

Anisotropic capillary surfaces with wetting energy

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Abstract

We study the stability of capillary surfaces for anisotropic energies having boundaries supported in horizontal planes. A wetting energy term for the surface to plane interface is included.

1 Introduction

The study of capillary phenomena for liquid surface interfaces has been an active area of research for hundreds of years beginning with the investigations of Leonardo da Vinci in 1490 and continuing with the work of Laplace, Young, Gauss, Rayleigh and others.

When immiscible materials come into contact, their interface forms so as to minimize a free surface energy subject to whatever volume constraints and boundary constraints are imposed by the physical environment. If the materials involved enter a solid (or liquid crystal) phase, then the classical isotropic surface tension must be replaced by an anisotropic one even if the surface material is homogeneous.

Let F be a smooth, positive function on S^2 . To a smooth, oriented, immersed surface $X : \Sigma \rightarrow \mathbf{R}^3$, we assign the free anisotropic energy

$$\mathcal{F}[X] := \int_{\Sigma} F(\nu) d\Sigma, \quad (1)$$

where $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \rightarrow S^2$ is the Gauss map of X , and $d\Sigma$ is the area form of the induced metric. We will assume that F satisfies a “convexity condition” in the following sense: Denote by DF and D^2F the gradient and Hessian of F on S^2 . Then we assume that at each point in S^2 the matrix $D^2F + F1$ is positive definite. Then, such an energy functional \mathcal{F} is sometimes referred to as a constant coefficient elliptic parametric functional. More general integrands where F may also depend on the position of the surface, may also be considered, and they are called elliptic parametric functionals (cf. [5]). For simplicity, we will restrict ourselves to the constant coefficient case.

It is known that the energy \mathcal{F} appearing in (1) possesses a canonical critical point which minimizes \mathcal{F} among closed surfaces enclosing a specific three dimensional volume ([3]). This critical point is given by the immersion $\chi : S^2 \rightarrow \mathbf{R}^3$ defined by $\chi(\nu) = DF(\nu) + F(\nu) \cdot \nu$, and is

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known as the *Wulff shape* which we will denote by W . In the special case where $F \equiv 1$, \mathcal{F} is the area functional and W is the round sphere of radius 1 with center at the origin.

The property that X is a critical point of \mathcal{F} for all compactly supported volume-preserving variations is characterized by the property that X has *constant anisotropic mean curvature*. This definition is a generalization of the idea of constant mean curvature (CMC) which arises from the area functional.

This paper is a continuation of [11]. As in [11], we will assume that F is a function of one variable, that is $F = F(\nu_3)$. In [11], the stability of volume constrained equilibrium surfaces for anisotropic energies with free boundary components constrained to lie in horizontal planes were treated neglecting wetting energy. This omission forces the surface to intersect the supporting planes in a right angles, thus reducing greatly the quantity of examples which need to be considered, since Alexandrov reflection already forces an equilibrium to be a surface of revolution. In [11] it was shown that, under some assumptions on W which are satisfied in the CMC case, any stable embedded equilibrium capillary surface between two horizontal planes with nonempty free boundary on these planes is either a sufficiently short cylinder or a suitable part of the Wulff shape.

If the wetting energy is taken into account, the total surface energy may be represented

$$\mathcal{E} := \mathcal{F} + \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1,$$

where \mathcal{A}_i are the wetted areas of the two planes, and ω_i are coupling constants which depend on the materials involved. Now, if the critical surface is required to lie between the planes, then *Alexandrov reflection can still be applied to show that an equilibrium surface must be rotationally symmetric* (Corollary 3.2). However, the contact angle between the surface and the supporting plane must be a prescribed constant (Proposition 3.2) and *any* region bounded by two horizontal planes in any rotationally invariant surface with constant anisotropic mean curvature (anisotropic Delaunay surface) can be considered as an equilibrium surface for such a problem. This makes a complete characterization of all stable domains difficult and therefore we will limit ourselves here to certain natural choices of the parameters ω_0 and ω_1 and certain natural choices of regions.

For example, if we assume that W is rotationally invariant with respect to the x_3 axis, and symmetric with respect to $\{x_3 = 0\}$, and that the curvature function of the generating curve of W with respect to the inward pointing normal is a non-decreasing function of arc length on $\{x_3 \geq 0\}$ as one moves in an upward direction. Then we have the following:

If $\omega_0 = \omega_1 > 0$ holds and X is an equilibrium capillary surface having free boundary on two horizontal planes, then the immersion is stable if and only if X is a part of an anisotropic Delaunay surface whose generating curve has no inflection point in its interior (Theorem 11.1).

We wish to explicitly take note of the papers of Athanassenas [1], Vogel [12], [13] and Zhou [15] which provided several motivating ideas for our present research. All of these papers treated the stability of capillary surfaces between parallel planes for the area functional. Zhou's paper [15] treated the stability of a liquid catenoidal bridge. In particular, it is shown in [15] that, for the area functional with wetting, stability is not a monotonic property; i.e. an unstable domain of the catenoid may be a proper subset of a stable one. This further complicates the stability analysis.

The paper is organized as follows:

- Section 2 contains preliminary material.

- Section 3 contains a derivation of the first and second variational formulas of rotationally symmetric anisotropic surface energies for drops having free boundaries on two parallel planes with the inclusion of wetting energy along the surface-plane interface.
- Section 4 derives a minimizing property of the part of the Wulff shape W between two parallel planes for the free boundary problem. This property was first shown by Winterbottom [14] for surfaces with free boundary on a plane. Our method is essentially the same as that used in [14].
- Section 5 contains a summary of results concerning anisotropic Delaunay surfaces. These surfaces were introduced in detail in [10] and play a fundamental role in our stability analysis.
- Section 6 specializes the results of Section 3 to the anisotropic Delaunay surfaces.
- Section 7 gives a spectral characterization for stability of domains in anisotropic Delaunay surfaces.
- Section 8, 9 and 10 treat respectively stability of anisotropic unduloids, nodoids and catenoids.
- Section 11 characterizes stable equilibria in the case when $\omega_0 = \omega_1 \geq 0$ holds.
- Section 12 treats the case when one of the ω_i 's is zero.
- Section 13 is an appendix containing results about eigenvalues for problems of Sturm-Liouville type which are used in the previous sections.

2 Basic assumption and preliminaries

We gather here some known facts concerning anisotropic surface energies which will be needed throughout the paper. The reader is referred to [10] for details.

Let $F : S^2 \rightarrow \mathbf{R}$ be a smooth, positive function. We associate with F the set

$$W(F) := \{\chi \in \mathbf{R}^3 \mid \sup_{\nu \in S^2} \frac{\langle \chi, \nu \rangle}{F(\nu)} = 1\}. \quad (2)$$

We will call $W(F)$ the *Wulff shape* of F . (In the literature, the domain in \mathbf{R}^3 interior to $W(F)$ is sometimes called the Wulff shape.) Unless otherwise stated, we will assume the following *convexity condition* holds: $W(F)$ is a smooth, uniformly convex surface. In this case, the map of the sphere defined by $\chi(\nu) := DF + F\nu$, gives an embedding of S^2 onto $W(F)$ whose Gauss map is ν , i.e. $\chi = \nu^{-1}$. It is not difficult to see that every uniformly convex surface $W \subset \mathbf{R}^3$ arises as the Wulff shape of the function $F(\nu) = q \circ \nu^{-1}$, where q is the support function of W and ν is its Gauss map.

The convexity condition implies that the principal curvatures of $W(F)$ with respect to the outward pointing normal, denoted by $-\mu_j$, $j = 1, 2$ are everywhere negative. In the case in which we will be most interested, when $W(F)$ is a surface of revolution with respect to the vertical axis, these principal curvatures are given by

$$1/\mu_2 = F(\nu_3) - \nu_3 F'(\nu_3), \quad 1/\mu_1 = (1 - \nu_3^2)F''(\nu_3) + 1/\mu_2. \quad (3)$$

To a smooth, oriented surface $X : \Sigma \rightarrow \mathbf{R}^3$, we assign the free anisotropic energy

$$\mathcal{F}[X] := \int_{\Sigma} F(\nu) d\Sigma,$$

where ν is the Gauss map of the immersion. Such an energy is sometimes referred to as a constant coefficient elliptic parametric functional. More general integrands where F may also depend on the position of the surface, may also be considered, and they are called elliptic parametric functionals (cf. [5]), but we will restrict ourselves to the constant coefficient case for simplicity.

For a compactly supported variation of X , $X_{\epsilon} = X + \epsilon \dot{X} + \mathcal{O}(\epsilon^2)$, the first variation formula

$$\partial_{\epsilon} \mathcal{F}[X]_{\epsilon=0} = - \int_{\Sigma} \Lambda \langle \dot{X}, \nu \rangle d\Sigma,$$

defines the anisotropic mean curvature function Λ . The convexity condition insures that the equation for prescribed Λ is an absolutely elliptic, quasilinear PDE in the sense of [6].

Useful local expressions for the anisotropic mean curvature can be obtained by letting DF and D^2F denote respectively the gradient and Hessian of F on S^2 . These tensors can be transplanted to any immersed oriented surface by using parallel translation. Then,

$$\Lambda = 2HF - \operatorname{div}_{\Sigma} DF = -\operatorname{trace}_{\Sigma}(D^2F + FI) \circ d\nu,$$

where H is the mean curvature of X and I is the identity endomorphism field on TS^2 .

When $F = F(\nu_3)$ and X is a surface of revolution with vertical axis, we have

$$\Lambda = \frac{k_1}{\mu_1} + \frac{k_2}{\mu_2},$$

where $k_i, i = 1, 2$ denote the principal curvatures of X .

3 Capillary problems

We consider the closed domain Ω in the three dimensional Euclidean space \mathbf{R}^3 with boundary $\partial\Omega$ consisting of a union of two horizontal planes Π_0 and Π_1 .

Let $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ be an embedding of a two dimensional orientable compact connected C^{∞} manifold Σ with finitely many boundary components which maps the interior of Σ into the interior of Ω . We denote by ν the outward pointing unit normal to X . We define the volume $V[X]$ of X as the usual volume of the three-dimensional domain enclosed by $X(\Sigma) \cup \Pi_1 \cup \Pi_2$. A variation $X_{\epsilon} : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ of X will be called an *admissible variation* if

$$V[X_{\epsilon}] = V[X]$$

for all ϵ .

The *wetting energy* is defined as follows. Set $C_i = X^{-1}(\Pi_i)$. Then, each C_i (possibly an empty set), $i = 0, 1$, defines a compact planar region which is the sum of the planar regions bounded by embedded curves $X(\partial\Sigma) \cap \Pi_i$. We let \mathcal{A}_i denote its area and set

$$\mathcal{W} := \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1,$$

where ω_i are coupling constants. In applications these constants depend on the composition of the materials involved.

We seek a condition that $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ is a critical point of the total energy

$$\mathcal{E} := \mathcal{F} + \mathcal{W}, \quad \mathcal{F}[X] := \int_{\Sigma} F(\nu) d\Sigma$$

for all admissible variations of X . This condition is equivalent to the condition that, for some constant Λ_0 , X is a critical point of a functional

$$\mathcal{E}_{\Lambda_0} := \mathcal{E} + \Lambda_0 V$$

for all variations $X_{\epsilon} : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ of X .

Let $\chi := DF + F\nu$. Here DF is the gradient of F on the sphere. We also consider DF as a vector field on an immersed surface via parallel translation in \mathbf{R}^3 . We may then consider χ as a map from an orientable surface into \mathbf{R}^3 .

Recall the first variation of V for a variation X_{ϵ} is given by

$$\delta V := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} V[X_{\epsilon}] = \int_{\Sigma} \langle \partial_{\epsilon}(X_{\epsilon})|_{\epsilon=0}, \nu \rangle d\Sigma,$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in \mathbf{R}^3 .

By an *admissible variation vector field* we will mean a smooth \mathbf{R}^3 valued map \dot{X} on Σ such that when \dot{X} is resolved into tangential and normal components as $\dot{X} = \xi + \psi\nu$, there holds

$$\int_{\Sigma} \psi d\Sigma = 0, \tag{4}$$

$$\langle \dot{X}, E_3 \rangle|_{\partial\Sigma} \equiv 0, \tag{5}$$

where $E_3 := (0, 0, 1)$. The space of admissible variation vector fields will be denoted by \mathcal{A} . The condition (4) is the infinitesimal condition that there exists a deformation

$$X_{\epsilon} = X + \epsilon\dot{X} + \mathcal{O}(\epsilon^2) \tag{6}$$

which preserves the volume while (5) is the infinitesimal condition that $X_{\epsilon}(\partial\Sigma) \subset \partial\Omega$ for all ϵ . In fact, a standard argument using the implicit function theorem shows that (4) and (5) are also sufficient to embed \dot{X} in a volume-preserving variation (6) with $X_{\epsilon}(\partial\Sigma) \subset \partial\Omega$ and $\partial_{\epsilon}(X_{\epsilon})|_{\epsilon=0} = \dot{X}$. In fact, the following lemma is proved by a modification of the proof of the existence of volume-preserving variations fixing the boundary given by Barbosa-do Carmo [2].

Lemma 3.1 *For any C^{∞} function $\psi : \Sigma \rightarrow \mathbf{R}$ satisfying*

(i) $\psi(w) = 0$ for $w \in \partial\Sigma$ where X is tangent to $\partial\Omega$,

and

(ii) $\int_{\Sigma} \psi d\Sigma = 0$,

there exists a vector field ξ which is tangent to X such that

$$X_{\epsilon} = X + \epsilon(\xi + \psi\nu) + \mathcal{O}(\epsilon^2)$$

is an admissible variation.

Thus \mathcal{A} may be thought of as the tangent space at X to the “manifold” \mathcal{I} of all immersions of Σ into Ω which contain the same volume as X and satisfy the free boundary condition that $\partial\Sigma$ is mapped into $\partial\Omega$.

For an admissible variation given by (6), the first variation of the anisotropic energy is given by

$$\begin{aligned}\delta\mathcal{F} &= \int_{\Sigma} \langle DF, -\nabla\psi + d\nu(\xi) \rangle + F(\operatorname{div}\xi - 2H\psi)d\Sigma \\ &= \int_{\Sigma} \psi(\operatorname{div}_{\Sigma}DF - 2HF)d\Sigma + \oint_{\partial\Sigma} -\psi\langle DF, n \rangle + F\langle \xi, n \rangle d\tilde{s} \\ &= -\int_{\Sigma} \psi\Lambda d\Sigma + \oint_{\partial\Sigma} \langle \chi \times \dot{X}, dX \rangle,\end{aligned}$$

where H is the mean curvature of X , n is the outward pointing unit normal along $\partial\Sigma$, $d\tilde{s}$ is the line element on $\partial\Sigma$, $dX = (\nu \times n)d\tilde{s}$, and Λ is the anisotropic mean curvature of X .

To compute the first variation of the wetting energy, we note that by the divergence theorem applied to the constant vector field E_3 , we have

$$\int_{\Sigma} \nu_3 d\Sigma = \mathcal{A}_0 - \mathcal{A}_1.$$

Thus, by using $E_3^T = \nabla x_3$ and $\nu_*\nabla x_3 = \nabla\nu_3$ and the divergence theorem, we obtain

$$\begin{aligned}\delta\mathcal{A}_0 - \delta\mathcal{A}_1 &= \delta \int_{\Sigma} \nu_3 d\Sigma \\ &= \int_{\Sigma} ((\delta\nu_3)d\Sigma + \nu_3\delta(d\Sigma)) \\ &= \int_{\Sigma} \langle -\nabla\psi + \nu_*\xi, E_3 \rangle + \nu_3(\operatorname{div}\xi - 2H\psi) d\Sigma \\ &= \oint_{\partial\Sigma} (-\psi\partial_n x_3) + \nu_3\langle \xi, n \rangle d\tilde{s}.\end{aligned}\tag{7}$$

Define a unit tangent t to $\partial\Sigma$ as $t := \nu \times n$. By considering variation vector fields \dot{X} which vanish near one of the boundary components, (7) gives

$$\delta\mathcal{A}_0 = \int_{C_0} \langle t \times E_3, \dot{X} \rangle d\tilde{s}, \quad \delta\mathcal{A}_1 = -\int_{C_1} \langle t \times E_3, \dot{X} \rangle d\tilde{s},$$

so that

$$\delta\mathcal{W} = \omega_0 \int_{C_0} \langle t \times E_3, \dot{X} \rangle d\tilde{s} - \omega_1 \int_{C_1} \langle t \times E_3, \dot{X} \rangle d\tilde{s}.$$

Combining these formulas gives the first variation formula

$$\delta\mathcal{E} = -\int_{\Sigma} \Lambda\psi d\Sigma + \oint_{C_0} \langle t \times (\chi + \omega_0 E_3), \dot{X} \rangle d\tilde{s} + \oint_{C_1} \langle t \times (\chi - \omega_1 E_3), \dot{X} \rangle d\tilde{s}.\tag{8}$$

Proposition 3.1 *An immersion $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ is a critical point of \mathcal{E} among all surfaces in \mathcal{I} if and only if there holds:*

$$\Lambda \equiv \Lambda_0, \quad \text{in } \Sigma, \quad (9)$$

for some constant Λ_0 and

$$\langle \chi, E_3 \rangle \equiv -(-1)^i \omega_i, \quad \text{on } C_i, \quad i = 0, 1. \quad (10)$$

Proof. Assume first that (9) and (10) hold. Because of (5), the surface integral in (8) vanishes for any admissible variation. Note that (10) implies that $t \times (\chi + (-1)^i \omega_i E_3)$ is proportional to E_3 along C_i and so for any admissible variation vector field the boundary integrals in (8) vanish.

