

ANISOTROPIC SURFACE ENERGY

MIYUKI KOISO AND BENNETT PALMER

ABSTRACT. We discuss several new results concerning free boundary problems for free anisotropic surface energies. The boundary terms contain wetting and line tension.

INTRODUCTION

An anisotropic surface energy is one that depends on the direction of a surface at each point. They were introduced by Georg Wulff to model the equilibrium shape of a crystal [16]. Whereas the surface energy of a liquid drop is isotropic, the ordered arrangement of molecules in a crystal mean that its interfacial energy depends on the surface direction. Some time after the discovery of liquid crystals in 1888, anisotropic surface energies were applied to study their interfacial surface energy also. The typical anchoring energy of a liquid crystal-liquid interface is of the form

$$\mathcal{F} = \int_{\Sigma} \gamma(n(X) \cdot \nu) d\Sigma,$$

where $n(X)$ is the director field, which gives the local orientation of the molecules, and ν is the surface normal. The director field is an \mathbf{RP}^2 valued function defined on the interior and boundary of the liquid crystal.

Here, we will consider the simple case where the energy density γ is a function of the surface normal ν alone. Our goal is twofold; first to point out the great similarity between the variational theory of surfaces with constant anisotropic mean curvature and the CMC (constant mean curvature) case and secondly to give an account of some new results for free boundary problems whose free surface energy is anisotropic.

1. ANISOTROPIC MEAN CURVATURE

Let γ be a “reasonable” function on the unit sphere S^2 . For an oriented surface $X : \Sigma \rightarrow \mathbf{R}^3$, we define the action

$$\mathcal{F}[\Sigma] := \int_{\Sigma} \gamma(\nu) d\Sigma,$$

where ν is the unit normal along X , and $d\Sigma$ is the volume form of Σ induced by X . Let

$$X_t = X + t\delta X + \mathcal{O}(t^2) \tag{1}$$

1991 *Mathematics Subject Classification*. Primary 49Q10, 76D45, Secondary 53C45.

Key words and phrases. anisotropic mean curvature, free boundary, Delaunay surface.

The first author is partially supported by Grant-in-Aid for Scientific Research (C) No. 19540217 of the Japan Society for the Promotion of Science, and Nara Women’s University Intramural Grant for Project Research. The second author was partially funded by Fellowship S-08154 from the Japan Society for the Promotion of Science.

be a smooth variation of an immersed surface $X : \Sigma \rightarrow \mathbf{R}^3$. Assume that δX has compact support in Σ . Then the anisotropic mean curvature function Λ is defined by the formula

$$\delta \mathcal{F}[X] := \partial_t \mathcal{F}[X_t]_{t=0} = - \int_{\Sigma} \Lambda \delta X \cdot \nu \, d\Sigma. \quad (2)$$

Therefore $\Lambda = 0$ characterizes critical points of the free energy. Recall that the algebraic volume enclosed by the surface can be defined as

$$\text{vol}[X] = \frac{1}{3} \int_{\Sigma} X \cdot \nu \, d\Sigma. \quad (3)$$

The first variation of the volume is given by

$$\delta \text{vol}[X] = \int_{\Sigma} \delta X \cdot \nu \, d\Sigma.$$

Therefore constant anisotropic mean curvature (CAMC) $\Lambda \equiv \text{constant}$ characterizes critical points of the free energy with the algebraic volume enclosed by the surface constrained to be a constant.

We will now compute the anisotropic mean curvature Λ assuming that the immersion is smooth. Write $\delta X = \psi \nu + \xi$ where ξ is a vector field on Σ . Then $\delta \nu = -\nabla \psi + d\nu(\xi)$ and $\delta d\Sigma = (-2H\psi + \text{div} \xi) d\Sigma$ give the first variations of the normal and area element. Let $D\gamma$ denote the gradient of γ on S^2 . Since the tangent planes $dX(T_p(\Sigma))$ and $T_{\nu(p)}S^2$ are parallel, we can regard $D\gamma$ as a vector field along X . Then we have,

$$\begin{aligned} \delta(\gamma d\Sigma) &= (D\gamma \cdot \delta \nu + \gamma (-2H\psi + \text{div} \xi)) d\Sigma \\ &= (D\gamma \cdot [-\nabla \psi + d\nu(\xi)] + \gamma (-2H\psi + \text{div} \xi)) d\Sigma. \end{aligned}$$

