- Can the previous result be extended to points where the curvature vanishes?
- Are there analogous results for other spaces? The case where $\mathbf{R}^4_1$ is replaced by S^4_1 would be particularly interesting.

Laguerre Geometry:

For an oriented surface $X : \Sigma \to \mathbf{R}^3$, the *middle sphere* at p is the sphere (or point sphere) of radius $\frac{1}{2}(\frac{1}{k_1} + \frac{1}{k_2})$ in oriented contact with the surface at $X(p)$.

This assignment defines the *Laguerre map*

$$Z : \Sigma \to \{\text{oriented spheres and point spheres}\} \approx \mathbf{R}^4_1 .$$
$$p \mapsto \text{middle sphere at } p$$



Facts about the Laguerre Gauss map:

- $Z : (\Sigma, III) \to \mathbf{R}^4_1$ is space-like and conformal.
- Z is marginally trapped..
- $$\mathrm{Area}(Y) = \int_\Sigma \frac{H^2 - K}{K}\, d\Sigma_X \quad = \quad \mathrm{Laguerre:\ functional\ of}\ X$$
- Z has zero mean curvature if and only if X is a Laguerre minimal surface.

Therefore, X is a minimizer for the Laguerre functional exactly when Z minimizes area among marginally trapped surfaces.

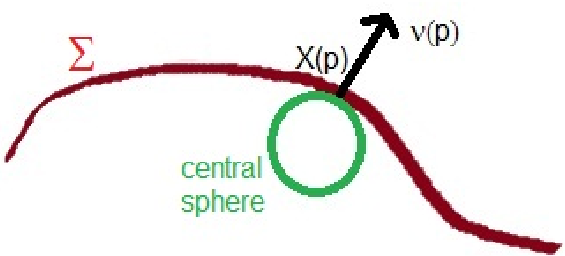
Conformal Geometry:

For an oriented surface $X : \Sigma \to \mathbf{R}^3$, the *central sphere* at p is the sphere (or plane) of radius $|H(p)|^{-1}$ in oriented contact with the surface at $X(p)$.

This assignment defines the *conformal Gauss map*

$$Y : \Sigma \to \{\text{oriented spheres and planes}\} \approx S^4_1 .$$

$$p \mapsto \text{central sphere at } p$$



Facts about the conformal Gauss map:

- $Y : (\Sigma, ds_X^2) \to S_1^4$ is space-like and conformal.
- Y is marginally trapped..
- $$\text{Area}(Y) = \int_\Sigma H^2 - K \, d\Sigma_X \quad = \quad \text{Willmore functional of } X$$
  $$+\text{topological constant}$$
- Y has zero mean curvature if and only if X is a Willmore surface.

Therefore, X is a minimizer for the Willmore functional exactly when Y minimizes area among marginally trapped surfaces.

For Further Reading I

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Thank You!