Conversely assume that $\delta\mathcal{E} = 0$ for all admissible variations. By considering admissible, compactly supported normal variations of X , one sees that (9) holds.

Suppose that (10) does not hold on some boundary arc $\alpha \subset C_0$. We may assume for convenience that $\langle \chi + \omega_0 E_3, E_3 \rangle > 0$ holds on some open arc $\alpha \subset C_0$. We can find a continuous map $Z : C_0 \rightarrow \mathbf{R}^3$ such that $\langle Z, E_3 \rangle \equiv 0$, $Z \equiv 0$ on $C_0 \setminus \alpha$ and $\langle t \times (\chi + \omega_0 E_3), Z \rangle \geq 0$ with strict inequality on some open arc. It is easy to see that Z may be extended to an admissible field on Σ . Writing $Z = \psi\nu + \xi$ it is clear that ψ can be extended to a function vanishing on C_1 and satisfying (4). Thus, we see that the first variation for this admissible field does not vanish which yields a contradiction. **q.e.d.**

Definition 3.1 We call an immersion $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ a *capillary surface* (for the energy \mathcal{E} , or, for F) if X satisfies (9) for some constant Λ_0 and (10) holds.

We will concentrate on the case where $F(\nu)$ ($\nu = (\nu_1, \nu_2, \nu_3)$) is a function of one variable, say $F = F(\nu_3)$.

Proposition 3.2 *Let $F = F(\nu_3)$ be an integrand for a rotationally symmetric anisotropic surface energy, and let Π_0, Π_1 be two horizontal planes $\{x_3 = \text{constant}\}$. Then, the angle between a capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_0 \cup \Pi_1)$ and the bounding planes $\Pi_0 \cup \Pi_1$ must be locally constant along the boundary in the sense that there exists a unique value c_i which depends only on F and ω_i , $i = 0, 1$, such that $\nu_3 = c_i$ holds on $\partial\Sigma \cap X^{-1}(\Pi_i)$.*

Proof. The boundary condition (10) becomes

$$\langle F'(\nu_3)E_3 + (F(\nu_3) - F'(\nu_3)\nu_3)\nu, E_3 \rangle = -(-1)^i \omega_i,$$

which can be expressed as

$$g(\nu_3) := (1 - \nu_3^2)F' + \nu_3 F = -(-1)^i \omega_i. \quad (11)$$

Since

$$g'(\nu_3) = (1 - \nu_3^2)F'' - \nu_3 F' + F = 1/\mu_1 > 0 \quad (12)$$

holds, g is monotonically increasing and is hence injective. **q.e.d.**

Corollary 3.1 *If the number $-(-1)^i \omega_i$ does not satisfy the inequality*

$$-F(-1) \leq -(-1)^i \omega_i \leq F(1), \quad (13)$$

then the solution of the free boundary problem with non-empty boundary component in Π_i does not exist.

Proof. Suppose there exists a solution of the free boundary problem with non-empty boundary component in Π_i . Then, in view of (11) and (12), there exists some $\nu_3 \in [-1, 1]$ such that ω_i satisfies

$$g(-1) \leq g(\nu_3) = -(-1)^i \omega_i \leq g(1),$$

which implies the inequality (13). **q.e.d.**

Corollary 3.2 *If a capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ for $F = F(\nu_3)$ is an embedding, then, X is a surface of revolution with vertical rotation axis, and its genus is zero.*

Proof. Since $D^2F + F1$ is positive definite, the equation $\Lambda = \text{constant}$ is absolutely elliptic in the sense that its linearization is elliptic at any sufficiently smooth surface (cf. [6]). Hence a maximum principle analogous to that for constant mean curvature surfaces holds for surfaces of constant anisotropic mean curvature. Therefore, if a capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ for $F = F(\nu_3)$ is an embedding, then, by the Alexandrov reflection methods and Proposition 3.2, X is a surface of revolution with vertical rotation axis, and its genus is zero. As for the application of the Alexandrov reflection methods to surfaces with boundary, we refer the readers to [7] and omit the details here. **q.e.d.**

Let X be an embedded capillary surface for $F = F(\nu_3)$. Then, from Corollary 3.2, X is represented as follows:

$$X : [s_0, s_1] \times S^1 \rightarrow \Omega, \quad X(s, \theta) = (x(s)e^{i\theta}, z(s)),$$

where s is an arc length parameter of the generating curve $(x(s), z(s))$ of X and we may assume that

$$X(\{s_0\} \times S^1) \subset \Pi_0, \quad X(\{s_1\} \times S^1) \subset \Pi_1, \quad z(s_0) < z(s_1)$$

holds. Since ν is the outward pointing unit normal of X , it holds that

$$\nu = (\nu_1, \nu_2, \nu_3) = (z'(s)e^{i\theta}, -x'(s)).$$

Set

$$(x', z')|_{C_i} = (\cos \eta_i, \sin \eta_i), \quad i = 0, 1.$$

The following result will explain the influence of the sign of the wetting energy on the angle between a capillary surface and the bounding planes, and it will be used in Section 11.

Lemma 3.2 *Assume that $F'(0) = 0$ holds. Then,*

$$\omega_0 > 0 \iff \cos \eta_0 > 0, \quad \omega_0 = 0 \iff \cos \eta_0 = 0, \quad \omega_0 < 0 \iff \cos \eta_0 < 0,$$

$$\omega_1 > 0 \iff \cos \eta_1 < 0, \quad \omega_1 = 0 \iff \cos \eta_1 = 0, \quad \omega_1 < 0 \iff \cos \eta_1 > 0$$

holds. Moreover, if $F(\nu_3)$ is an even function, then

$$\omega_0 = \omega_1 \iff \cos \eta_0 = -\cos \eta_1$$

holds. Further more, if $F \equiv 1$ holds, then

$$\cos \eta_i = (-1)^i \omega_i$$

holds.

Proof. By assumption and (11), $g(0) = 0$ holds. Therefore,

$$\omega_0 > 0 \iff g(\nu_3)|_{C_0} < 0 \iff \nu_3|_{C_0} < 0 \iff x'|_{C_0} > 0,$$

$$\omega_1 > 0 \iff g(\nu_3)|_{C_1} > 0 \iff \nu_3|_{C_1} > 0 \iff x'|_{C_1} < 0$$

holds.

If $F(\nu_3)$ is an even function, then $g(\nu_3)$ is an odd function. Therefore,

$$\omega_0 = \omega_1 \iff -\nu_3|_{C_0} = \nu_3|_{C_1} \iff x'|_{C_0} = -x'|_{C_1}$$

holds. **q.e.d.**

Proposition 3.3 *Let \mathcal{F} be a rotationally symmetric anisotropic surface energy. Let Π_0, Π_1 be two horizontal planes. And let $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \Pi_0 \cup \Pi_1)$ be a capillary surface. Then for admissible variations with variation vector field $\xi + \psi\nu$, the second variation of energy is given by*

$$\delta^2 \mathcal{E} = - \int_{\Sigma} \psi L[\psi] d\Sigma + \oint_{\partial\Sigma} \psi B[\psi] d\tilde{s} =: \mathcal{I}[\psi], \quad (14)$$

where L is the self-adjoint Jacobi operator

$$L[\psi] := \operatorname{div}(A\nabla\psi) + \langle A d\nu, d\nu \rangle \psi, \quad (15)$$

$$A := (D^2 F + F1)|_{\nu},$$

and

$$B[\psi] = \begin{cases} \langle n, A\nabla\psi \rangle - (-1)^i (\cot \eta_i)(k_1/\mu_1)\psi, & \sin \eta_i \neq 0, \\ \psi, & \sin \eta_i = 0 \end{cases}$$

on C_i .

Proof. We will denote by “ \cdot ” the derivative with respect to the variation parameter. The first variation formula (8) above gives,

$$\delta \mathcal{E} = - \int_{\Sigma} \Lambda \psi d\Sigma + \oint_{\partial\Sigma} \langle dX \times (\chi + (-1)^i \omega_i E_3), \dot{X} \rangle$$

using $\dot{X} = \xi + \psi\nu$. This is valid for any surface, so we can take the variation of the previous formula at an equilibrium surface $\Lambda \equiv \Lambda_0$, to obtain

$$\begin{aligned} \delta^2 \mathcal{E} &= - \int_{\Sigma} \psi \dot{\Lambda} d\Sigma + \Lambda_0 \delta \int_{\Sigma} \psi d\Sigma + \oint_{\partial\Sigma} \langle d\dot{X} \times (\chi + (-1)^i \omega_i E_3), \dot{X} \rangle \\ &\quad + \oint_{\partial\Sigma} \langle dX \times \dot{\chi}, \dot{X} \rangle + \oint_{\partial\Sigma} \langle dX \times (\chi + (-1)^i \omega_i E_3), \ddot{X} \rangle. \end{aligned}$$

The second integral vanishes since the variation is volume-preserving to all orders. As we saw in [10],

$$\dot{\Lambda} = L[\psi] + \langle \nabla \Lambda, \xi \rangle$$

holds. Now since $\Lambda \equiv \Lambda_0$,

$$\dot{\Lambda} = L[\psi]$$

holds. We can therefore express the previous formula for $\delta^2 \mathcal{E}$ as

$$\delta^2 \mathcal{E} =: - \int_{\Sigma} \psi L[\psi] dA + I + II + III,$$

where I, II, III represent the boundary integrals.

For an admissible variation, we have $\langle \dot{X}, E_3 \rangle \equiv 0$ along $\partial\Sigma$. Differentiating in the t direction gives $\langle d\dot{X}(t), E_3 \rangle \equiv 0$ along $\partial\Sigma$. The condition (10) implies that $\langle (\chi + (-1)^i \omega_i E_3), E_3 \rangle \equiv 0$ along $\partial\Sigma$ also. It follows that $I = 0$ since $\langle d\dot{X} \times (\chi + (-1)^i \omega_i E_3), \dot{X} \rangle \equiv 0$ since the three factors are coplaner.

Similarly, $III = 0$. If the variation maps $\partial\Sigma$ into $\partial\Omega$ to all orders, then $\langle \ddot{X}, E_3 \rangle \equiv 0$ and so $\langle dX \times (\chi + (-1)^i \omega_i E_3), \ddot{X} \rangle \equiv 0$ since all three factors are coplaner.

We now consider II . Recall that $t = \nu \times n$. And so $dX = t d\tilde{s}$ holds on $\partial\Sigma$. Note that

$$\dot{\chi} = d\chi(\dot{\nu}) = (D^2 F + F1)(-\nabla\psi + \nu_* \xi) = A(-\nabla\psi + \nu_* \xi).$$

Hence, we have

$$\begin{aligned} \langle t \times \dot{\chi}, \dot{X} \rangle &= -\langle t \times \dot{X}, \dot{\chi} \rangle \\ &= -\langle \psi n + (t \times \xi), A(-\nabla\psi + \nu_* \xi) \rangle \\ &= \psi \langle n, A\nabla\psi \rangle - \psi \langle n, A\nu_* \xi \rangle. \end{aligned}$$

In the third line we have used that $t \times \xi$ is normal to the surface. At any point of $\partial\Sigma$ where $\sin \eta_i$ vanishes, ψ vanishes and so the integrand of II vanishes. Now we consider any point of $\partial\Sigma$ where $\sin \eta_i$ does not vanish. On C_i , since

$$\nu = (\sin \eta_i \cos \theta, \sin \eta_i \sin \theta, -\cos \eta_i), \quad n = -(-1)^i (\cos \eta_i \cos \theta, \cos \eta_i \sin \theta, \sin \eta_i),$$

we write $E_3 = -(\cos \eta_i)\nu - (-1)^i (\sin \eta_i)n$. Then from $0 = \langle \dot{X}, E_3 \rangle$, we obtain $\langle \xi, n \rangle = -(-1)^i \psi \cot \eta_i$. Therefore, on C_i ,

$$\langle n, A\nu_* \xi \rangle = \langle \nu_* An, \xi \rangle = -\frac{k_1}{\mu_1} \langle n, \xi \rangle = (-1)^i \frac{k_1}{\mu_1} \psi \cot \eta_i. \quad (16)$$

Combining these formulas shows that (14) holds. **q.e.d.**

Definition 3.2 A capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ is said to be *stable* if the second variation $\delta^2\mathcal{E}$ of \mathcal{E} is nonnegative for all admissible variations of X , otherwise it is said to be unstable.

The analytic condition for stability is thus

Proposition 3.4 *A capillary surface $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ is stable if and only if*

$$\inf\left(-\int_{\Sigma}\psi L[\psi]d\Sigma+\oint_{\partial\Sigma}\psi B[\psi]d\tilde{s}\right)\geq 0$$

holds, where the infimum is taken over all smooth functions ψ which satisfy

(i) $\psi(w) = 0$ for $w \in \partial\Sigma$ where X is tangent to $\partial\Omega$,

and

(ii) $\int_{\Sigma}\psi d\Sigma = 0$.

4 Minimizing property of the Wulff shape

The method of this section is essentially the same as that used by Winterbottom [14] which proves a minimizing property of the part of the Wulff shape with free boundary on a plane.

Let \mathcal{F} be an anisotropic energy defined by a rotationally symmetric elliptic integrand $F = F(\nu_3)$. We let W be its Wulff shape. Let W_1 be a region in W which is included in the closed domain Ω bounded by two horizontal planes Π_j , $j = 0, 1$, with Π_0 being the lower of the two. If χ denotes the position vector of W , then we have seen that W_1 is in equilibrium for the functional

$$\mathcal{E} = \mathcal{F} + \omega_0\mathcal{A}_0 + \omega_1\mathcal{A}_1,$$

where ω_0, ω_1 are constants which are defined as follows: If $\Pi_j \cap W =: C_j$ is not empty, then $\omega_j := -(-1)^j \langle \chi, E_3 \rangle|_{C_j}$. If $\Pi_j \cap W$ is the empty set, then any real number can be chosen as ω_j . Applying the divergence theorem to the position vector X gives, $\mathcal{E}(W_1) = 3V(W_1)$ where V denotes the enclosed 3-volume.

For any compact embedded surface S with finitely many boundary components on $\Pi_0 \cup \Pi_1$, we let \underline{S} be the closed surface consisting of S together with the planer regions enclosed by $S \cap (\Pi_0 \cup \Pi_1)$. If \underline{S} encloses volume (the usual volume with respect to the Lebesgue measure) V , then we say that S encloses volume V .

We will show:

Theorem 4.1 *Assume that both of $\Pi_0 \cap W$ and $\Pi_1 \cap W$ are circles. Among all embedded surfaces S with free boundary on $\Pi_0 \cup \Pi_1$ (or without boundary) which enclose the same volume as W_1 , there holds*

$$\mathcal{E}(W_1) \leq \mathcal{E}(S). \tag{17}$$

Consequently, W_1 is stable.

More generally, we can show the following result by the same method we will use below to prove Theorem 4.1.

Theorem 4.1' *Assume that both of $\Pi_0 \cap W$ and $\Pi_1 \cap W$ are circles. Let Π'_i , $i = 0, 1$ be any pair of horizontal planes with Π'_0 , being the lower one. Consider the following energy functional for embedded surfaces with free boundary on $\Pi'_0 \cup \Pi'_1$ (or without boundary):*

$$\mathcal{E}' = \mathcal{F} + \omega_0 \mathcal{A}'_0 + \omega_1 \mathcal{A}'_1,$$

where \mathcal{A}'_j is the area of the planar region in Π'_j bounded by the boundary components of the considered surface. Assume that S is such an surface and $S \cup (\Pi'_0 \cup \Pi'_1)$ encloses the same volume as W_1 . Then there holds

$$\mathcal{E}(W_1) \leq \mathcal{E}'(S).$$

Also in the case where one of $\Pi_0 \cap W$, $\Pi_1 \cap W$ is the empty set, W_1 has a minimizing property for the energy functional in the sense of the following result. It is proved in a similar way to the proof of Theorem 4.1, and the detail is left to the reader.

Theorem 4.2 *Assume that $\Pi_0 \cap W$ (resp. $\Pi_1 \cap W$) is the empty set. Among all embedded surfaces S with free boundary (possibly the empty set) on Π_1 (resp. Π_0), and $S \cap \Pi_0$ (resp. $S \cap \Pi_1$) is the empty set, and enclose the same volume as W_1 , there holds*

$$\mathcal{E}(W_1) \leq \mathcal{E}(S). \quad (18)$$

Consequently, W_1 is stable.

Remark 4.1 If one of $\Pi_0 \cap W$, $\Pi_1 \cap W$ is the empty set, in general, it is not true that W_1 is a minimizer of the energy among all embedded surfaces S with free boundary on $\Pi_0 \cup \Pi_1$ enclosing the same volume as W_1 . In fact, the following example gives a counter example: Denote by h the distance between Π_0 and Π_1 . We consider a half sphere S_1 with boundary on $\Pi_0 \cup \Pi_1$ and a part K_1 of a cylinder with boundary on $\Pi_0 \cup \Pi_1$ both of which enclose volume V_0 . Then, it is easy to check that the area of S_1 is smaller than the area of K_1 if and only if $(2/3)^4 \pi h^3 > V_0$ holds. Therefore, in the case where $1 < h < (3/2)^{1/9}$ holds, a half sphere with radius 1 is not an energy minimizer for $F \equiv 1$ and $\omega_0 = \omega_1 = 0$. Since the Wulff shape for $F \equiv 1$ is the sphere of radius 1, this gives the counter example we wanted to have.