It is straightforward to check that $d\nu D\gamma = \nabla(\gamma \circ \nu)$ holds. Using this, we get

$$\delta(\gamma d\Sigma) = ([-D\gamma \cdot \nabla \psi - 2H\gamma\psi] + \text{div}(\gamma\xi)) d\Sigma.$$

Integrating this over Σ and applying Stokes' theorem gives,

$$\delta \mathcal{F} = \oint_{\partial \Sigma} (\gamma \xi - \psi D\gamma) \cdot \eta \, dL + \int_{\Sigma} \psi (\text{div} D\gamma - 2H\gamma) \, d\Sigma, \quad (4)$$

where η is the outward pointing normal to $\partial \Sigma$. Assuming that δX has compact support, the first integral vanishes and comparison with (2) leads to

$$-\Lambda = \text{div} D\gamma - 2H\gamma. \quad (5)$$

We will impose the following *convexity condition* on γ : the map

$$\chi : S^2 \rightarrow \mathbf{R}^3, \quad \nu \mapsto D\gamma(\nu) + \gamma(\nu)\nu,$$

defines a smooth, closed surface $W := \chi(S^2)$. The surface W is called the *Wulff shape*. For example, when $\|\cdot\|$ is a smooth norm on \mathbf{R}^3 , with dual norm $\|\cdot\|_*$ and \mathcal{F} is defined by

$$\mathcal{F}[\Sigma] := \int_{\Sigma} \|\nu\| \, d\Sigma, \quad (6)$$

where ν is the usual normal to the oriented surface Σ , then W is just the unit sphere in the dual norm: $W = \{x \mid \|x\|_* = 1\}$. If an arbitrary smooth, *even* function $\gamma(\nu)$ which satisfies the convexity condition, is given on S^2 , then the positive degree one homogeneous extension of γ defines a norm on \mathbf{R}^3 . The condition that γ be even is, in fact, required

in the applications to nematic liquid crystals since the director field is only \mathbf{RP}^2 valued, [13].

Wulff's Theorem roughly states that for any closed surface S enclosing the same 3-volume as W , there holds $\mathcal{F}[W] \leq \mathcal{F}[S]$, i.e. W solves the isoperimetric problem for the anisotropic energy \mathcal{F} . There are many different versions of this result which cover different classes of functionals and different interpretations of what is meant by a surface. In the smooth case discussed above, the first proof appears to have first been given by Chandrasekhar [2] in a paper related to nematic liquid crystals.

When the convexity condition holds, the differential equation for constant (or more generally prescribed) anisotropic mean curvature, is absolutely elliptic in the sense of Hopf. This means that the equation possesses a Maximum Principle analogous to the well known one for constant mean curvature surfaces. One can infer from this, for example, that if two surfaces have the same constant anisotropic mean curvature and they have the same position and normal at a point and one of them lies on one side of the other in a neighborhood of this point, then the two surfaces agree near the point in question. Since the Maximum Principle is the most important analytic tool for studying constant mean curvature, it is not surprising that many results for CMC surfaces find natural extensions to CAMC surfaces. The following recent result generalizes the famous Alexandrov theorem.

Theorem 1 (He-Li-Ma-Ge, [6]). *The only closed embedded CAMC surfaces in \mathbf{R}^3 are given by the Wulff shape W and its rescalings.*

Also, Giga and Zhai have recently given the following version of the Hopf theorem. Let $\tilde{\gamma}$ denote the positive degree one homogeneous extension of γ to $\mathbf{R}^3 \setminus 0$ and write $\tilde{\gamma}(p) =: |p| + \epsilon \tilde{\gamma}_1(p)$.

Theorem 2 (Giga-Zhai, [4]). *There exist numbers $r = r(c_1, c_2, c_3)$ with the following property: If $\inf_{S^2} \tilde{\gamma} \geq c_1 > 0$, $\sup_{S^2} |\nabla \tilde{\gamma}_1| \leq c_2$, $\sup_{S^2} |\nabla^2 \tilde{\gamma}_1| \leq c_3$ hold and if $|\epsilon| < r$, then the CAMC immersion of the sphere into \mathbf{R}^3 is unique.*

There is also an extension of the Ruh-Vilms theorem to CAMC surfaces, [7]. Let $\Sigma \rightarrow \mathbf{R}^3$ be a smooth oriented CAMC surface with anisotropic mean curvature Λ and Gauss map ν . For maps $f : \Sigma \rightarrow S^2$, define an energy by

$$E_\nu[f] := \int_\Sigma (D^2\gamma + \gamma I)_\nu \nabla f \cdot \nabla f \, d\Sigma.$$

Then ν is a critical point of this energy, i. e. $\delta E_\nu[\nu] = 0$. An even stronger result states that if the first Dirichlet eigenvalue of Σ for the eigenvalue problem associated with the second variation of the functional $\mathcal{F} + \Lambda \cdot \text{vol}$ is non negative for all compactly supported variation (which is always locally the case), then ν is the absolute minimizer of the energy E_ν among all maps f as above with $f \equiv \nu$ on $\partial\Sigma$.

2. THE FUNCTIONAL

Let $|\cdot|_H$ and $|\cdot|_V$ be smooth norms on \mathbf{R}^2 which we will refer to as the horizontal and vertical norm. It is assumed that the vertical norm is invariant under reflection through the coordinate axes so that

$$|(x, y)|_V = (|x|, |y|)|_V \tag{7}$$

holds for all $(x, y) \in \mathbf{R}^2$. We define a norm $\|\cdot\|$ on \mathbf{R}^3 by $\|(x_1, x_2, x_3)\| := |((x_1, x_2)|_H, x_3)|_V$. This norm will define an anisotropic energy functional as in (6). As noted above, the Wulff shape W for this functional is the unit sphere in the dual norm. All horizontal slices of W are homothetic. We will say such W (or such an anisotropic energy functional) to be of *product type*.

We will also use the horizontal norm to define a one dimensional anisotropic energy. For a smooth planar curve C with unit normal N , we define

$$\mathcal{L}[C] := \int_C |N|_H dL,$$

where dL is the line element of C . The one dimensional Wulff shape for this energy is the unit sphere in the dual norm $|\cdot|_H^*$ which we call Ω . The reason for choosing this particular definition for the line tension will be apparent soon.

Let κ be the curvature of the curve C with respect to the inward directed normal and let κ_Ω denote the curvature of Ω also with respect to the inward pointing normal. On C , this curvature is evaluated at the point on Ω where the normals to the two curves agree. The first variation formula for \mathcal{L} reads,

$$\delta\mathcal{L} = \int_C \frac{\kappa}{\kappa_\Omega} \delta C \cdot N dL.$$

3. SOME EXAMPLES OF CAMC SURFACES

The classical CMC surfaces of revolution are known as Delaunay surfaces. We will review here a construction of *anisotropic Delaunay surfaces* which was found in [7].

First, consider a Wulff shape which is axially symmetric with vertical axis. The generating curve of W will be expressed in the form $v \mapsto (u(v), v)$. We seek a second curve $(x(s), z(s))$ which will be the generator of a surface Σ with constant anisotropic mean curvature Λ . By choosing the orientation correctly, we can assume that $\Lambda \leq 0$ holds. It turns out, that all solutions of this problem can be produced as follows. The horizontal coordinates of the generating curves are related by the quadratic equation,

$$2ux + \Lambda x^2 = c, \tag{8}$$

where c is a flux parameter. Once $x(u(v))$ is found, the vertical coordinate can be found from

$$z := \int_{v_0}^v x_u(v) dv. \tag{9}$$

Note that the equation (8) is universal for all axially symmetric Wulff shapes and that the anisotropy only affects the equation (9). This construction extends to CAMC hypersurfaces of \mathbf{R}^{n+1} by replacing (8) with the equation $nux^{n-1} + \Lambda x^n = c$.

The surfaces generated by (8) and (9) are similar in structure to the well known Delaunay surfaces. They fall into six classes depending on the choices of the parameters Λ and c : plane, anisotropic catenoid, homothety of the Wulff shape, cylinder, anisotropic unduloid, and anisotropic nodoid (See Figure 1).

The case where W is axially symmetric corresponds to the case where the horizontal norm is the usual one in the plane. It turns out, that if this norm can be replaced by an arbitrary horizontal norm, then (generalized) Delaunay surfaces for the new functional



Wulff shape (left) and anisotropic catenoid (right)



Anisotropic unduloid (left) and anisotropic nodoid (right)

FIGURE 1. A Wulff shape and anisotropic Delaunay surfaces

are obtained by taking the product of the (x, z) curves obtained above with the unit circle in the new horizontal norm. Some examples are shown below (Figure 2).

Recently Chen and Kamien, [3] studied a different type of anisotropic Delaunay surface which models the shape of nematic liquid crystal films. In their model, the director field of the nematic is prescribed to lie along the meridian curves.