Remark 4.2 We can give a generalization of Theorem 4.2 analogous to the generalization of Theorem 4.1. However, we will omit its statement.

Proof of Theorem 4.1. Let v be the Gauss map of W_1 . Note that $S^2 \setminus v(W_1)$ has two components. We denote by Ω_j the component containing $(-1)^{j+1} E_3$. For $N \in S^2 \setminus \{(-1)^{j+1} E_3\}$, let $\hat{N}^{(j)}$ be the point on $v(C_j)$ which lies on the geodesic through N and E_3 . As before, for $N \in S^2$, let $\chi(N) = DF(N) + F(N) \cdot N$ and define the function

$$g(N) = \begin{cases} F(N), & N \in v(W_1), \\ \langle \chi(\hat{N}^{(j)}), N \rangle, & N \in \Omega_j \setminus \{(-1)^{j+1} E_3\}, \\ \langle \chi(\hat{E}_1^{(j)}), (-1)^{j+1} E_3 \rangle, & N = (-1)^{j+1} E_3. \end{cases}$$

Note that g is continuous on S^2 since $F(N) = \langle \chi(\hat{N}^{(j)}), N \rangle$ on C_j . The use of E_1 in the last part of the definition is somewhat arbitrary since $\langle \chi(\hat{N}^{(j)}), E_3 \rangle$ has the same value for all N .

The proof of the theorem will follow from:

Claim 1. Let \mathcal{G} be the anisotropic energy functional associated with the function g defined above. Then for any S as in the statement of the theorem, there holds

$$\mathcal{G}(\underline{S}) \leq \mathcal{E}(S). \quad (19)$$

In addition, equality holds for W_1 .

Claim 2. \underline{W}_1 is the Wulff shape of the functional \mathcal{G} , i.e.

$$\underline{W}_1 = W(g), \quad (20)$$

as defined in (2).

Assuming both these claims for now, we let S be as in the statement of the theorem. Then applying Wulff's theorem ([3]), we have

$$\mathcal{E}(W_1) = \mathcal{G}(\underline{W}_1) \leq \mathcal{G}(\underline{S}) \leq \mathcal{E}(S),$$

and the result follows.

Before proving the claims we note the useful inequalities:

$$\langle \chi(N), \nu \rangle \leq F(\nu), \quad \forall \nu, N \in S^2, \quad (21)$$

$$\langle \xi, \nu \rangle \leq F(\nu), \quad \forall \nu \in S^2, \quad \xi \in B_F, \quad (22)$$

where B_F denotes the three dimensional region interior to $W(F)$. These inequalities follow immediately from the definitions in Section 2 and the fact that $\chi(\cdot)$ is the inverse of the Gauss map of W .

Also, we consider the function on S^2 defined by

$$\alpha(\nu) := \langle \chi(\nu), E_3 \rangle = F'(\nu_3) + \nu_3/\mu_2.$$

As noted above $\partial_{\nu_3} \alpha = 1/\mu_1 > 0$ and so α is an increasing function of ν_3 .

We now prove the claims. Let N denote the Gauss map of S . We have

$$\mathcal{G}(\underline{S}) = \mathcal{G}(S \cap \{N \in v(W_1)\}) + \mathcal{G}(S \cap \{N \notin v(W_1)\}) + \mathcal{G}(\underline{S} \cap (\Pi_1 \cup \Pi_2)).$$

We have

$$\mathcal{G}(S \cap \{N \in v(W_1)\}) = \mathcal{F}(S \cap \{N \in v(W_1)\}). \quad (23)$$

Note that on the set $\{N \notin v(W_1)\}$, we have $g(N) \leq F(N)$ by (21) and the definition of g . Thus,

$$\mathcal{G}(S \cap \{N \notin v(W_1)\}) \leq \mathcal{F}(S \cap \{N \notin v(W_1)\}). \quad (24)$$

Finally, on $\underline{S} \cap \Pi_j$, we have $N \equiv (-1)^{j+1} E_3$ and so

$$\mathcal{G}(\underline{S} \cap \Pi_j) = (-1)^{j+1} \langle \chi(\hat{E}_1^{(j)}), E_3 \rangle \mathcal{A}_j(S). \quad (25)$$

By combining (23), (24) and (25), we obtain (19). Note that when $S = W_1$, $\{N \notin v(W_1)\}$ is empty and so equality holds in (19). This proves the first claim.

We move on to (20). Let $\xi \in W_1$. Then in particular $\xi \in W$ and so

$$\sup_{N \in v(W_1)} \frac{\langle \xi, N \rangle}{g(N)} = \sup_{N \in S^2} \frac{\langle \xi, N \rangle}{F(N)} = 1.$$

To show that $\xi \in W(g)$, it remains to show that $\langle \xi, N \rangle \leq g(N)$ for all $N \notin v(W_1)$. We write $\xi = \chi(\nu)$. We assume that $N \in \Omega_1$ and we write

$$N = \sqrt{1 - N_3^2 + (\hat{N}_3^{(1)})^2} \hat{N}^{(1)} + (N_3 - \hat{N}_3^{(1)})E_3. \quad (26)$$

Note that $(N_3 - \hat{N}_3^{(1)}) \geq 0$ holds. Because $\xi \in W$, we have

$$\langle \xi, \hat{N}^{(1)} \rangle \leq F(\hat{N}^{(1)}) = \langle \chi(\hat{N}^{(1)}), \hat{N}^{(1)} \rangle. \quad (27)$$

By monotonicity of α , we have

$$\langle \xi, E_3 \rangle = \alpha(\nu) \leq \alpha(\hat{N}^{(1)}) = \langle \chi(\hat{N}^{(1)}), E_3 \rangle. \quad (28)$$

Using (26) and the last two inequalities, we have

$$\begin{aligned} \langle \xi, N \rangle &= \sqrt{1 - N_3^2 + (\hat{N}_3^{(1)})^2} \langle \xi, \hat{N}^{(1)} \rangle + (N_3 - \hat{N}_3^{(1)}) \langle \xi, E_3 \rangle \\ &\leq \sqrt{1 - N_3^2 + (\hat{N}_3^{(1)})^2} \langle \chi(\hat{N}^{(1)}), \hat{N}^{(1)} \rangle + (N_3 - \hat{N}_3^{(1)}) \langle \chi(\hat{N}^{(1)}), E_3 \rangle \\ &= \langle \chi(\hat{N}^{(1)}), N \rangle \\ &= g(N). \end{aligned}$$

The case when $N \in \Omega_2$ is similar.

We next show that $\underline{W}_1 \cup \Pi_j \subset W(g)$. Let $\xi \in \underline{W}_1 \cap \Pi_1$. Note that $\underline{W}_1 \cap \Pi_1 \subset B_F$ and therefore $\langle \xi, \nu \rangle \leq F(\nu)$ for all $\nu \in S^2$. Therefore $\langle \xi, \nu \rangle \leq g(\nu)$ for all $\nu \in v(W_1)$.

Now let $N \notin v(W_1)$. We assume first again that $N \in \Omega_1$. Note that the inequality (27) still holds since $\xi \in B_F$. Also, for $\xi \in \underline{W}_1 \cap \Pi_1$, $\langle \chi(\hat{N}^{(1)}), E_3 \rangle = \langle \xi, E_3 \rangle$ for all $N \in \Omega_1$. Therefore we have equality in (28). If we express N using (26), then following the same steps as above yields $\langle \xi, N \rangle \leq g(N)$. Similar arguments cover the cases $N \in \Omega_0$ and $\xi \in \underline{W}_1 \cap \Pi_0$.

We therefore have that \underline{W}_1 is contained in $W(g)$. Since \underline{W}_1 is a closed, convex surface, $\underline{W}_1 = W(g)$ holds. **q.e.d.**

5 Anisotropic Delaunay surfaces

We will refer to a surface of revolution with constant anisotropic mean curvature for a rotationally symmetric energy functional as an *anisotropic Delaunay surface*. Such surfaces were studied in detail by the authors in [10]. We will summarize important results about them which will be required in the following sections.

Consider an anisotropic Delaunay surface Σ parameterized by

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)),$$

where $(x(s), z(s))$ is the arc length parameterization of the generating curve, and $x(s) \geq 0$ holds for all s . We have identified \mathbf{R}^3 with $\mathbf{C} \times \mathbf{R}$ in the formula above. The Gauss map of the surface X is given by

$$\nu = (z'(s)e^{i\theta}, -x'(s)).$$

We choose the orientation of the generating curve so that ν points “outward” from the surface. We denote by $X[s_1, s_2]$ the restriction $X|_{[s_1, s_2] \times S^1}$ of X . There is a natural map from the surface to the Wulff shape W defined by the requirement that the oriented tangent planes to both surfaces agree at corresponding points. Thus, if W is parameterized as

$$\chi(\sigma, \theta) = (u(\sigma)e^{i\theta}, v(\sigma)),$$

then at corresponding points the outward pointing unit normals must agree and we have

$$x' = u_\sigma, \quad z' = v_\sigma. \quad (29)$$

In [10], we showed that the profile curve (x, z) satisfies the equation

$$\frac{2xz'}{\mu_2} + \Lambda x^2 = c, \quad (30)$$

where Λ is the anisotropic mean curvature and c is a real constant called the flux parameter. Also, $-\mu_2$ is the principal curvature of the Wulff shape in the θ direction. Since W is a surface of revolution, we have $\mu_2 = \mu_2(\nu_3) = \mu_2(-u_\sigma) = \mu_2(-x')$ by (29). Computing the principal curvature $-\mu_2 = -v_\sigma/u$, (30) can be expressed as

$$2ux + \Lambda x^2 = c. \quad (31)$$

The orientation of an anisotropic Delaunay surface may be chosen so that $\Lambda \leq 0$ holds and then the anisotropic Delaunay surfaces fall into six cases as follows:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane*.
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid*.
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to vertical translation and homothety)*.
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *cylinder of radius $(\mu_2|_{\nu_3=0}|\Lambda|)^{-1}$* .
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid*.
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid*.

Any surface in each case above is complete, and it has similar properties to the corresponding CMC surface in the sense of the following Lemma.

Lemma 5.1 ([10], [11]) *(i) The generating curve $C : (x(s), z(s))$ of an anisotropic catenoid is a graph over an interval of the z -axis, and $z'(s) \neq 0$ for all s . C is perpendicular to the horizontal line at a unique point.*

(ii) Let $(x(s), z(s))$, $(x \geq 0)$, be the generating curve of an anisotropic unduloid or an anisotropic nodoid. Then, there is a unique local maximum B and a unique local minimum $N > 0$ of x , which we will call a bulge and a neck respectively.

(iii) The generating curve $C : (x(s), z(s))$ of an anisotropic unduloid is a graph over the z -axis, and $z'(s) > 0$ for all s . C is a periodic curve with respect to the vertical translation, and the region from a neck to the next neck (and/or a bulge to the next bulge) gives one period. Therefore, C has a unique inflection point (x, z) between each neck and the next bulge, which satisfies $x = \sqrt{c/(-\Lambda)}$.

(iv) The curvature of the generating curve C of an anisotropic nodoid has a definite sign. C is a non-embedding periodic curve with respect to the vertical translation. The region from a neck to the next neck (and/or a bulge to the next bulge) gives one period.

Moreover, for an anisotropic catenoid, we can prove the following:

Lemma 5.2 *The generating curve $C : (x(s), z(s))$ of an anisotropic catenoid is a graph over the whole z -axis.*

This result will be proved at the end of this section.

For our stability analysis and our proof of Lemma 5.2, we will need a representation formula for the profile curves which is summarized in the following result from [10].

Proposition 5.1 ([10]) *Let W be the Wulff shape of a rotationally symmetric anisotropic surface energy \mathcal{F} . Let*

$$\sigma \mapsto (u(\sigma), v(\sigma)), \quad \sigma \in (-\infty, \infty),$$

be the profile curve of W , where σ is the arc length. Then

$$\mu_2^{-1} v_\sigma - u = 0$$

holds. Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be a surface with constant anisotropic mean curvature $\Lambda \leq 0$, and let the Gauss map of X coincide with that of W at $s = s(\sigma)$. Then X is given as follows.

(i) *When X is an anisotropic catenoid,*

$$x = c/(2u)$$

for some nonzero constant c .

(ii) *When X is an anisotropic unduloid,*

$$x = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c > 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{\sigma | u \geq \sqrt{-\Lambda c}\}$.

(iii) *When X is an anisotropic nodoid,*

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

for some constants $c < 0$ and $\Lambda < 0$, where $x = x(u(\sigma))$ is defined in $\{-\infty < \sigma < \infty\}$.

In all cases above, z is given by

$$z = \int^u v_u x_u du. \tag{32}$$

Conversely, for a Wulff shape W defined as above, define x and z as in (i) \sim (iii) and (32). Then $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic Delaunay surface which satisfies

$$2\mu_2^{-1}z_s x + \Lambda x^2 = c,$$

where s is the arc length of (x, z) , and Λ is supposed to be zero for Case (i). Moreover, X has the same regularity as that of W .

Proof of Lemma 5.2 Assume that z is bounded. Then (since $dz = x_u v_u du = -u^{-2} v_u du$), $u^{-2} v_u du$ must be integrable near $u = 0$. Since v is a monotone function of u , this implies that $v_u/u \rightarrow 0$ as $u \rightarrow 0$. And so, $v_{uu}|_{u=0} = 0$ holds, which implies that the curvature of the generating curve (u, v) of the Wulff shape W is zero at $u = 0$. This contradicts the assumption about W . **q.e.d.**

6 Second variation for anisotropic Delaunay surfaces

In the rest of the paper, we will study the stability of embedded capillary surfaces $(\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ for $F = F(\nu_3)$. In view of Corollary 3.2, they are parts of anisotropic Delaunay surfaces. Since we have already determined the stability of parts of the Wulff shape (§4) and the cylinder ([11]), we will restrict ourselves to considering the stability of parts of the anisotropic unduloids, nodoids and catenoids. In all of these cases, a capillary surfaces $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$ has two boundary components. In this section, we will give some formulas for the second variation for these surfaces, and we will give a criterion for the stability by decomposing each admissible variation into its rotationally symmetric part and the remainder.

We may represent X as

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad s_1 \leq s \leq s_2.$$

Let

$$X_\epsilon = X + \epsilon(\xi + \psi\nu) + \mathcal{O}(\epsilon^2)$$

be a variation of X . Then,

$$\begin{aligned} L[\psi] &= x^{-1}\{(\mu_1^{-1}x\psi_s)_s + (\mu_2^{-1}x^{-1}\psi_\theta)_\theta\} + \langle Adv, d\nu \rangle \psi \\ &= x^{-1}\{(\mu_1^{-1}x\psi_s)_s + \mu_2^{-1}x^{-1}\psi_{\theta\theta}\} + \{\mu_1^{-1}(x''z' - x'z'')^2 + \mu_2^{-1}(x^{-1}z')^2\}\psi. \end{aligned} \quad (33)$$

Note that on C_i , $(x', z') = (\cos \eta_i, \sin \eta_i)$ holds. Therefore, in case where $\sin \eta_i \neq 0$,

$$\begin{aligned} B[\psi] &= \langle n, A\nabla\psi \rangle - (-1)^i (\cot \eta_i)(k_1/\mu_1)\psi \\ &= \langle n, \mu_1^{-1}\psi_s \frac{\partial}{\partial s} + \mu_2^{-1}x^{-2}\psi_\theta \frac{\partial}{\partial \theta} \rangle - (-1)^i (\cot \eta_i)((x''z' - x'z'')/\mu_1)\psi \\ &= -(-1)^i \mu_1^{-1}\{\psi_s + (\cot \eta_i)(x''z' - x'z'')\psi\} \\ &= -(-1)^i \mu_1^{-1}\{\psi_s + (x'/z')(x''z' - x'z'')\psi\} \\ &= -(-1)^i \mu_1^{-1}\{\psi_s - (z''/z')\psi\} \quad \text{on } C_i. \end{aligned} \quad (34)$$

First we consider rotationally symmetric variations. If X_ϵ is a rotationally symmetric admissible variation of X , then X_ϵ is represented as a variation C_ϵ of the generating curve

$$C(s) = (x(s), z(s)), \quad s_1 \leq s \leq s_2,$$

of X , here, the parameters $s = s_1, s_2$ correspond respectively to the curves C_0 and C_1 . Set

$$C_\epsilon = C + \epsilon(\phi C' + \psi \tilde{\nu}) + \mathcal{O}(\epsilon^2),$$

where

$$C' = (x', z'), \quad \tilde{\nu} = (z', -x').$$

Set

$$\begin{aligned} \tilde{L}[\psi] &:= (\mu_1^{-1} x \psi')' + Q \psi, \quad Q := Q_1 \cdot x, \quad Q_1 := \langle A d\nu, d\nu \rangle = \frac{(k_1)^2}{\mu_1} + \frac{(k_2)^2}{\mu_2}, \\ B_1[\psi] &:= \begin{cases} \mu_1^{-1} x(\psi' + (\cot \eta_i) k_1 \psi) = \mu_1^{-1} x\{\psi' - (z''/z')\psi\}, & \text{if } z' \neq 0, \\ x\psi, & \text{if } z' = 0 \end{cases} \end{aligned} \quad (35)$$

on C_i . For a surface of revolution generated by $(x(s), z(s))$, k_1 and k_2 are given by

$$k_1 = x''z' - x'z'', \quad k_2 = -x^{-1}z'.$$

Then, the second variation of the energy is given, from (14), (33) and (34), by

$$\mathcal{I}[\psi] = 2\pi \left(- \int_{s_1}^{s_2} \psi \tilde{L}[\psi] ds + [\psi B_1[\psi]]_{s_1}^{s_2} \right). \quad (36)$$