Chad Kuhn and the second author have generalized the construction of anisotropic Delaunay surfaces given above to obtain CAMC surfaces which are invariant under a helicoidal motion. An example is shown in Figure 3 below.

4. CAPILLARY SURFACES

Consider a non liquid drop of a fixed volume constrained to lie between two horizontal planes as shown in Figure 4. The total energy consists of the free energy \mathcal{F} , the *wetting energy* $\omega \cdot \mathcal{A} := \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1$, where \mathcal{A}_i is the area in the plane Π_i wetted by the drop,



Wulff shape (left) and generalized anisotropic catenoid (right)



Generalized anisotropic unduloid (left) and generalized anisotropic nodoid (right)

FIGURE 2. A Wulff shape (product type) and generalized anisotropic Delaunay surfaces

and the *line tension* $\tau \cdot \mathcal{L} = \tau_0 \mathcal{L}_0 + \tau_1 \mathcal{L}_1$:

$$\mathcal{E} := \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}. \quad (10)$$

The wetting energy is an energy cost assigned to the interface between the material of the drop and the planes. Because particles on the boundary curves C_i are in contact with three phases, the curves should have their own energy cost. In the isotropic case \mathcal{L} is defined to be the length of the boundary curve. Line tension was introduced by Gibbs. It is known to only contribute to the shape of drops of size in the range of a few microns.

For an oriented surface Σ , there is a map $\chi : \Sigma \rightarrow W$ which assigns to a point in the surface the unique point in W where the normals to the two surfaces agree. The Euler-Lagrange equations for the energy (10) are

$$\Lambda \equiv \text{constant in } \Sigma, \quad \chi \cdot E_3 = (-1)^{i+1}(\omega_i + \tau_i \kappa / \kappa_\Omega) \text{ on } C_i. \quad (11)$$

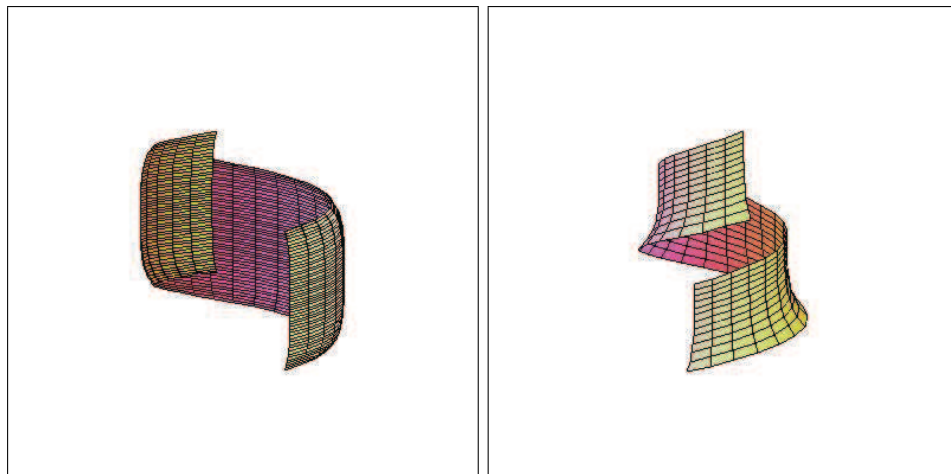


FIGURE 3. Positively and negatively curved parts of CAMC helicoidal surface. The Wulff shape is $(x_1^2 + x_2^2)^2 + x_3^4 = 1$. The value of the anisotropic mean curvature is $\Lambda = -1$.

For an anisotropic Delaunay surface Σ , the boundary condition takes the simpler form

$$\chi \cdot E_3 = (-1)^{i+1}(\omega_i + \tau_i/x_i), \quad (12)$$

on C_i , $i = 0, 1$, where x_i is the radius of C_i . Then, if a part of Σ bounded by two horizontal planes is chosen, two continua pairs of constants (ω_i, τ_i) can always be found so that the Euler-Lagrange equations hold. We will call an equilibrium for (10) which is an anisotropic Delaunay, a Delaunay capillary surface. In some cases, these surfaces are the only equilibria.

Theorem 3 ([11]). *Assume that the free energy in (10) is rotationally symmetric and that $\tau_i \geq 0$, $i = 0, 1$ holds. Then every capillary surface which is embedded into the region between the two planes is a Delaunay capillary surface.*

In particular, for a part of the Wulff shape \hat{W} between two horizontal planes Π_i , $i = 0, 1$, we can choose constants (ω_i, τ_i) such that $\chi \cdot E_3 = (-1)^{i+1}(\omega_i + \tau_i/x_i)$, and so \hat{W} is an equilibrium surface for the energy (10). A surface which consists of the part of a rescaling of W which lies above a horizontal plane will be referred to as a *sessile drop*.

A result known as Winterbottom's theorem [15], states that for $\tau_i = 0$, \hat{W} is the absolute minimizer of $\mathcal{F} + \omega \cdot \mathcal{A}$ among all surfaces enclosing the same volume and having free boundary on the supporting planes Π_i . Our definition of the functional \mathcal{L} is essentially the only one possible if we wish to preserve the property that the domains \hat{W} are in equilibrium for suitable choices of the parameters τ_i and ω_i . For the choices of the parameters for which \hat{W} is in equilibrium, the surface is not, in general, a minimizer. We do, however have the following result.

Theorem 4 ([11]). *(i) Assume that the anisotropic energy functional \mathcal{F} is of product type. Assume $\tau_i < 0$, $i = 0, 1$ and that \hat{W} is in equilibrium for (10). Then \hat{W} is the absolute minimizer of the energy among all symmetric surfaces enclosing the same*

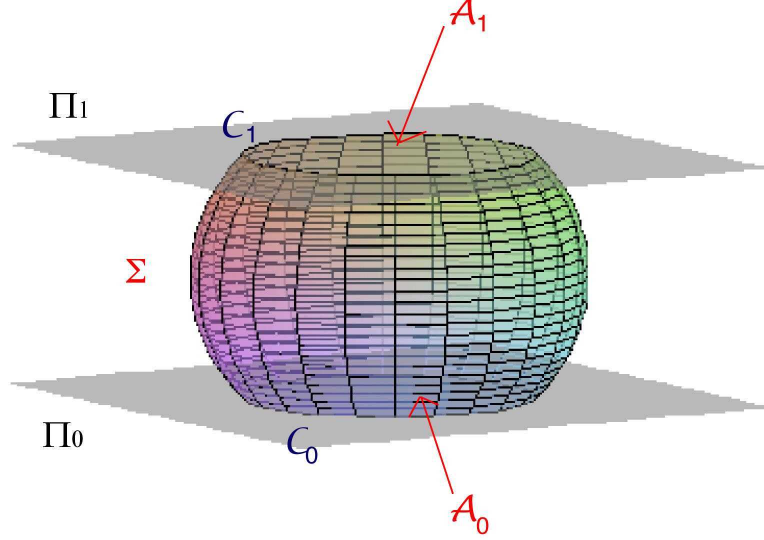


FIGURE 4. Configuration

volume and having free boundary components on $\Pi_0 \cup \Pi_1$. (Here a symmetric surface means a surface all of whose horizontal cross sections are rescalings of Ω).

(ii) Assume $\tau \geq 0$. If a minimizer exists for the problem of minimizing the energy (10) among all surfaces enclosing a fixed volume and having one boundary component which lies on Π_0 , then the minimizer is a sessile drop.

Assuming that we have an equilibrium surface, the second variation of energy is given by

$$\delta^2 \mathcal{E} = - \int_{\Sigma} \psi J[\psi] d\Sigma + \oint_{\partial \Sigma} \psi B_{\tau}[\psi] dL. \quad (13)$$

Here, J is the Jacobi operator given by

$$J[\psi] = \delta \Lambda = \operatorname{div}[A \nabla \psi] + \langle A \cdot d\nu, d\nu \rangle \psi, \quad (14)$$

where $A := D^2 \gamma + \gamma I$. The boundary operator B_{τ} is given by

$$B_{\tau}[\psi] = A \left(\nabla \psi + \frac{\nu_3}{n_3} \psi d\nu(n) \right) \cdot n - \tau \frac{1}{n_3} \left(\left[\frac{1}{\kappa_{\Omega}} \left(\frac{\psi}{n_3} \right)_L \right]_L + \frac{\kappa^2 \psi}{\kappa_{\Omega} n_3} \right), \quad (15)$$

where n_3 is the third component to the conormal. An equilibrium surface is said to be *stable* if for all smooth functions ψ with zero mean value, the second variation is non negative.