Next we consider general variations. Let

$$X_\epsilon = X + \epsilon(\xi + \psi \nu) + \mathcal{O}(\epsilon^2)$$

be an any admissible variation of X . Define functions φ and ζ as

$$\varphi := (2\pi)^{-1} \int_0^{2\pi} \psi d\theta, \quad \zeta := \psi - \varphi.$$

We remark that μ_1, μ_2, Q_1 , and φ are independent of θ , and ζ has a period 2π with respect to the variable θ and satisfies the following equalities.

$$\int_0^{2\pi} \zeta d\theta = \int_0^{2\pi} \zeta_s d\theta = 0. \quad (37)$$

Note that, from (14), (33), (34) and (37), we have

$$\mathcal{I}[\psi] = \begin{cases} \int_{s_1}^{s_2} \int_0^{2\pi} \left\{ \frac{(\psi_s)^2}{\mu_1} + \frac{x^{-2}(\psi_\theta)^2}{\mu_2} - Q_1 \psi^2 \right\} x d\theta ds - \int_0^{2\pi} [\mu_1^{-1} x(z''/z')(\varphi^2 + \zeta^2)]_{s_1}^{s_2} d\theta, & \text{if } z' \neq 0, \\ \int_{s_1}^{s_2} \int_0^{2\pi} \left\{ \frac{(\psi_s)^2}{\mu_1} + \frac{x^{-2}(\psi_\theta)^2}{\mu_2} - Q_1 \psi^2 \right\} x d\theta ds, & \text{if } z' = 0. \end{cases} \quad (38)$$

We obtain

Lemma 6.1

$$\int_0^{2\pi} \frac{(\psi_s)^2}{\mu_1} + \frac{x^{-2}(\psi_\theta)^2}{\mu_2} - Q_1 \psi^2 d\theta \geq \int_0^{2\pi} \frac{(\zeta_s)^2}{\mu_1} + \left(\frac{x^{-2}}{\mu_2} - Q_1 \right) \zeta^2 d\theta + 2\pi \left\{ \frac{(\varphi_s)^2}{\mu_1} - Q_1 \varphi^2 \right\}, \quad (39)$$

here equality holds if and only if $\zeta(s, \theta) \equiv a(s) \cos \theta + b(s) \sin \theta$ for some a, b which are independent of θ .

Before proving Lemma 6.1, we recall the Wirtinger's inequality (cf. [6, pp.50–51]):

Lemma 6.2 For any function $\zeta(\theta)$ with period 2π which satisfies $\int_0^{2\pi} \zeta d\theta = 0$, the following inequality holds:

$$\int_0^{2\pi} (\zeta')^2 d\theta \geq \int_0^{2\pi} \zeta^2 d\theta, \quad (40)$$

here the equality holds if and only if $\zeta(\theta) \equiv a \cos \theta + b \sin \theta$ for some a, b which are independent of θ .

Proof of Lemma 6.1. By using (37) and Lemma 6.2, we see

$$\begin{aligned} \int_0^{2\pi} \frac{(\psi_s)^2}{\mu_1} + \frac{x^{-2}(\psi_\theta)^2}{\mu_2} - Q_1 \psi^2 d\theta &= \int_0^{2\pi} \frac{(\varphi_s + \zeta_s)^2}{\mu_1} + \frac{x^{-2}(\zeta_\theta)^2}{\mu_2} - Q_1 (\varphi + \zeta)^2 d\theta \\ &= \int_0^{2\pi} \frac{(\zeta_s)^2}{\mu_1} + \frac{x^{-2}(\zeta_\theta)^2}{\mu_2} - Q_1 \zeta^2 d\theta \\ &\quad + 2 \int_0^{2\pi} \frac{\zeta_s \varphi_s}{\mu_1} - Q_1 \zeta \varphi d\theta + \int_0^{2\pi} \frac{(\varphi_s)^2}{\mu_1} - Q_1 \varphi^2 d\theta \\ &= \int_0^{2\pi} \frac{(\zeta_s)^2}{\mu_1} + \frac{x^{-2}(\zeta_\theta)^2}{\mu_2} - Q_1 \zeta^2 d\theta + \int_0^{2\pi} \frac{(\varphi_s)^2}{\mu_1} - Q_1 \varphi^2 d\theta \\ &\geq \int_0^{2\pi} \frac{(\zeta_s)^2}{\mu_1} + \left(\frac{x^{-2}}{\mu_2} - Q_1 \right) \zeta^2 d\theta + 2\pi \left\{ \frac{(\varphi_s)^2}{\mu_1} - Q_1 \varphi^2 \right\}. \end{aligned}$$

q.e.d.

Set

$$C_*^\infty[s_1, s_2] := \{u \in C^\infty([s_1, s_2]) | u(t) = 0 \text{ for } t \in \partial[s_1, s_2] \text{ where } z'(t) = 0.\},$$

$$C_*^\infty(\Sigma) := \{\psi \in C^\infty(\Sigma) | \psi \text{ vanishes at any point in } \partial\Sigma \text{ where } X \text{ is tangent to } \partial\Omega.\}.$$

For any function $\zeta \in C_*^\infty[s_1, s_2]$ and for any function $\zeta \in C_*^\infty(\Sigma)$, set

$$J[\zeta] := (\mu_1^{-1} x \zeta_s)_s - x(\mu_2^{-1} x^{-2} - Q_1) \zeta,$$

$$\mathcal{J}[\zeta] = - \int_{s_1}^{s_2} \zeta J[\zeta] ds + [\zeta B_1[\zeta]]_{s_1}^{s_2}.$$

And, for any function $\varphi \in C_*^\infty[s_1, s_2]$, set

$$\tilde{\mathcal{I}}[\varphi] = - \int_{s_1}^{s_2} \varphi \tilde{L}[\varphi] ds + [\varphi B_1[\varphi]]_{s_1}^{s_2}. \quad (41)$$

By using (38) and Lemma 6.1, we immediately obtain the following.

Lemma 6.3 For any function $\psi \in C_*^\infty(\Sigma)$, set

$$\varphi := (2\pi)^{-1} \int_0^{2\pi} \psi \, d\theta, \quad \zeta := \psi - \varphi.$$

Then,

$$\mathcal{I}[\psi] \geq \int_0^{2\pi} \mathcal{J}[\zeta] \, d\theta + 2\pi \tilde{\mathcal{I}}[\varphi]$$

holds. Here the equality holds if and only if $\zeta(s, \theta) \equiv a(s) \cos \theta + b(s) \sin \theta$ for some a, b which are independent of θ .

We will show:

Proposition 6.1 X is stable if and only if both of the following (i) and (ii) hold.

(i) For any function $u \in C_*^\infty[s_1, s_2]$,

$$\mathcal{J}[u] = - \int_{s_1}^{s_2} u J[u] \, ds + [u B_1[u]]_{s_1}^{s_2} \geq 0$$

holds.

(ii) For any function $\varphi \in C_*^\infty[s_1, s_2]$ satisfying $\int_{s_1}^{s_2} \varphi x \, ds = 0$,

$$\tilde{\mathcal{I}}[\varphi] = - \int_{s_1}^{s_2} \varphi \tilde{L}[\varphi] \, ds + [\varphi B_1[\varphi]]_{s_1}^{s_2} \geq 0$$

holds, that is, X is stable for rotationally symmetric variations.

Proof. We will prove Proposition 6.1 for the case where $z' \neq 0$ holds at any point of $\partial\Sigma$. In the case where $z' = 0$ at some points of $\partial\Sigma$, the proof is similar.

If both of (i) and (ii) hold, then, by Lemma 6.3, the stability of X holds.

We will prove the converse. First, suppose (ii) does not hold. Then, there exists a function $\varphi(s)$ on $[s_1, s_2]$, such that

$$\int_{s_1}^{s_2} \varphi x \, ds = 0, \quad \int_{s_1}^{s_2} \left\{ \frac{(\varphi')^2}{\mu_1} - Q_1 \varphi^2 \right\} x \, ds - [\mu_1^{-1} x (z''/z') \varphi^2]_{s_1}^{s_2} < 0$$

holds. Set

$$\psi(s, \theta) := \varphi(s).$$

Then,

$$\int_{\Sigma} \psi \, d\Sigma = 0, \quad \mathcal{I}[\psi] < 0$$

holds, and therefore X is unstable. Next, suppose (i) does not hold. Then, there exists a function $u(s)$ on $[s_1, s_2]$, such that

$$\int_{s_1}^{s_2} \left\{ \frac{(u')^2}{\mu_1} + \left(\frac{x^{-2}}{\mu_2} - Q_1 \right) u^2 \right\} x \, ds - [\mu_1^{-1} x (z''/z') u^2]_{s_1}^{s_2} < 0$$

holds. Set

$$\psi(s, \theta) := u(s) \sin \theta.$$

Then,

$$\int_{\Sigma} \psi \, d\Sigma = 0,$$

and, from (38),

$$\begin{aligned} \mathcal{I}[\psi] &= \int_{s_1}^{s_2} \int_0^{2\pi} \left\{ \frac{(u' \sin \theta)^2}{\mu_1} + \frac{x^{-2}(u \cos \theta)^2}{\mu_2} - Q_1 u^2 \sin^2 \theta \right\} x \, d\theta ds - \int_0^{2\pi} [\mu_1^{-1} x(z''/z') u^2]_{s_1}^{s_2} \sin^2 \theta \, d\theta \\ &= \pi \left(\int_{s_1}^{s_2} \left\{ \frac{(u')^2}{\mu_1} + \frac{x^{-2} u^2}{\mu_2} - Q_1 u^2 \right\} x ds - [\mu_1^{-1} x(z''/z') u^2]_{s_1}^{s_2} \right) < 0 \end{aligned}$$

holds, and therefore again X is unstable. **q.e.d.**

7 Eigenvalue problems associated with the second variation

Let $X : (\Sigma, \partial\Sigma) \rightarrow (\Omega, \partial\Omega)$,

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad s_1 \leq s \leq s_2,$$

be a capillary surface.

Let us define some function spaces associated with X . Denote by $L^2(\Sigma)$, (resp. $L^2([s_1, s_2])$), the usual Hilbert space completion of $C^\infty(\Sigma)$, (resp. $C^\infty([s_1, s_2])$), with respect to the norm defined by the inner product

$$(\psi_1, \psi_2)_{L^2} = \int_{\Sigma} \psi_1 \psi_2 \, d\Sigma, \quad (\varphi_1, \varphi_2)_{L^2} = \int_{s_1}^{s_2} \varphi_1 \varphi_2 \, ds,$$

respectively. We denote by $H^1(\Sigma)$, (resp. $H^1([s_1, s_2])$), the completion of $C^\infty(\Sigma)$, (resp. $C^\infty([s_1, s_2])$), with respect to the norm defined by the inner product

$$(\psi_1, \psi_2)_{H^1} = \int_{\Sigma} (\psi_1 \psi_2 + \nabla \psi_1 \nabla \psi_2) \, d\Sigma, \quad (\varphi_1, \varphi_2)_{H^1} = \int_{s_1}^{s_2} (\varphi_1 \varphi_2 + \varphi_1' \varphi_2') \, ds,$$

respectively. Here $\nabla \psi_1 \nabla \psi_2$ denotes the inner product of the gradient of ψ_1 and that of ψ_2 with respect to the Riemannian metric of Σ induced by X .

In view of Proposition 6.1, we will consider three eigenvalue problems associated with the second variation of the energy as follows:

$$L[\psi] = -\lambda \psi, \quad B[\psi]|_{\partial\Sigma} = 0, \quad \psi \in H^1([s_1, s_2] \times S^1) = H^1(\Sigma), \quad (42)$$

$$\tilde{L}[\varphi] = -\tilde{\lambda} x \varphi, \quad B_1[\varphi]|_{\partial[s_1, s_2]} = 0, \quad \varphi \in H^1([s_1, s_2]), \quad (43)$$

$$J[u] = -\lambda^J x u, \quad B_1[u]|_{\partial[s_1, s_2]} = 0, \quad u \in H^1([s_1, s_2]). \quad (44)$$

We denote the eigenvalues of L by $\lambda_1[s_1, s_2] < \lambda_2[s_1, s_2] \leq \lambda_3[s_1, s_2] \leq \dots$. Since \tilde{L} (resp. J) is a Sturm-Liouville operator, the eigenvalue problem (43) (resp. (44)) has a discrete spectrum, all

eigenvalues are real, and the multiplicity of each eigenvalue is one. We denote these eigenvalues by $\tilde{\lambda}_1[s_1, s_2] < \tilde{\lambda}_2[s_1, s_2] < \tilde{\lambda}_3[s_1, s_2] < \dots$ (resp. $\lambda_1^J[s_1, s_2] < \lambda_2^J[s_1, s_2] < \dots$). Then, each eigenfunction belonging to $\tilde{\lambda}_i[s_1, s_2]$ (resp. $\lambda_i^J[s_1, s_2]$) has exactly $i - 1$ zeros in (s_1, s_2) . And each eigenfunction of the problem (42), (43), or (44) belonging to the first eigenvalue has a definite sign in the interior of its domain.

Denote the support function of X by q , that is

$$q := \langle X, \nu \rangle = xz' - x'z.$$

Lemma 7.1

$$L[\nu_1] = L[\nu_2] = L[\nu_3] = 0, \quad (45)$$

$$L[z'] = x^{-2}\mu_2^{-1}z' \quad (46)$$

$$\tilde{L}[x'] = 0, \quad (47)$$

$$J[z'] = 0, \quad (48)$$

$$\tilde{L}[q] = -\Lambda x, \quad B_1[q] = -\frac{xz}{\mu_1 z'} k_1 = \frac{zk_1}{\mu_1 k_2} \quad (\text{for } z' \neq 0), \quad (49)$$

$$B[\nu_1] = B[\nu_2] = 0, \quad (50)$$

$$B[\nu_3] = (-1)^i \mu_1^{-1} (z')^{-1} k_1, \quad B_1[x'] = x \mu_1^{-1} (z')^{-1} k_1 = -\frac{k_1}{\mu_1 k_2} \quad (\text{for } z' \neq 0), \quad (51)$$

$$B[z'] = 0 = B_1[z']. \quad (52)$$

Proof. The Jacobi operator measures the infinitesimal variation of the anisotropic mean curvature. Since a translation does not change the anisotropic mean curvature, we have, for any constant vector $\mathbf{a} \in \mathbf{R}^3$, $L[\langle \nu, \mathbf{a} \rangle] = 0$. In particular, we have $L[\nu_i] = 0$, ($i = 1, 2, 3$). Therefore,

$$0 = L[-\nu_3] = L[x'] = x^{-1} \tilde{L}[x'].$$

Set

$$u := z'.$$

Then we have

$$0 = L[\nu_2] = L[z' \sin \theta] = [x^{-1} \{(\mu_1^{-1} x u_s)_s - \mu_2^{-1} x^{-1} u\} + Q_1 u] \sin \theta = x^{-1} (\sin \theta) J[u],$$

$$L[z'] = x^{-1} (\mu_1^{-1} x u_s)_s + Q_1 u = x^{-2} \mu_2^{-1} u.$$

Moreover we compute

$$B_1[z'] = z'' - (z''/z')z' = 0 \quad (\text{for } z' \neq 0),$$

$$B[z'] = -(-1)^i x^{-1} B_1[z'] = 0, \quad \text{on } C_i,$$

$$B[\nu_1] = B[z' \cos \theta] = -(-1)^i x^{-1} B_1[z'] \cos \theta = 0, \quad \text{on } C_i,$$

$$B[\nu_2] = B[z' \sin \theta] = -(-1)^i x^{-1} B_1[z'] \sin \theta = 0, \quad \text{on } C_i,$$

$$B[\nu_3] = B[-x'] = (-1)^i \mu_1^{-1} \{x'' - (z''/z')x'\} = (-1)^i \mu_1^{-1} (z')^{-1} (x''z' - x'z'').$$

The other equalities are obtained by some simple direct computations. **q.e.d.**

Proposition 7.1 *Assume that $z'(s) \neq 0$ for all $s \in (s_1, s_2)$. Then, X is stable if and only if X is stable for rotationally symmetric variations.*

Proof. In view of (48) and (52), z' is an eigenfunction of the problem (44), and the corresponding eigenvalue is zero. If $z'(s) \neq 0$ for all $s \in (s_1, s_2)$, then $\lambda_1^J[s_1, s_2] = 0$ holds. And so (i) in Proposition 6.1 is satisfied. Therefore, by Proposition 6.1, X is stable if and only if X is stable for rotationally symmetric variations. **q.e.d.**

When the problem (43) has zero eigenvalues, denote by E the eigenspace of zero eigenvalues, and by E^\perp its orthogonal complement in $L^2([s_1, s_2])$. In view of Proposition 7.1, the following lemma is proved by a modification of the proof of Theorem 1.3 in [8].