For liquid sessile drops, Widom showed that spherical caps with negative line tension are stable with respect to rotationally symmetric deformations. Their instability when more general deformations are allowed, was shown in [12]. Despite this, the existence of liquid drops with negative line tension is contentious. According to [5], the wavelengths of destabilizing variations may fall below the length scale where the surface tension model is valid. Therefore the mathematical instability may not be physically relevant. Some experimental evidence supports the existence of liquid drops with negative line tension.

In the anisotropic case, we have the following.

Theorem 5 ([11]). *If Σ is a Delaunay capillary surface and at least one τ_i is negative, then Σ is unstable.*

We also give the following characterization of stable sessile drops. By the previous result, we can assume that the line tension is non negative. In the case where the free energy is axially symmetric and the angle of intersection of the sessile drop S with the supporting plane is not 90° , there is a unique cone \mathcal{C} which is tangent to S along C_0 . Let V denote the volume of S and let $V_{\mathcal{C}}$ denote the volume of the cone. For some $R \in \mathbf{R}$, $S \subset RW$ holds. Denote by \underline{x} the radius of the boundary circle of S .

Theorem 6 ([11]). *Let W be a rotationally symmetric Wulff shape which is symmetric with respect to the plane $z = 0$ and let S be a sessile drop with $\tau \geq 0$.*

Then, S is stable if and only if

$$\tau \leq \frac{3V_{\mathcal{C}}V}{\pi R \underline{x}(V_{\mathcal{C}} - \sigma(\nu_3)V)}, \quad (16)$$

holds, where

$$\sigma(\nu_3) := \begin{cases} +1, & \nu_3 > 0, \\ 0, & \nu_3 = 0, \\ -1, & \nu_3 < 0. \end{cases}$$

Corollary 1. *Let S be a sessile drop for a functional of product type and let \bar{S} be the surface obtained by replacing the level curves of the height function z by circles of radius $x(z)$. Then S is stable if and only if (16) holds where all of the quantities are measured using \bar{S} .*

5. THE CASE $\tau = 0$

Here we will briefly describe some new results in the case when the line tension is zero.

With one boundary component, the only capillary surfaces are sessile drops, which are energy minimizing by Winterbottom's theorem. The simplest interesting problem is then the case of two boundary components and no wetting: $\omega_0 = 0 = \omega_1$. In the isotropic case, it was shown independently by Athanassenas and Vogel, that the only stable capillary surfaces are cylinders whose volumes are sufficiently large in comparison to their heights. In [8], this result was extended to the case of a rotationally invariant anisotropic functionals whose Wulff shape satisfies the curvature condition given below:

(*) the curvature of the generating curve of the Wulff shape is an increasing function of its arc length measured in the upwards direction from the plane $x_3 = 0$.

Recent numerical simulations [1] indicate that the conclusion is no longer true without the condition (*). It appears that for some functionals having energy density of the form $\gamma = 1 + e\nu_3^2$, with $e < 0$, stable parts of anisotropic unduloids appear when the volume is small, but not too small. These unduloids have a bulge on one supporting planes and a neck on the other. For larger volumes, the minimizers are again cylinders.

We also obtained

Theorem 7 ([1]). *Any convex anisotropic capillary surface is stable.*

and

Theorem 8 ([1]). *If the generating curve of an anisotropic capillary surface contains two or more inflection points, then it is unstable.*

5.1. **The case $\omega_0 = \omega_1 \geq 0$.** For a rotationally symmetric Wulff shape W , in the case of equal nonnegative wetting constants, the following two results completely determine the stable equilibria when the condition (*) holds. We let $\mu_i(0)$, $i = 1, 2$ denote the principal curvatures of W at the bulge $x_3 = 0$ with respect to the inward pointing normal. Here we let μ_1 denote the curvature of the generating curve of W .

Theorem 9 ([8], [9]). *Assume that the Wulff shape W is rotationally symmetric and satisfies the condition (*). Let Σ be a capillary surface with free boundary on two horizontal planes for the functional \mathcal{F} with $\omega_0 = \omega_1 =: \omega \geq 0$.*

(i) *If $\omega = 0$, then Σ is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder of height h and radius R , which is perpendicular to $\Pi_0 \cup \Pi_1$ which satisfies*

$$\frac{\mu_1(0)}{\mu_2(0)} h^2 \leq (\pi R)^2 .$$

(ii) *If $\omega > 0$ holds, then Σ is stable if and only if Σ is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.*

We call an anisotropic capillary surface *spanning* if its intersection with both supporting planes is a circle of positive radius.