Lemma 7.2 *Assume that $z'(s) \neq 0$ holds for all $s \in (s_1, s_2)$. For convenience, we denote $\tilde{\lambda}_i[s_1, s_2]$ by $\tilde{\lambda}_i$.*

- (I) *If $\tilde{\lambda}_1 \geq 0$, then X is stable.*
- (II) *If $\tilde{\lambda}_1 < 0 < \tilde{\lambda}_2$, then there exists a uniquely determined function $\phi \in C_*^\infty[s_1, s_2]$ satisfying $\tilde{L}[\phi] = x$ and $B_1[\phi]|_{\partial[s_1, s_2]} = 0$, and the following (II-1) and (II-2) hold.*
 - (II-1) *If $\int_{s_1}^{s_2} \phi x ds \geq 0$, then X is stable.*
 - (II-2) *If $\int_{s_1}^{s_2} \phi x ds < 0$, then X is unstable.*
- (III) *If $\tilde{\lambda}_2 = 0$, then, for an eigenfunction φ_2 belonging to 0, the following (III-A) and (III-B) hold:*
 - (III-A) *If $\int_{s_1}^{s_2} \varphi_2 x ds \neq 0$, then X is unstable.*
 - (III-B) *If $\int_{s_1}^{s_2} \varphi_2 x ds = 0$, then there exists a uniquely determined function $\phi \in E^\perp \cap C_*^\infty[s_1, s_2]$ satisfying $\tilde{L}[\phi] = x$ and $B_1[\phi]|_{\partial[s_1, s_2]} = 0$, and the following (III-B1) and (III-B2) hold.*
 - (III-B1) *If $\int_{s_1}^{s_2} \phi x ds \geq 0$, then X is stable.*
 - (III-B2) *If $\int_{s_1}^{s_2} \phi x ds < 0$, then X is unstable.*
- (IV) *If $\tilde{\lambda}_2 < 0$, then X is unstable.*

Lemma 7.3 *Assume that $z'(s) \neq 0$ holds for all $s \in (s_1, s_2)$. Then, $\tilde{\lambda}_1[s_1, s_2] < 0$.*

Proof. We know, from Lemma 7.1, $L[\nu_1] = 0$ and $B[\nu_1] = 0$. Therefore, 0 is an eigenvalue of the problem (42). Since the number of the nodal domains is two, $\lambda_j = 0$ for some $j \geq 2$. Therefore, $\mathcal{I}[\psi] < 0$ holds for some $\psi \in C_*^\infty(\Sigma)$. Hence, from Lemma 6.3,

$$0 > \mathcal{I}[\psi] \geq \int_0^{2\pi} \mathcal{J}[\zeta] d\theta + 2\pi\tilde{\mathcal{I}}[\varphi]$$

holds, where

$$\varphi := (2\pi)^{-1} \int_0^{2\pi} \psi d\theta, \quad \zeta := \psi - \varphi.$$

On the other hand, using the same method as that was used in the proof of Proposition 7.1, $\mathcal{J}[\zeta] \geq 0$ holds. Therefore,

$$0 > \mathcal{I}[\psi] \geq 2\pi\tilde{\mathcal{I}}[\varphi]$$

holds, which implies that $\tilde{\lambda}_1[s_1, s_2] < 0$ holds. **q.e.d.**

Lemma 7.4 *Assume that X is not a part of a cylinder. Assume also that $z'(s) \neq 0$ holds for all $s \in (s_1, s_2)$. If X has no inflection point on $s_1 \leq s \leq s_2$, then $\tilde{\lambda}_i := \tilde{\lambda}_i[s_1, s_2]$ satisfies $\tilde{\lambda}_1 < 0 < \tilde{\lambda}_2$.*

Proof. Since Lemma 7.3 implies $\tilde{\lambda}_1 < 0$, we need only to prove that $\tilde{\lambda}_2 > 0$ holds.

We will prove this only for the case where X is a part of an anisotropic unduloid. For the case where X is a part of an anisotropic nodoid or catenoid, the proof is similar. We may assume that $s = 0$ corresponds to either a bulge or a neck. Let $s = s_I$, $s = -s_J$ ($s_I > 0$ and $s_J > 0$) correspond to the successive inflection points of the extension of X .

For $\hat{s} \in (s_1, s_2)$, denote by $\bar{\lambda}_1([s_1, \hat{s}])$ the first eigenvalue of the problem

$$\tilde{L}[\varphi] = -\lambda\varphi \text{ in } [s_1, \hat{s}], \quad B_1[\varphi]|_{s=s_1} = 0, \quad \varphi(\hat{s}) = 0. \quad (53)$$

And denote by $\underline{\lambda}_1([\hat{s}, s_2])$ the first eigenvalue of the problem

$$\tilde{L}[\varphi] = -\lambda\varphi \text{ in } [\hat{s}, s_2], \quad \varphi(\hat{s}) = 0, \quad B_1[\varphi]|_{s=s_2} = 0. \quad (54)$$

By applying the min-max principle, we have

$$\begin{aligned} \bar{\lambda}_1([s_1, \hat{s}]) &= \min \left\{ \left(- \int_{s_1}^{\hat{s}} \varphi \tilde{L}[\varphi] ds \right) \left(\int_{s_1}^{\hat{s}} \varphi^2 ds \right)^{-1} \mid \varphi \in C^\infty([s_1, \hat{s}]) - \{0\}, B_1[\varphi]|_{s=s_1} = 0, \varphi(\hat{s}) = 0 \right\} \\ &= \min \left\{ \left(- \int_{s_1}^{\hat{s}} \varphi \tilde{L}[p] ds \right) \left(\int_{s_1}^{\hat{s}} \varphi^2 ds \right)^{-1} \mid \varphi \in C^0([s_1, \hat{s}]) - \{0\}, \right. \\ &\quad \left. \varphi \text{ is piecewise } C^1 \text{ and piecewise } C^2 \text{ on } [s_1, \hat{s}], \right. \\ &\quad \left. B_1[\varphi]|_{s=s_1} = 0, \varphi(\hat{s}) = 0 \right\} \end{aligned} \quad (55)$$

(cf. [4]). By using (55) and the corresponding result on $\underline{\lambda}_1$, we can easily observe the monotonicity of $\bar{\lambda}_1$ and $\underline{\lambda}_1$ in the following sense.

Claim 1 If $s_1 < \sigma_0 < \sigma_1 < s_2$, then

$$\begin{aligned} \bar{\lambda}_1([s_1, \sigma_0]) &> \bar{\lambda}_1([s_1, \sigma_1]), \\ \underline{\lambda}_1([\sigma_0, s_2]) &< \underline{\lambda}_1([\sigma_1, s_2]) \end{aligned} \quad (56)$$

hold.

On the other hand, we will show the following:

Claim 2 For any $s \in (-s_J, 0)$,

$$\bar{\lambda}_1([s, 0]) > 0 \quad (57)$$

holds. And for any $s \in (0, s_I)$,

$$\underline{\lambda}_1([0, s]) > 0 \quad (58)$$

holds.

Proof of Claim 2. We will prove (57). The proof of (58) is similar. Assume $-s_0 \in (-s_J, 0)$. Let $\varphi \in C^\infty([-s_0, 0])$ be a non-constant function satisfying

$$B_1[\varphi]|_{s=-s_0} = 0, \quad \varphi(0) = 0.$$

Since x' vanishes only at $s = 0$ on $[-s_0, 0]$ and $x''(0) \neq 0$ holds, we can define a function $\zeta = \varphi/x'$ on $[-s_0, 0]$. We compute

$$\begin{aligned} - \int_{-s_0}^0 \varphi \tilde{L}[\varphi] ds &= \int_{-s_0}^0 x \mu_1^{-1} (\zeta')^2 (x')^2 ds - \zeta^2 x' B_1[x']|_{s=-s_0} \\ &= \int_{-s_0}^0 x \mu_1^{-1} (\zeta')^2 (x')^2 ds - [\zeta^2 x' x k_1 (\mu_1 z')^{-1}]|_{s=-s_0}, \end{aligned}$$

here we used (51). If $s = 0$ corresponds to a bulge, then, $x'(-s_0) > 0$ and $k_1(-s_0) < 0$ hold. If $s = 0$ corresponds to a neck, then $x'(-s_0) < 0$ and $k_1(-s_0) > 0$ hold. Therefore,

$$- \int_{-s_0}^0 \varphi \tilde{L}[\varphi] ds > 0$$

holds, which implies that $\bar{\lambda}_1([-s_0, 0]) > 0$ holds.

Now, suppose that $\tilde{\lambda}_2 := \tilde{\lambda}_2[s_1, s_2] \leq 0$ holds. We will derive a contradiction. Let e be an eigenfunction belonging to $\tilde{\lambda}_2$, that is,

$$\tilde{L}[e] = -\tilde{\lambda}_2 e, \quad B_1[e]|_{\partial[s_1, s_2]} = 0$$

hold. Then, $e(\hat{s}) = 0$ holds only for a unique $\hat{s} \in (s_1, s_2)$. Since e does not vanish in (s_1, \hat{s}) , $e|_{[s_1, \hat{s}]}$ is an eigenfunction belonging to $\bar{\lambda}_1([s_1, \hat{s}])$. Therefore,

$$\bar{\lambda}_1([s_1, \hat{s}]) = \tilde{\lambda}_2 \leq 0. \quad (59)$$

Similarly,

$$\underline{\lambda}_1([\hat{s}, s_2]) = \tilde{\lambda}_2 \leq 0. \quad (60)$$

Assume that $\hat{s} \leq 0$ holds. Then, from (57), (59) and (56), we get a contradiction. Similarly, $\hat{s} > 0$ does not hold. **q.e.d.**

8 Stability of anisotropic unduloid

Let us assume that $X : \mathbf{R} \times S^1 \rightarrow \mathbf{R}^3$, $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic unduloid for a rotationally symmetric integrand $F = F(\nu_3)$ and $F(\nu_3)$ is an even function. We may assume that $s = 0$ corresponds to either a bulge or a neck of X and $z(0) = 0$. Let s_I be the smallest positive number which corresponds to an inflection point of X .

First we will consider the case where $s = 0$ corresponds to a bulge. Let s_N be the smallest positive number which corresponds to a neck of X .

The following result determines the stability of all symmetric regions of X with respect to a bulge. It is obtained from Lemmas 8.1, 8.2, 8.3, 8.4, 8.6, and 8.5 below.

Theorem 8.1 *Assume that $F(\nu_3)$ is an even function. Let X be an anisotropic unduloid with a bulge occurring when $s = 0$.*

- (i) *If $0 < s_0 \leq s_I$, then $X[-s_0, s_0]$ is stable.*
- (ii) *If $s_0 > s_I$, then $X[-s_0, s_0]$ is unstable.*

Lemmas 8.1 and 8.2 below give the proof of (ii) of Theorem 8.1.

Lemma 8.1 *If $s_I < s_0 < s_N$, then $X[-s_0, s_0]$ is unstable.*

Proof. By using (41) and Lemma 7.1, we get

$$\tilde{\mathcal{I}}[x'] = [xx' \mu_1^{-1}(z')^{-1} k_1]_{-s_0}^{s_0}.$$

Since

$$z' > 0, \quad x'(s_0) = -x'(-s_0) < 0, \quad k_1(s_0) = k_1(-s_0) > 0,$$

we have

$$\tilde{\mathcal{I}}[x'] < 0.$$

Moreover, by the assumption,

$$\int_{-s_0}^{s_0} x'x \, ds = (1/2)[x^2]_{-s_0}^{s_0} = 0$$

holds. Therefore, from Proposition 6.1, $X[-s_0, s_0]$ is unstable. **q.e.d.**

Lemma 8.2 *If $s_0 \geq s_N$, then $X[-s_0, s_0]$ is unstable.*

Proof. For $X[-s_N, s_N]$, x' gives an eigenfunction belonging to the second eigenvalue $\tilde{\lambda}_2^0[-s_N, s_N] = 0$ of the eigenvalue problem

$$\tilde{L}[\varphi] = -\tilde{\lambda}^0 x \varphi, \quad \varphi|_{\partial[s_1, s_2]} = 0, \quad \varphi \in H^1[s_1, s_2] \quad (61)$$

for $[s_1, s_2] = [-s_N, s_N]$. Note that the problem (61) corresponds to the fixed boundary problem. If $s_0 > s_N$, then $\tilde{\lambda}_2^0[-s_0, s_0] < \tilde{\lambda}_2^0[-s_N, s_N] = 0$ holds. Therefore, there exists an admissible variation X_ϵ of $X[-s_0, s_0]$ which satisfies $X_\epsilon = X$ for $s = \pm s_0$ and gives a negative second variation of the energy. Hence, $X[-s_0, s_0]$ is unstable. If $s_0 = s_N$, then, since $z''(s_N) = 0$, it follows that $\tilde{\lambda}_2[-s_0, s_0] < \tilde{\lambda}_2^0[-s_0, s_0] = 0$ from a similar method to that used to obtain Lemma 13.3 (i). Hence, by Lemma 7.2, $X[-s_0, s_0]$ is unstable. **q.e.d.**

Lemma 8.3 *$\tilde{\lambda}_2[-s_I, s_I] = 0$ holds and $X[-s_I, s_I]$ is stable.*

Proof. Note that x' vanishes only at $s = 0$. Therefore, from (47) and (51), x' is an eigenfunction belonging to the second eigenvalue 0. Since, at any inflection point, $x = \sqrt{c/(-\Lambda)}$ holds, we have

$$\int_{-s_I}^{s_I} x'x \, ds = (1/2)[x^2]_{-s_I}^{s_I} = 0.$$

Hence, this is the case (III-B) of Lemma 7.2. We know, from Lemma 7.1, $\tilde{L}[q] = -\Lambda x$ and $B_1[q]|_{\partial[-s_I, s_I]} = 0$ hold. Since $q = xz' - x'z > 0$ holds for $-s_I \leq s \leq s_I$, $\int_{-s_I}^{s_I} qx \, ds > 0$ holds. Now the stability follows from $\Lambda < 0$. **q.e.d.**

Let w be the even Jacobi field which satisfies

$$\tilde{L}[w] = 0, \quad w(0) = 1, \quad w'(0) = 0.$$

Since x' is an eigenfunction belonging to the second eigenvalue zero of the Dirichlet boundary value problem in $-s_N \leq s \leq s_N$, there exists a unique value $\alpha \in (0, s_N)$ such that

$$w(s) \begin{cases} > 0, & 0 \leq s < \alpha, \\ = 0, & s = \alpha, \\ < 0, & \alpha < s \leq s_N. \end{cases}$$

Set

$$\rho := x\mu_1^{-1}.$$

w is represented as

$$w(s) = \begin{cases} c_1 x'(s) + c_2 x'(s) \int_{s_I}^s \frac{ds}{\rho(x')^2}, & 0 \leq s \leq s_N, \\ -c_1 x'(s) + c_2 x'(s) \int_{-s_I}^s \frac{ds}{\rho(x')^2}, & -s_N \leq s \leq 0, \end{cases} \quad (62)$$

where c_1, c_2 are determined as follows. We observe

$$\begin{aligned} \lim_{s \rightarrow +0} \left(x'(s) \int_{s_I}^s \frac{ds}{\rho(x')^2} \right) &= \lim_{s \rightarrow +0} \frac{-1}{\rho x''} = -\mu_1(0)(x(0)x''(0))^{-1} > 0, \\ \lim_{s \rightarrow -0} \left(x'(s) \int_{-s_I}^s \frac{ds}{\rho(x')^2} \right) &= \lim_{s \rightarrow -0} \frac{-1}{\rho x''} = -\mu_1(0)(x(0)x''(0))^{-1} > 0. \end{aligned}$$

Hence, from $w(0) = 1$ and $x'(0) = 0$,

$$c_2 = -(\mu_1(0))^{-1} x(0)x''(0) > 0 \quad (63)$$

holds. c_1 is determined from $w'(0) = 0$.

Lemma 8.4 *Assume that $\alpha \leq s_I$ holds. Then, for any $s_0 \in (0, \alpha)$, $X[-s_0, s_0]$ is stable.*

Proof. Since $w(s) > 0$ for all $s \in [-s_0, s_0]$, any function $\phi : [-s_0, s_0] \rightarrow \mathbf{R}$ is represented as $\phi = \zeta w$. we compute

$$\tilde{I}[\phi] = - \int_{-s_0}^{s_0} \phi \tilde{L}[\phi] ds + [\phi B_1[\phi]]_{-s_0}^{s_0} = \int_{-s_0}^{s_0} x\mu_1^{-1}(\zeta')^2 w^2 ds + [\zeta^2 w B_1[w]]_{-s_0}^{s_0}. \quad (64)$$

We divide the situation into the following two cases:

Case(1) $B_1[w]|_{s=s_0} = -B_1[w]|_{s=-s_0} \geq 0$.

Case(2) $B_1[w]|_{s=s_0} = -B_1[w]|_{s=-s_0} < 0$.

By Lemma 7.4, $\tilde{\lambda}_1[-s_0, s_0] < 0 < \tilde{\lambda}_2[-s_0, s_0]$ holds. Therefore, Case(1) does not occur. Therefore Case(2) holds. On the other hand, since $0 < s_0 < s_I$, from (49),

$$B_1[q]|_{s=s_0} > 0, \quad B_1[q]|_{s=-s_0} < 0$$

holds. Therefore, we can take $\beta > 0$ so that $\varphi := q + \beta w$ satisfies

$$B_1[\varphi]|_{s=\pm s_0} = 0.$$

In this case,

$$\tilde{L}[\varphi] = \tilde{L}[q] = -\Lambda x > 0, \quad \int_{-s_0}^{s_0} \varphi x \, ds = \int_{-s_0}^{s_0} (q + \beta w)x \, ds > 0$$

holds because $q, \beta, w, x > 0$. Hence, by Lemma 7.2, $X[-s_0, s_0]$ is stable. **q.e.d.**

Similarly, we obtain the following:

Lemma 8.5 *Assume that $\alpha > s_I$ holds. Then, for any $s_0 \in (0, s_I)$, $X[-s_0, s_0]$ is stable.*

The following lemma completes the proof of Theorem 8.1.

Lemma 8.6 *Assume that $\alpha \leq s_0 < s_I$ holds. Then $X[-s_0, s_0]$ is stable.*

Before proving Lemma 8.6, we compute, for $0 < s < s_N$,

$$w' = c_1 x'' + c_2 x'' \int_{s_I}^s \frac{ds}{x \mu_1^{-1} (x')^2} + \frac{c_2}{x \mu_1^{-1} x'}, \quad (65)$$

$$z' w' - z'' w = (c_1 + c_2 \int_{s_I}^s \frac{ds}{x \mu_1^{-1} (x')^2}) (x'' z' - x' z'') + \frac{c_2 z' \mu_1}{x x'} = \frac{1}{x x'} (w k_1 x + c_2 \mu_1 z'), \quad (66)$$

$$B_1[w] = \frac{x \mu_1^{-1}}{z'} (z' w' - z'' w) = \frac{1}{\mu_1 x' z'} (w k_1 x + c_2 \mu_1 z'). \quad (67)$$

Proof of Lemma 8.6. Note that, for $0 < s < s_I$, $k_1(s) < 0$ and $z''(s) < 0$ hold. Since $\alpha \leq s_0 < s_I$, from (66), at $s = s_0$,

$$B_1[w]|_{s=s_0} < 0 \quad (68)$$

holds. On the other hand, from (49), $B_1[q]|_{s=s_0} > 0$ holds. Therefore, we can take $\beta > 0$ so that $\varphi := q + \beta w$ satisfies $B_1[\varphi]|_{s=\pm s_0} = 0$.

First we will prove the following:

Claim 8.1

$$\varphi(s_0) > 0$$

holds.

Proof. We compute

$$\begin{aligned} \varphi' &= x z'' - x'' z + \beta \left(c_1 x'' + c_2 x'' \int_{s_I}^s \frac{\mu_1}{x (x')^2} ds + \frac{c_2 \mu_1}{x x'} \right) \\ &= x z'' + \frac{x''}{x'} (-x' z + \beta w) + \frac{\beta c_2 \mu_1}{x x'}. \end{aligned} \quad (69)$$

From (49) and (66), we have at $s = s_0$,

$$0 = B_1[\varphi] = \frac{x k_1}{\mu_1 x' z'} (-x' z + \beta w) + \frac{\beta c_2}{x'}. \quad (70)$$

By using (69) and (70), at $s = s_0$, we obtain

$$\varphi' = xz'' - x'' \cdot \frac{\beta c_2 \mu_1 z'}{xx'k_1} + \frac{\beta c_2 \mu_1}{xx'} = z'' \left(x - \frac{\beta c_2 \mu_1}{xk_1} \right) < 0. \quad (71)$$

Recall

$$B_1[\varphi] = x\mu_1^{-1}(\varphi' - \frac{z''}{z'}\varphi).$$

Since $B_1[\varphi]|_{s=s_0} = 0$, by using (71), we obtain

$$\varphi(s_0) = ((z'/z'')\varphi')|_{s=s_0} > 0.$$

q.e.d.

In view of Lemmas 7.2 and 7.4, in order to prove Lemma 8.6, it is sufficient to prove the following:

Claim 8.2

$$\int_{-s_0}^{s_0} \varphi x \, ds > 0$$

holds.

Proof. Note that

$$\tilde{L}[\varphi] = \tilde{L}[q + \beta w] = \tilde{L}[q] + \beta \tilde{L}[w] = -\Lambda x$$

holds. Therefore,

$$\int_0^{s_0} \varphi x \, ds = \int_0^{s_0} (q + \beta w)x \, ds = \int_0^{s_0} qx \, ds + \beta/(-\Lambda) \int_0^{s_0} w \tilde{L}[\varphi] \, ds \quad (72)$$

holds. We compute

$$\int_0^{s_0} w \tilde{L}[\varphi] \, ds = [x(\mu_1)^{-1}(w\varphi' - w'\varphi)]_0^{s_0} = [wB_1[\varphi] - \varphi B_1[w]]_0^{s_0} = -(\varphi B_1[w])|_{s=s_0}, \quad (73)$$

here we used $w''(0) = z''(0) = 0$. By using (68), Claim 8.1, and (73), we obtain

$$\int_0^{s_0} w \tilde{L}[\varphi] \, ds > 0. \quad (74)$$

Note that φx is an even function. Since $q > 0$ in $0 \leq s \leq s_0$, $\beta > 0$, and $\Lambda < 0$ hold, from (72) and (74), we obtain the desired result. **q.e.d.**

We will now consider the stability of anisotropic unduloids which are symmetric with respect to a neck situated on the plane $z = 0$. Thus, we assume that $s = 0$ corresponds to a neck of X . Let s_B be the smallest positive number which corresponds to a bulge of X .

Proposition 8.1 *Assume that $F(\nu_3)$ is an even function. Let X be an anisotropic unduloid with a neck occurring when $s = 0$. If $s_0 > s_I$, then $X[-s_0, s_0]$ is unstable.*

Proof. First we assume that $s_I < s_0 < s_B$ holds. By using (41) and Lemma 7.1, we obtain

$$\tilde{\mathcal{I}}[x'] = [xx'\mu_1^{-1}(z')^{-1}k_1]_{-s_0}^{s_0}.$$

Since

$$z' > 0, \quad x'(s_0) = -x'(-s_0) > 0, \quad k_1(s_0) = k_1(-s_0) < 0,$$

we have

$$\tilde{\mathcal{I}}[x'] < 0.$$

Moreover, by the assumption,

$$\int_{-s_0}^{s_0} x'x \, ds = (1/2)[x^2]_{-s_0}^{s_0} = 0$$

holds. Therefore, from Proposition 6.1, $X[-s_0, s_0]$ is unstable.

Next we assume that $s_0 \geq s_B$ holds. Then, the instability of $X[-s_0, s_0]$ can be proved by a similar way to the proof of Lemma 8.2. **q.e.d.**

Lemma 8.7 (i) $\tilde{\lambda}_2[-s_I, s_I] = 0$ holds.

(ii) $X[-s_I, s_I]$ is stable if and only if $\int_{-s_I}^{s_I} qx \, ds \geq 0$ holds.

Proof. Note that x' vanishes only at $s = 0$. Therefore, from (47) and (51), x' is an eigenfunction belonging to the second eigenvalue 0. Since, at any inflection point, $x = \sqrt{c/(-\Lambda)}$ holds, we have

$$\int_{-s_I}^{s_I} x'x \, ds = (1/2)[x^2]_{-s_I}^{s_I} = 0.$$

Hence, this is the case (III-B) in Lemma 7.2. We have, from Lemma 7.1, $\tilde{L}[q] = -\Lambda$ and $B_1[q]|_{\partial[-s_I, s_I]} = 0$. Therefore, $X[-s_I, s_I]$ is stable if and only if $\int_{-s_I}^{s_I} qx \, ds \geq 0$ holds. **q.e.d.**

We will compute $\int_{-s_I}^{s_I} qx \, ds$.

$$\begin{aligned} \int_{-s_I}^{s_I} qx \, ds &= \int_{-s_I}^{s_I} (xz' - x'z)x \, ds = 3 \int_0^{s_I} x^2 \, dz - (x^2z)|_{s=s_I} \\ &\geq z(s_I) \left\{ \left(3 \min_{0 \leq s \leq s_I} x \right)^2 - (x(s_I))^2 \right\} = z(s_I) \left\{ 3(x(0))^2 - (x(s_I))^2 \right\}. \end{aligned}$$

Therefore, at least, if the surface is near to a cylinder, then $\int_{-s_I}^{s_I} qx \, ds > 0$ and so it is stable. Letting $z' = 1$ in (30), we have

$$x(0) = (-\Lambda)^{-1} \left\{ (\mu_2^{-1}|_{\nu_3=0}) - \sqrt{(\mu_2^{-1}|_{\nu_3=0})^2 + \Lambda c} \right\}.$$

Recall that

$$x(s_I) = \sqrt{c/(-\Lambda)}$$

holds (cf. Lemma 5.1). Set

$$A := \mu_2^{-1}|_{\nu_3=0}.$$

Then,

$$(-\Lambda)^2 \left\{ 3(x(0))^2 - (x(s_I))^2 \right\} = 3(A - \sqrt{A^2 + \Lambda c})^2 + \Lambda c = 2(A - 2\sqrt{A^2 + \Lambda c})(A - \sqrt{A^2 + \Lambda c}),$$

and $A - \sqrt{A^2 + \Lambda c} > 0$ holds. So, if $A - 2\sqrt{A^2 + \Lambda c} \geq 0$, then the surface is stable. We have obtained the following.

Lemma 8.8 *If*

$$c \geq (3/4)((\mu_2^{-1}|_{\nu_3=0})^2|\Lambda|)^{-1},$$

then $X[-s_I, s_I]$ is stable.

Remark 8.1 (i) Because of (3), $\mu_2^{-1}|_{\nu_3=0} = F|_{\nu_3=0}$. Therefore, for CMC surfaces, $\mu_2^{-1}|_{\nu_3=0} = 1$.
(ii)

$$c < ((\mu_2^{-1}|_{\nu_3=0})^2|\Lambda|)^{-1}$$

holds, and $c = ((\mu_2^{-1}|_{\nu_3=0})^2|\Lambda|)^{-1}$ gives a cylinder (cf. Section 5). Lemma 8.8 implies that, if the anisotropic unduloid X is “sufficiently near” to the cylinder, then $X[-s_I, s_I]$ is stable.

Example 8.1 We now give an example that shows that when $s = 0$ corresponds to a neck of an anisotropic unduloid, the stability of $X[-s_I, s_I]$ for a fixed value of Λ depends on the value of c . This is in marked contrast to the case where the bulge is located at $s = 0$.

We consider anisotropic unduloids for the functional whose Wulff shape is generated by the curve $u^4 + v^2 = 1$, (see Figure 4 below). The curvature of this Wulff shape is positive except at points where $u = 0$. However the image $X[-\infty, \infty]$ is contained in a band centered around the circle $v = 0$ so the Wulff shape may be perturbed near the points where $u = 0$ to a uniformly convex one without changing the values of the calculations given below.

For $\Lambda = -1/2$, let

$$I(c) = \int_0^{s_I} qx \, ds.$$

As shown above, $X[-s_I, s_I]$ is stable if and only if $I(c) \geq 0$ holds. Numerical calculations give

$$I(0.25) = -.03530099746, \quad I(0.50) = -.0835067556, \quad I(0.8) = -.0182154406,$$

$$I(0.82557) = -.0001892275, \quad I(0.9) = .068234681, \quad I(1.0) = .204462164.$$

This indicates that $X[-s_I, s_I]$ is stable if and only if $c \geq c_0 \approx 0.82557$ holds.

9 Stability of anisotropic nodoid

Let us assume that $X : \mathbf{R} \times S^1 \rightarrow \mathbf{R}^3$, $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic nodoid for a rotationally symmetric integrand $F = F(\nu_3)$ and $F(\nu_3)$ is an even function. We may assume that $s = 0$ corresponds to a bulge of X and $z(0) = 0$. Let s_I (resp. s_N) be the smallest positive number which corresponds to a point where z' vanishes (resp. a neck of X).

The following result will be shown, which means that any embedded capillary surface $X[-s_0, s_0]$ of the type described above between two parallel planes is stable.

Theorem 9.1 *Assume that $F(\nu_3)$ is an even function. Let X be an anisotropic nodoid with a bulge occurring when $s = 0$. Then, for any $s_0 \in (0, s_I]$, $X[-s_0, s_0]$ is stable.*

Let w be the even Jacobi field which satisfies

$$\tilde{L}[w] = 0, \quad w(0) = 1, \quad w'(0) = 0.$$

Lemma 9.1 *There exists a unique value $\alpha \in (0, s_I)$ such that*

$$w(s) \begin{cases} > 0, & 0 \leq s < \alpha, \\ = 0, & s = \alpha, \\ < 0, & \alpha < s \leq s_N. \end{cases} \quad (75)$$

Proof. Denote by $\lambda_j^D[s_1, s_2]$ the j th eigenvalue of the problem

$$L[\psi] = -\lambda^D \psi, \quad \psi|_{\partial[s_1, s_2]} = 0, \quad \psi \in H^1([s_1, s_2] \times S^1) = H^1(\Sigma). \quad (76)$$

Denote by $\tilde{\lambda}_j^D[s_1, s_2]$ the j th eigenvalue of the problem

$$\tilde{L}[\varphi] = -\tilde{\lambda}^D x \varphi, \quad \varphi|_{\partial[s_1, s_2]} = 0, \quad \varphi \in H^1([s_1, s_2]). \quad (77)$$

And denote by $\lambda_j^{J,D}[s_1, s_2]$ the j th eigenvalue of the problem

$$J[\eta] = -\lambda^{J,D} x \eta, \quad \eta|_{\partial[s_1, s_2]} = 0, \quad \eta \in H^1([s_1, s_2]). \quad (78)$$

Since, because of (47), x' is an eigenfunction belonging to $\tilde{\lambda}_2^D[-s_N, s_N] = 0$, there exists a unique value $\alpha \in (0, s_N)$ such that (75) holds.

On the other hand, from (45), $L[\nu_1] = 0$. Since $z'|_{\partial X[-s_I, s_I]} = 0$ and the number of the nodal domains of ν_1 is two, $\lambda_2^D[-s_I, s_I] \leq 0$ holds. Therefore, $\lambda_1^D[-s_I, s_I] < 0$ holds. Let ψ be an eigenfunction belonging to $\lambda_1^D[-s_I, s_I]$. We know, from Lemma 6.3,

$$0 > \mathcal{I}[\psi] \geq \int_0^{2\pi} \mathcal{J}[\zeta] d\theta + 2\pi \tilde{\mathcal{I}}[\varphi]$$

holds. However, from (48), z' is an eigenfunction belonging to $\lambda_1^{J,D}[-s_I, s_I] = 0$, and so

$$\mathcal{J}[\zeta] \geq 0$$

holds. Therefore,

$$0 > 2\pi \tilde{\mathcal{I}}[\varphi],$$

which implies that $\tilde{\lambda}_1^D[-s_I, s_I] < 0$. Consequently we have $\alpha \in (0, s_I)$. **q.e.d.**

Proof of Theorem 9.1. In view of Lemmas 7.4 and 9.1, the result follows by a similar way to the proofs of Lemmas 8.4, 8.6. **q.e.d.**

10 Stability of the anisotropic catenoid

We consider an anisotropic catenoid \mathcal{C} for a rotationally symmetric even integrand $F = F(\nu_3)$. \mathcal{C} is parametrized as $X : \mathbf{R} \times S^1 \rightarrow \mathbf{R}^3$, $X(s, \theta) = (x(s)e^{i\theta}, z(s))$. We may assume that $s = 0$ corresponds to the neck of X and $z(0) = 0$. We denote by N the neck size $x(0)$.

We will show the following criterion for the stability:

Theorem 10.1 *Let $X[s_1, s_2]$, $s_1 < s_2$ be a domain in an anisotropic catenoid. Then $X[s_1, s_2]$ is stable for the free boundary problem if and only if the expression*

$$\frac{\pi}{6} \left\{ 9 \left(\int_{s_1}^{s_2} x^2 dz \right)^2 - 5(z(s_2) - z(s_1)) \int_{s_1}^{s_2} x^4 dz \right\} \quad (79)$$

is non negative.

This is a generalization of a result which was proved by Zhou [15] in the isotropic case. In fact, in the isotropic case, (79) reduces to (1) in Zhou [15] in the following way: If the (isotropic) catenoid is given by $(x, z) = (\cosh z, z)$, then the normal is $\nu = (\coth z e^{i\alpha}, -\tanh z) =: (\sin \theta e^{i\alpha}, \cos \theta)$. Since $\tanh z = \cos \theta$, we have $dz / \cosh^2 z = \sin \theta d\theta$, so $dz = d\theta / \sin \theta$. Using $1/x = \sin \theta$, it is easy to transform the integrals in Zhou [15] into the ones appearing in (79).

Later in this section, we will show Theorem 10.1 by a different method than the one given in Zhou [15].

We note that in [15], Zhou produced examples of domains in an (isotropic) catenoid which show that the property of monotonicity does not hold for stability of domains for the free boundary problem with wetting. Specifically, an unstable domain may be a subset of a stable one. We do however have the following.