Theorem 10 ([10]). *Assume that the Wulff shape W is rotationally symmetric and satisfies the condition (*).*

(I) *Assume $0 < \omega < \bar{\omega} := \text{maximum height on } W$. Then, there exist constants $0 < V_0 < V_1$ such that*

(i) *For volumes $V_0 \leq V < V_1$, there exists a unique stable spanning capillary surface with volume V , height h and wetting constant ω , and the surface is an anisotropic unduloid. This surface has inflection points on the boundary exactly when $V = V_0$.*

(ii) *For $V = V_1$, there exists a unique stable capillary surface with volume V , height h and wetting constant ω , and the surface is homothetic to a part of the Wulff shape.*

(iii) *For $V_1 < V$, there exists a unique stable capillary surface with volume V , height h and wetting constant ω , and the surface is an anisotropic nodoid.*

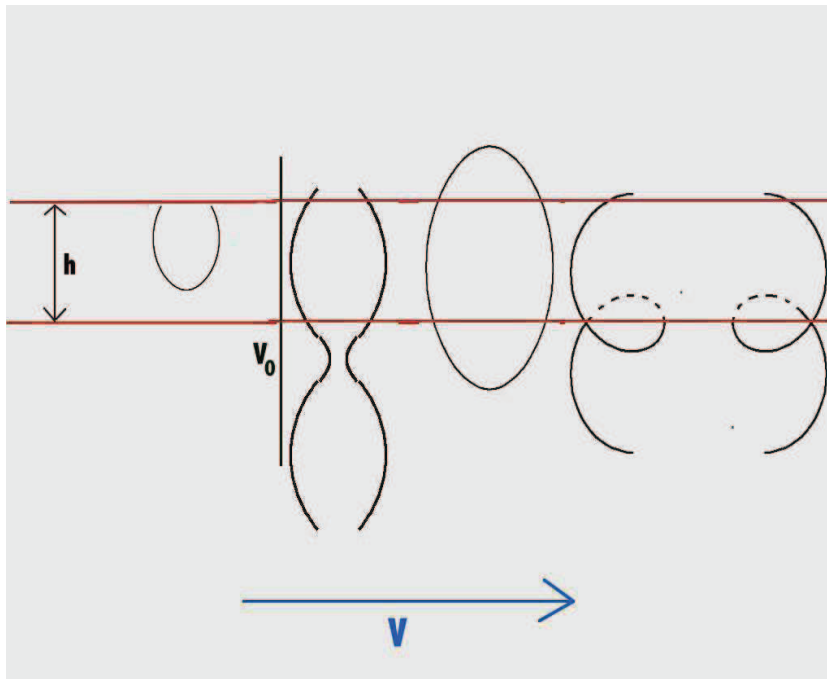


FIGURE 5. Stable capillary surfaces between two parallel planes for $\omega_0 = \omega_1 > 0$

(II) Assume $\omega = \bar{\omega}$. Then, there exists a constant $V_0 > 0$ such that, for $V_0 < V$, there exists a unique stable capillary surface with volume V , height h and wetting constant ω , and the surface is an anisotropic nodoid.

We conjecture that below the volume V_0 there is a pitchfork bifurcation. As the volume is decreased, the family of stable symmetric surfaces with no inflection points on their generating curves ((x, z) curves in Section 3) continue as a family of unstable symmetric surfaces with two interior inflection points on their generating curves. This branch of surfaces is joined by two unstable branches, one of which is just the reflection through a horizontal plane of the other, whose generating curves have exactly one interior inflection point. It was stated by Vogel [14], that such a bifurcation occurs in the CMC case.

When the condition (*) on the functional is dropped, numerical simulations [1], indicate that non convex, stable critica occur for comparatively small volumes while for large volumes, the stable equilibria are again convex.

5.2. The case $\omega_0 = \omega_1 < 0$. We assume that the Wulff shape W is rotationally symmetric, and that the wetting constants ω_0, ω_1 are equal and negative: $\omega := \omega_0 = \omega_1 < 0$. In this case, we do not have such a beautiful geometric characterization of stable capillary surfaces as the case of nonnegative wetting which was given in the previous subsection. In this subsection, we will give some characteristic results for the negative wetting case.