Corollary 10.1 *Let X be an immersion of an anisotropic catenoid. For s_1 fixed and all $s > 0$ sufficiently small, $X[s_1, s_1 + s]$, $X[s_1 - s, s_1]$ are stable, while for all $s > 0$ sufficiently large, $X[s_1, s_1 + s]$, $X[s_1 - s, s_1]$ are unstable.*

The proof of this result will be given after the proof of Theorem 10.1.

In view of Lemmas 7.2, 7.4, in order to determine the stability of $X[s_1, s_2]$, it is sufficient to know the sign of the integral $\int_{s_1}^{s_2} \phi x ds$ for a function ϕ satisfying $\tilde{L}[\phi] = x$ and $B_1[\phi]|_{\partial[s_1, s_2]} = 0$. Here, instead of using Lemma 7.4, we will prove that $\lambda_2[s_1, s_2]$ vanishes directly by comparing it with the corresponding eigenvalue for a part of the Wulff shape, and, by using this, we will show that $\tilde{\lambda}_2[s_1, s_2] > 0$ holds. First, we will prove an interesting fact that the anisotropic Gauss map of the anisotropic catenoid is conformal, which will be used to prove $\lambda_2[s_1, s_2] = 0$.

Proposition 10.1 *The anisotropic Gauss map*

$$\chi : \mathcal{C} \rightarrow W$$

$$p \mapsto DF|_{\nu(p)} + F\nu(p)$$

is conformal. If $\sigma \mapsto (u(\sigma), v(\sigma))$ is the arc length parameterization of the profile curve of W , then the metrics $ds_{\mathcal{C}}^2$ on \mathcal{C} and ds_W^2 on W are related by

$$ds_{\mathcal{C}}^2 = \frac{c^2}{4} u^{-4} ds_W^2,$$

here c is the “flux parameter” given by $2ux = c$.

Proof. With the usual notation,

$$ds_C^2 = ds^2 + x^2 d\theta^2 = ds^2 + \left(\frac{c}{2u}\right)^2 d\theta^2.$$

We have $x_\sigma = x_u u_\sigma = -(c/2)u^{-2}u_\sigma$ and $z_\sigma = z_s s_\sigma = v_\sigma x_u = -(c/2)u^{-2}v_\sigma$. Thus $ds = \sqrt{x_\sigma^2 + z_\sigma^2} d\sigma = (|c|/2)u^{-2}d\sigma$. Using this above gives

$$ds_C^2 = \left(\frac{c^2}{4}\right)u^{-4}d\sigma^2 + \left(\frac{c^2}{4}\right)u^{-2}d\theta^2 = \left(\frac{c^2}{4}\right)u^{-4}(d\sigma^2 + u^2 d\theta^2) = \left(\frac{c^2}{4}\right)u^{-4}ds_W^2.$$

q.e.d.

Remark 10.1 It can be shown that, in general, the Gauss map of an anisotropic Delaunay surface is harmonic.

Proposition 10.2 *Let $X : \mathbf{R} \times S^1 \rightarrow \mathbf{R}^3$ be an anisotropic catenoid. Then, for any $s_1 < s_2$, $\lambda_2[s_1, s_2] = 0$ holds.*

Proof. We know, from Lemma 7.1, $L[\nu_1] = 0$ and $B[\nu_1] = 0$ hold. Therefore, 0 is an eigenvalue of the problem (42). Since the number of the nodal domains is two, $\lambda_j := \lambda_j[s_1, s_2] = 0$ for some $j \geq 2$. Therefore, in order to prove $\lambda_2 = 0$, it is sufficient to show $\lambda_2 \geq 0$.

The image of $X[s_1, s_2]$ under the anisotropic Gauss map is a domain $\chi[\sigma_1, \sigma_2]$ in W which is the minimizer for an appropriate free boundary problem by Theorem 4.1. If L^W denotes the Jacobi operator for the Wulff shape, then we claim that, acting on functions $f = f(s)$,

$$L = \left(\frac{4}{c^2}\right)u^4 L^W \tag{80}$$

holds. Since $4u^4/c^2 = -K_\Sigma/K_W$, this generalizes the classical identity

$$L = \Delta - 2K = (-K)(\hat{\Delta} + 2)$$

between the stability operator of a minimal surface and the Laplacian $\hat{\Delta}$ on S^2 .

We may rescale the immersion so that $c = 2$ holds. Note that by the previous proposition $\sigma_s = u^2 = -k_1/\mu_1 = k_2/\mu_2$ holds. Using this, we obtain

$$\begin{aligned} L[f] &= \frac{1}{x} \left(\frac{x}{\mu_1} f_s \right)_s + \left(\frac{k_1^2}{\mu_1} + \frac{k_2^2}{\mu_2} \right) f = u \sigma_s \left(\frac{x}{\mu_1} \sigma_s f_\sigma \right)_\sigma + u^4 (\mu_1 + \mu_2) \\ &= u^4 \left(\frac{1}{u} \left(\frac{u}{\mu_1} f_\sigma \right)_\sigma - 2H_W f \right) = u^4 L^W[f]. \end{aligned}$$

Also the boundary operator is transformed according to

$$\begin{aligned} B[f] &= -(-1)^i \mu_1^{-1} \left(f_s - \left(\frac{z_{ss}}{z_s} f \right) \right) = -(-1)^i \mu_1^{-1} \left(\sigma_s f_\sigma - \left(\frac{v_{\sigma\sigma}}{v_\sigma} \right) \sigma_s f \right) \\ &= u^2 B^W[f], \end{aligned}$$

using $v_\sigma = z_s$, where B^W is the boundary operator for the Wulff shape.

Therefore, if $\lambda_2 < 0$ holds, then there exists f_i , $i = 1, 2$ on $[\sigma_1, \sigma_2] \times S^1$ with $L^W[f_i] + u^{-4}\lambda_i f_i = 0$ which satisfies $B^W[f_i] = 0$ on the boundary. Also, since the eigenvalues are distinct and the boundary value problem is self-adjoint, the $\{f_1, f_2\}$ may be assumed to be orthonormal in $L^2([\sigma_1, \sigma_2] \times S^1, d\Sigma)$.

For any smooth g on $[\sigma_1, \sigma_2] \times S^1$ we can find a linear combination $f = af_1 + bf_2$ which is orthogonal to g on $L^2([\sigma_1, \sigma_2] \times S^1, d\Sigma_W)$, where $d\Sigma_W$ is the area form of the Wulff shape W . We then have

$$\begin{aligned} - \int f L^W[f] d\Sigma_W &= \int f(a\lambda_1 u^{-4} f_1 + b\lambda_2 u^{-4} f_2) d\Sigma_W \\ &= \int (af_1 + bf_2)(a\lambda_1 f_1 + b\lambda_2 f_2) d\Sigma \\ &= (a^2\lambda_1 + b^2\lambda_2) < 0. \end{aligned}$$

We thus obtain that $\chi[\sigma_1, \sigma_2]$ is unstable which clearly contradicts Theorem 4.1. **q.e.d.**

Corollary 10.2 *Under the same assumption in Proposition 10.2, $\tilde{\lambda}_2[s_1, s_2] > 0$ holds.*

Proof. It is clear that, from Proposition 10.2, $\tilde{\lambda}_2 := \tilde{\lambda}_2[s_1, s_2] \geq 0$ holds. Assume now that $\tilde{\lambda}_2 = 0$ holds. Let φ be an eigenfunction belonging to $\tilde{\lambda}_2$. Note that by Lemma 7.1, x' and q are linearly independent solutions of $\tilde{L} = 0$. Hence, φ is represented as $\varphi = ax' + bq$, where a, b are constants. However, from (84) below, $B_1[ax' + bq]|_{s_1} = B_1[ax' + bq]|_{s_2} = 0$ implies that $a = b = 0$, which is a contradiction. **q.e.d.**

Let

$$V(s) := \pi \int_0^s x^2 dz = \pi \int_0^s x^2 z' ds$$

be \pm the volume inside $X[0, s]$.

Lemma 10.1 *An even, rotationally symmetric solution of $L[\phi] = 1$ is given by*

$$\phi = \frac{1}{2\delta} \left[\{x^2 z - (3/\pi)V\} x' + x^2 q \right] = \frac{1}{2\delta} \{x^3 z' - (3/\pi)V x'\},$$

where

$$\delta = -N^2 x''(0)/\mu_1(0) < 0. \quad (81)$$

Proof. Note that by Lemma 7.1, we have $L[x'] = 0 = L[q]$. The solution of $L[\phi] = 1$ can be found by variation of parameters using the odd Jacobi field x' and the even Jacobi field q . As usual, one looks for unknown functions α, β satisfying

$$\phi = \alpha x' + \beta q, \quad \alpha' x' + \beta' q = 0.$$

Using the notation of Chapter 6, the equation $L[\phi] = 1$, is equivalent to the equation

$$\left(\frac{x}{\mu_1} \phi' \right)' + Q\phi = x.$$

Straightforward calculations then yield that α, β satisfy

$$\begin{pmatrix} x' & q \\ x'' & q' \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_1 \end{pmatrix}. \quad (82)$$

Because both x' and q are solutions of $L = 0$,

$$x'q' - x''q = (\mu_1\delta)/x \quad (83)$$

for a constant δ , so that the previous system of equations has solution $\alpha' = -(qx)/\delta$, $\beta' = (xx')/\delta$. Thus, $\beta = x^2/(2\delta)$ and

$$\begin{aligned} \alpha &= -\frac{1}{\delta} \int_0^s x(xz' - x'z) ds = -\frac{1}{\delta} \left\{ \frac{V(s)}{\pi} - \frac{1}{2} \left(zx^2 - \int_0^s x^2 dz \right) \right\} \\ &= -\frac{1}{\delta} \left(\frac{V(s)}{\pi} - \frac{zx^2}{2} + \frac{V(s)}{2\pi} \right) = \frac{1}{2\delta} \left(x^2z - \frac{3V(s)}{\pi} \right). \end{aligned}$$

The numerical value (81) of the constant δ can be obtained by setting $s = 0$ in (83) using $x'(0) = 0$. **q.e.d.**

Proof of Theorem 10.1. By using the formulas (cf. Lemma 7.1)

$$B_1[x'] = -\frac{k_1}{\mu_1 k_2}, \quad B_1[q] = \frac{zk_1}{\mu_1 k_2}, \quad (84)$$

we get

$$B_1[\phi] = \alpha B_1[x'] + \beta B_1[q] = \frac{3V}{2\delta\pi} \frac{k_1}{\mu_1 k_2}. \quad (85)$$

Using the formulas (84) and (85), we seek a solution of $L[g] = \text{constant}$ on $[s_1, s_2]$ with $B_1[g] = 0$ on $\partial[s_1, s_2]$, of the form

$$g = a_1\phi + a_2q + a_3x'.$$

Since $\frac{k_1}{\mu_1 k_2}$ does not vanish on \mathcal{C} , we have

$$a_1 \left(\frac{3V(s_i)}{2\pi\delta} \right) + a_2 z(s_i) - a_3 = 0, \quad i = 1, 2. \quad (86)$$

A calculation yields, using (86),

$$\begin{aligned} \int_{s_1}^{s_2} gx ds &= \frac{a_1}{2\delta} \left(\frac{5}{2} \int_{s_1}^{s_2} x^4 dz - \frac{3}{2\pi} x^2 V|_{s_1}^{s_2} \right) + a_2 \left(\frac{3}{2} \int_{s_1}^{s_2} x^2 dz - \frac{1}{2} zx^2|_{s_1}^{s_2} \right) + a_3 \left(\frac{x^2}{2} |_{s_1}^{s_2} \right) \\ &= \frac{5a_1}{4\delta} \int_{s_1}^{s_2} x^4 dz + \frac{3a_2}{2} \int_{s_1}^{s_2} x^2 dz. \end{aligned} \quad (87)$$

A solution of (86) is given by

$$(a_1, a_2, a_3) = \left(\frac{-2\pi\delta}{3} (z(s_2) - z(s_1)), V(s_2) - V(s_1), V(s_2)z(s_1) - V(s_1)z(s_2) \right). \quad (88)$$

Note that if $s_2 > s_1$, then $z(s_2) > z(s_1)$. Then $a_1 > 0$ holds so that $L[g] = a_1$ is a *positive* constant. (87) and (88) gives

$$\int_{s_1}^{s_2} g x ds = \frac{\pi}{6} \left\{ 9 \left(\int_{s_1}^{s_2} x^2 dz \right)^2 - 5(z(s_2) - z(s_1)) \int_{s_1}^{s_2} x^4 dz \right\}. \quad (89)$$

This with Lemma 7.2, Corollary 10.2 gives the desired result. **q.e.d.**

Proof of Corollary 10.1. Let $z_1 = z(s_1)$ and let

$$\Gamma(z_1, z) := 9 \left(\int_{z_1}^z x^2 d\tilde{z} \right)^2 - 5(z - z_1) \int_{z_1}^z x^4 d\tilde{z}.$$

The first statement follows by applying the mean value theorem for integrals. We can find ξ_1, ξ_2 between z_1 and z such that

$$x^2(\xi_1)(z - z_1) = \int_{z_1}^z x^2 d\tilde{z}, \quad x^4(\xi_2)(z - z_1) = \int_{z_1}^z x^4 d\tilde{z},$$

and hence

$$\Gamma(z_1, z) = (z - z_1)^2 (9x^4(\xi_1) - 5x^4(\xi_2)) > 0$$

holds for $0 < |z_1 - z| \approx 0$.

If the anisotropic catenoid is normalized so that $c = 2$, then $xz'/\mu_2 = 1$ holds and $1/\mu_2$ has a positive lower bound and is also bounded above. We then obtain, for all $p > 1$,

$$\int_{s=s_1}^{s=\pm\infty} x^p dz = \int_{s=s_1}^{s=\pm\infty} x^p z' ds = \int_{s=s_1}^{s=\pm\infty} \mu_2 x^{p-1} ds = \pm\infty$$

since x has a positive lower bound and because the anisotropic catenoid is complete. Note also that since $x = 1/u$, we have $x \rightarrow \infty$ as $u \rightarrow 0$.

In view of Lemma 5.2, the height function z is unbounded. Using l'Hopital's rule,

$$\lim_{z \rightarrow \pm\infty} \frac{\left(\int_{z_1}^z x^2 d\tilde{z} \right)^2}{\int_{z_1}^z x^4 d\tilde{z}} = \lim_{z \rightarrow \pm\infty} \frac{2 \int_{z_1}^z x^2 d\tilde{z}}{x^2} = \lim_{z \rightarrow \pm\infty} \frac{x}{x_z} = \lim_{u \rightarrow 0} \frac{v_u}{u}, \quad (90)$$

using that $x = 1/u$ and $x_z = u_v$. Since $v(u)$ is even, we have $v_u(u = 0) = 0$ and so, $v_u/u = \mathcal{O}(1)$ as $u \rightarrow 0$. This gives

$$\limsup_{z \rightarrow \pm\infty} \frac{\left(\int_{z_1}^z x^2 d\tilde{z} \right)^2}{(z - z_1) \int_{z_1}^z x^4 d\tilde{z}} = 0,$$

and so $\Gamma(z_1, z) < 0$ holds for all z with $|z| \gg 0$. **q.e.d**

Example 10.1 We consider an anisotropic catenoid \mathcal{C} obtained from the Wulff shape $W = \{((x_1)^2 + (x_2)^2)^2 + (x_3)^2 = 1\}$ having flux parameter $c = 2$. (See Figures 1, 2 and 4.) This Wulff shape has strictly positive curvature except at the points where $x_3 \neq \pm 1$. The integral appearing in (89) can be transformed into an integral over a domain in $W = \{u^4 + v^2 = 1\}$, using

$$x = 1/u, \quad dz = x_u v_u du = (-u^{-2})(-2u^3(1 - u^4)^{-1/2}) du = 2u(1 - u^4)^{-1/2} du.$$

We obtain

$$\int_0^s gx \, ds = \left(\frac{\pi}{6}\right)\Gamma, \quad (91)$$

where

$$\Gamma := 9 \left(\int_1^t \frac{2}{u\sqrt{1-u^4}} \, du \right)^2 - 5 \int_1^t \frac{2u}{\sqrt{1-u^4}} \, du \int_1^t \frac{2}{u^3\sqrt{1-u^4}} \, du, \quad t := u(s).$$

Note that the height function is bounded so that the ends of \mathcal{C} are planer (Figure 2). A plot of the function is shown in Figure 3.

11 Characterization of stable equilibria with $\omega_0 = \omega_1 \geq 0$

In this section we will treat equilibria for rotationally symmetric energy functionals satisfying some additional conditions. We will be able to obtain a complete characterization of the stable critical points when $\omega_0 = \omega_1 \geq 0$ holds. We will assume that the anisotropic energy functional \mathcal{F} is chosen such that its Wulff shape W satisfies:

- W1. W is a smooth, uniformly convex surface of revolution with vertical rotation axis.
- W2. W is invariant with respect to reflection through the horizontal plane $x_3 = 0$.
- W3. The curvature function of the generating curve of W with respect to the inward pointing normal is a non-decreasing function of arc length on $\{x_3 \geq 0\}$ as one moves in an upward direction.