We will consider the part $X[-s_0, s_0]$ of an anisotropic unduloid

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad -s_0 \leq s \leq s_0,$$

where s is arc length of the generating curve $\Gamma : (x(s), z(s))$. We assume that $s = 0$ corresponds to a neck and that $s = \pm s_I$ ($s_I > 0$) correspond to two successive inflection points of Γ . We will consider the case $0 < s_0 \leq s_I$. Recall that (x, z) is given by the following formulas (see Section 3):

$$2ux + \Lambda x^2 = c, \quad z = \int_{v_0}^v x_u(v) dv.$$

In [9], we have seen that, numerical computations show that the stability of $X[-s_I, s_I]$ for a fixed value of Λ depends on the value of c . However, in the same paper, we proved the following result which shows that, if the anisotropic unduloid X is sufficiently near to the cylinder, then $X[-s_I, s_I]$ is stable.

Lemma 1 ([9], Lemma 8.8). *If*

$$\sqrt{3}N \geq x^*,$$

then $X[-s_I, s_I]$ is stable. Here $N := x(0)$ is the neck size of the unduloid X , and $x^ := x(s_I)$ is the radius of the circle at an inflection point.*

Recently, we improved this result and obtained the following:

Theorem 11. *Let N, x^* be the same as those in Lemma 1. If*

$$\sqrt{3}N \geq x^*$$

holds, then, for any $s_0 \in (0, s_I]$, the part of the unduloid $X[-s_0, s_0]$ is stable.

REFERENCES

- [1] Arroyo, J., Koiso, M., Palmer, B., *Stability of non liquid bridges*, preprint (2009).
- [2] Chandrasekhar, S., *Surface tension of liquid crystals*, Molecular Crystals **2** (1966), 71-80.
- [3] Chen, B. G., Kamien, R. D., *Nematic films and radially anisotropic Delaunay surfaces*, The European Physical Journal E, Volume 28, Issue 3 (2009), 315-329.
- [4] Giga, Y., Zhai, J., *Uniqueness of constant weakly anisotropic mean curvature immersion of \mathbf{S}^2 in \mathbf{R}^3* , Advances in Differential Equations **14** (2009), 601-619.
- [5] Guzzardi, L., Rosso, R., Virga, E. G., *Residual stability of sessile droplets with negative line tension*, Physics Review E, **73** 021602.1-021602.14, (2006).
- [6] He, Y., Li, H., Ma, H., Ge, J., *Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures*, eprint arXiv: 0712.0694, 2007 - arxiv.org. To appear in Indiana Univ. Math. J.
- [7] Koiso, M., Palmer, B., *Geometry and stability of surfaces with constant anisotropic mean curvature*, Indiana Univ. Math. J. **54** (2005), 1817-1852.
- [8] Koiso, M., Palmer, B., *Stability of anisotropic capillary surfaces between two parallel planes*, Calculus of Variations and Partial Differential Equations **25** (2006), 275-298.
- [9] Koiso, M., Palmer, B., *Anisotropic capillary surfaces with wetting energy*, Calculus of Variations and Partial Differential Equations **29** (2007), 295-345.
- [10] Koiso, M., Palmer, B., *Uniqueness theorems for anisotropic capillary surfaces*, SIAM Journal on Mathematical Analysis **39** (2007), 721-741.
- [11] Koiso, M., Palmer, B., *Equilibria for anisotropic surface energies with wetting and line tension*, preprint (2009).
- [12] Sekimoto, K., Oguma, R., Kawasaki, K., *Morphological stability analysis of partial wetting*, Annals of Physics **176**, Issue 2 (1987), 359-392.
- [13] Virga, E. G., Variational Theories for Liquid Crystals, Chapman and Hall, London (1994).

- [14] Vogel, T. I., *Comments on radially symmetric liquid bridges with inflected profiles*, Discrete and Continuous Dynamical Systems, Supplementary volume (2005), 862-867.
- [15] Winterbottom, W. L., *Equilibrium shape of a small particle in contact with a foreign substrate*, Acta Metal. **15** (1967), 303-310.
- [16] Wulff, G., *Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflächen*, Zeitschrift für Krystallographie und Mineralogie **34** (1901), 449-530.

(M. K.) DEPARTMENT OF MATHEMATICS, NARA WOMEN'S UNIVERSITY & PRESTO, JST, KITA-UOYA NISHIMACHI, NARA 630-8506, JAPAN,
E-mail address: `koiso@cc.nara-wu.ac.jp`

(B. P.) DEPARTMENT OF MATHEMATICS, IDAHO STATE UNIVERSITY, POCA TELLO, ID 83209, U.S.A.,
E-mail address: `palmbenn@isu.edu`