We will show

Theorem 11.1 *Let X be an embedded equilibrium capillary surface with free boundary on two horizontal planes for the functional*

$$\mathcal{E} := \mathcal{F} + \omega\mathcal{A}_0 + \omega\mathcal{A}_1,$$

with $\omega \geq 0$ and with the Wulff shape for the functional satisfying the conditions W1 – W3 stated above. Assume that X is an embedding and that its image is contained in the region between the planes. If $\omega = 0$, then X is stable if and only if the surface is a sufficiently short cylinder or a half of the Wulff shape. If $\omega > 0$ holds, then X is stable if and only if X is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.

Figure 5 Shows the generating curve of an anisotropic unduloid for the functional whose Wulff shape is given in Figure 4. The figure shows the largest stable, symmetric region. A part of the corresponding unduloid is shown in Figure 6

The rest of this section will be devoted to the proof of Theorem 11.1.

The case $\omega = 0$ was treated in [11] and the result stated above is proved there.

Assume that $\omega > 0$ holds. Let $X[s_1, s_2] : [s_1, s_2] \times S^1 \rightarrow \Omega$, $X(s, \theta) = (x(s)e^{i\theta}, z(s))$, be an embedded capillary surface. In view of Lemma 3.2,

$$x'(s_1) = -x'(s_2) > 0 \quad (92)$$

holds. Hence, the image of X is either a part of the Wulff shape, or a part of an anisotropic unduloid or nodoid which has a bulge in its interior. In the case where the surface is part of the Wulff shape, we know that its generating curve contains no inflection points and we also know from Section 4 that the surface minimizes the functional so it is clearly stable. We will ignore this surface from now on and treat only anisotropic unduloids and nodoids.

Let C be the portion of the generating curve of the anisotropic Delaunay surface between the two bounding planes. There are five cases:

Case(I) C has no inflection point.

Case(II) C has only one inflection point.

Case(III) Both endpoints of C are inflection points, and C has no inflection point in its interior.

Case(IV) C has exactly two inflection points in its interior.

Case(V) C has at least three inflection points.

In Cases(I) and (III), the stability of the surface follows from Theorems 8.1(i) and 9.1. In Case(IV), the instability follows from Theorem 8.1(ii). In Case(V), from (92), C has at least three zeros of x' in its interior. Since $\tilde{L}[x'] = 0$, the second Dirichlet eigenvalue $\lambda_2^0[s_1, s_2]$ for the Jacobi operator \tilde{L} is negative. Hence, $\tilde{\lambda}_2[s_1, s_2] < \lambda_2^0[s_1, s_2] < 0$, which implies that the surface $X[s_1, s_2]$ is unstable. Only Case(II) remains to be considered and this is the only case where the condition W3 will be used.

Lemma 11.1 *Assume that $X : \mathbf{R} \times S^1 \rightarrow \mathbf{R}^3$ is an anisotropic unduloid for a rotationally symmetric integrand $F = F(\nu_3)$ and $F(\nu_3)$ is an even function. Assume also that $s = 0$ corresponds to a bulge of X . Denote by s_I, s_N the smallest positive numbers which correspond to an inflection point, a neck of X , respectively. Let w be the even Jacobi field defined by (62). Denote by α the smallest positive number s with $w(s) = 0$. Set*

$$f(s) := \begin{cases} c_2 x'(s) \int_{s_I}^s \frac{ds}{\rho(x')^2}, & 0 \leq s \leq s_N, \\ -2c_1 x'(s) + c_2 x'(s) \int_{-s_I}^s \frac{ds}{\rho(x')^2}, & -s_N \leq s \leq 0. \end{cases} \quad (93)$$

Then,

$$\alpha < s_I \iff f'(0) > 0 \quad (94)$$

holds. In particular, if the Wulff shape satisfies the condition W3, then $\alpha < s_I$ holds.

Proof. Since

$$0 = w'(0) = c_1 x''(0) + f'(0), \quad x''(0) < 0$$

holds,

$$c_1 > 0 \iff f'(0) > 0 \quad (95)$$

holds. On the other hand, since $x'(s) < 0$ ($0 < s < s_N$) holds,

$$c_1 > 0 \iff w(s) < 0 \ (s_I \leq \forall s \leq s_N) \iff \alpha < s_I \quad (96)$$

holds. (95) with (96) implies (94). Moreover, we proved in [11, Lemma 5.3] that, if the Wulff shape satisfies W3, then $f'(0) > 0$ holds. **q.e.d.**

Now assume Case(II). We may assume that $s = 0$ corresponds to a bulge and $z(0) = 0$. Also we may assume that the considered embedding $X[s_1, s_2]$ satisfies $-s_I < s_1 < 0 < s_I < s_2 < s_N$ where s_I corresponds to an inflection point and s_N corresponds to a neck.

Set

$$\psi(s) := w(s) + ax'(s),$$

here a is a real constant and w , as before, is the even Jacobi field.

We would like to define the constant a so that ψ has (at least) one zero in $s_1 < s < s_2$ and

$$\frac{\psi'}{\psi} > \frac{z''}{z'} \quad \text{at } s = s_1, \quad \frac{\psi'}{\psi} < \frac{z''}{z'} \quad \text{at } s = s_2 \quad (97)$$

holds.

Assume for now that we can prove this. Then, ψ is an eigenfunction belonging to the j th eigenvalue $\hat{\lambda}_j = 0$ of the following eigenvalue problem for some $j \geq 2$:

$$\tilde{L}[\varphi] = -\lambda x\varphi, \quad \left(\varphi' - \frac{\psi'(s_1)}{\psi(s_1)}\varphi\right)\Big|_{s=s_1} = 0, \quad \left(\varphi' - \frac{\psi'(s_2)}{\psi(s_2)}\varphi\right)\Big|_{s=s_2} = 0, \quad \varphi \in H^1([s_1, s_2]). \quad (98)$$

Therefore, from (97), by using Lemma 13.3 (ii) below,

$$\tilde{\lambda}_2[s_1, s_2] < \hat{\lambda}_2 \leq \hat{\lambda}_j = 0$$

holds. Therefore, $X[s_1, s_2]$ is unstable.

By a similar computation to the derivation of (66) in Section 8, we have

$$z'\psi' - z''\psi = \frac{1}{xx'}(\psi k_1 x + c_2 \mu_1 z'). \quad (99)$$

We will find a so that

$$\psi > 0 \quad \text{and} \quad \psi k_1 x + c_2 \mu_1 z' > 0 \quad \text{at } s = s_1, \quad (100)$$

$$\psi < 0 \quad \text{and} \quad \psi k_1 x + c_2 \mu_1 z' < 0 \quad \text{at } s = s_2 \quad (101)$$

are satisfied. Then, (97) is satisfied and we are done.

Because of (92), $\mu_1|_{s=s_1} = \mu_1|_{s=s_2}$ and $z'(s_1) = z'(s_2)$ hold. Also, $k_1(s_1) < 0$ and $k_1(s_2) > 0$ hold. Therefore,

$$(100)\&(101) \iff \psi(s_1) > 0, \psi(s_2) < 0, |\psi k_1 x|_{s=s_2} > c_2 \mu_1 z'|_{s=s_2} = c_2 \mu_1 z'|_{s=s_1} > |\psi k_1 x|_{s=s_1} \quad (102)$$

holds.

Recall the representation formula stated in Proposition 5.1. We compute

$$k_1 = x''z' - x'z'' = (u_{\sigma\sigma}v_\sigma - u_\sigma v_{\sigma\sigma})\sigma_s,$$

$$\sigma_s = (x_u)^{-1} = \frac{-\Lambda\sqrt{u^2 + \Lambda c}}{\sqrt{u^2 + \Lambda c} \pm u}.$$

Hence,

$$k_1 x = -\sqrt{u^2 + \Lambda c} (u_{\sigma\sigma} v_\sigma - u_\sigma v_{\sigma\sigma})$$

holds. Since

$$\mu_1 = |u_{\sigma\sigma} v_\sigma - u_\sigma v_{\sigma\sigma}|,$$

we have

$$|k_1 x|_{s=s_1} = |k_1 x|_{s=s_2}, \quad |k_1 x| = \sqrt{u^2 + \Lambda c} \mu_1. \quad (103)$$

By using (102) and (103), we have

$$(100)\&(101) \iff -\psi(s_2) > \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_2} = \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_1} > \psi(s_1) > 0 \quad (104)$$

Now we will find a which satisfies the right hand side of (104), that is,

$$-(w(s_2) + ax'(s_2)) > \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_2} = \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_1} > w(s_1) + ax'(s_1) > 0. \quad (105)$$

Set

$$P := x'(s_1).$$

Then,

$$x'(s_2) = -P,$$

and

$$P > 0$$

holds. Set

$$A := \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_2} = \frac{c_2 z'}{\sqrt{u^2 + \Lambda c}} \Big|_{s=s_1}.$$

We will show

$$-w(s_1) > w(s_2) \quad (106)$$

holds. Once this is done, we can take a so that

$$\frac{A - w(s_1)}{P} > a > \max \left\{ \frac{A + w(s_2)}{P}, -\frac{w(s_1)}{P} \right\}. \quad (107)$$

This a satisfies (105), and the instability of $X[s_1, s_2]$ holds.

We will show (106). We will use the expression of w in terms of the generating curve (u, v) of the Wulff shape.

By Lemma 11.1,

$$f'(0) > 0 \iff c_1 > 0 \iff w(s) < 0 \ (s_I \leq \forall s \leq s_N) \iff \alpha < s_I \quad (108)$$

holds. Set $\sigma_I := \sigma(s_I)$. It is easily seen that

$$f(s) = \begin{cases} c_2 u_\sigma \int_{\sigma_I}^\sigma \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma, & 0 < s < s_I, \\ -c_2 u_\sigma \int_{\sigma_I}^\sigma \frac{-u_{\sigma\sigma} v_\sigma + u_\sigma v_{\sigma\sigma}}{(u_\sigma)^2 \sqrt{u^2 + \Lambda c}} d\sigma, & s_I < s < s_N \end{cases}$$

holds. Therefore,

$$f(-s_1) > 0, \quad f(s_2) = -f(s_1)$$

holds. Hence, we obtain

$$\begin{aligned} w(s_1) &= w(-s_1) = c_1 x'(-s_1) + f(-s_1) = -c_1 x'(s_1) + f(-s_1), \\ w(s_2) &= c_1 x'(s_2) + f(s_2) = -c_1 x'(s_1) - f(-s_1). \end{aligned}$$

Therefore,

$$-w(s_1) - w(s_2) = 2c_1 x'(s_1)$$

holds. Since $x'(s_1) > 0$,

$$-w(s_1) > w(s_2) \iff c_1 > 0 \tag{109}$$

holds. By using (108) and (109),

$$-w(s_1) > w(s_2) \iff f'(0) > 0$$

holds. In [11], it was shown that if the generating curve (u, v) of the Wulff shape satisfies the curvature condition W3, then $f'(0) > 0$ holds. This completes the proof. **q.e.d.**

12 Stability for the case where one of ω_i is equal to zero

In the previous section, we determined the stability (or instability) for each critical point in terms of its geometric property when the Wulff shape satisfies certain conditions and $\omega_0 = \omega_1 \geq 0$ holds. In this section, we will study the stability for equilibria in the case where either ω_0 or ω_1 is equal to zero. In this case, there is no such a complete characterization of the stable critical points, which will be shown below.

Before stating the main theorem of this section, we will show:

Proposition 12.1 *Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be an immersion of an anisotropic Delaunay surface for a rotationally symmetric even integrand $F = F(\nu_3)$. Let $X[-s_0, s_0]$, $s_0 > 0$ be a symmetric subdomain with either a neck or a bulge occurring when $s = 0$. Then $X[-s_0, s_0]$ is stable if and only if $X[0, s_0]$ is stable and $\tilde{\lambda}_2[-s_0, s_0] \geq 0$ holds.*

Proof. We may assume that $z(0) = 0$. Suppose first that $X[-s_0, s_0]$ is stable. Then clearly $\tilde{\lambda}_2[-s_0, s_0] \geq 0$ holds. If a volume preserving, energy decreasing variation of $X[0, s_0]$ existed which maintained the free boundary conditions, then this variation could be reflected across the plane $\{x_3 = 0\}$ to produce a volume preserving, energy decreasing variation of $X[-s_0, s_0]$, so $X[0, s_0]$ is stable.

We now assume $X[0, s_0]$ is stable and $\tilde{\lambda}_2[-s_0, s_0] \geq 0$ holds. If $\tilde{\lambda}_1[0, s_0] \geq 0$ then let ψ_1 be a corresponding eigenfunction. Note that ψ_1 does not change sign in $[0, s_0]$. Since $z''(0) = 0$ we

have $0 = B_1[\psi_1]_{s=0} = (x/\mu_1)\psi_1'(0)$. The even reflection $\tilde{\psi}_1$ of ψ_1 satisfies $(\tilde{L} + \tilde{\lambda}_1[0, s_0])[\tilde{\psi}_1] = 0$ on $[-s_0, s_0]$ and $\tilde{\psi}_1$ does not change sign. It follows that $\tilde{\lambda}_1[-s_0, s_0] \geq 0$ and so $X[-s_0, s_0]$ is stable.

Now suppose $\tilde{\lambda}_1[0, s_0] < 0$ holds. We then know, by stability of $X[0, s_0]$ and Lemma 7.2, that $0 \leq \tilde{\lambda}_2[0, s_0]$ holds and there is a function ϕ satisfying $\tilde{L}[\phi] = x$ on $[0, s_0]$, $B_1[\phi] = 0$ on $\partial[0, s_0]$ and

$$\int_0^{s_0} \phi x \, ds \geq 0$$

holds. As before, we have $0 = B_1[\phi]_{s=0} = (x/\mu_1)\phi'(0)$. The even extension $\tilde{\phi}$ of ϕ thus satisfies $\tilde{L}[\tilde{\phi}] = x$ on $[-s_0, s_0]$, $B_1[\tilde{\phi}] = 0$ on $\partial[-s_0, s_0]$. Also,

$$\int_{-s_0}^{s_0} \phi x \, ds = 2 \int_0^{s_0} \phi x \, ds \geq 0.$$

Therefore $X[-s_0, s_0]$ is stable. **q.e.d.**

Now we give the main results of this section.

Theorem 12.1 *Assume that the Wulff shape satisfies the conditions W1-W3 of the previous section and that X is an equilibrium surface for the functional*

$$\mathcal{E} := \mathcal{F} + \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1,$$

with $\omega_0 = 0$ and $\omega_1 > 0$. Assume also that X is an embedding and that its image lies between the two supporting planes. Then the image of X is part of an anisotropic Delaunay surface with generating curve C .

- (i) *If C has no interior inflection point, then it is stable.*
- (ii) *If C has exactly one inflection point in its closure and it is an interior point, then there is no conclusion. More precisely, both stable and unstable such surfaces of this type exist.*
- (iii) *If C has more than one inflection points in its closure, then it is unstable.*

The following Propositions 12.2, 12.3 give examples for each of the two cases in Theorem 12.1 (ii). They will be proved later in this section.

Proposition 12.2 *Assume that the Wulff shape satisfies the same assumptions as in Theorem 12.1. Let s_I be the smallest positive number which corresponds to an inflection point of X . Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be an anisotropic unduloid with a bulge occurring when $s = 0$. Then, for small $\epsilon > 0$, $X[0, s_I + \epsilon]$ is stable.*

Remark 12.1 Proposition 12.2 with Lemma 8.1 shows that we cannot omit the condition $\tilde{\lambda}_2[-\hat{s}, \hat{s}] \geq 0$ in Proposition 12.1.

Proposition 12.3 *Assume that the Wulff shape satisfies the same assumptions as in Theorem 12.1. Let $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ be an anisotropic unduloid with either a neck or a bulge occurring when $s = 0$. Denote by T the half period of $x(s)$. Then, for $\epsilon_1, \epsilon_2 \in \mathbf{R}$ with small $|\epsilon_1|$ and $|\epsilon_2|$, $X[\epsilon_1, T + \epsilon_2]$ is unstable.*

The rest of this section will be devoted to the proof of these results.

Proof of Theorem 12.1. There are the following four cases:

Case(I) C has no inflection points in its closure.

Case(II) The upper endpoint of C is an inflection point, and C has no inflection point in its interior.

Case(III) C has exactly one inflection point in its closure and it is an interior point.

Case(IV) C contains at least two inflection points in its closure.

Since Propositions 12.2, 12.3 prove (ii), we need to examine only Cases(I), (II), and (IV).

We can again assume that the surface is of the form $X[0, s_1]$ where X is part of the Wulff shape or a suitable part of an anisotropic unduloid or nodoid having a neck or bulge on the plane $z = 0$ which corresponds to the value $s = 0$. If the surface is part of the Wulff shape, then as in Theorem 4.1, it is automatically stable and there are no inflection points. We will ignore this case from now on.

In Cases(I) and (II), $s = 0$ must correspond to a bulge. (If $s = 0$ were a neck then because $\omega_1 > 0$ holds, the generating curve would necessarily contain an inflection point in its interior.) The stability of $X[0, s_1]$ then follows from Proposition 12.1, Theorems 8.1 (i) and 9.1.

We will show that, in Case(IV), the surface is unstable. In this case, the surface must be an anisotropic unduloid. We will denote by T the half period of $x(s)$.

If a bulge is located at $s = 0$, then since $\omega_1 > 0$ holds, the surface contains at least one neck at $s = T$ and one bulge at $s = 2T$ in its interior. Then, since $L[x'] = 0$, the second Dirichlet eigenvalue of the Jacobi operator must be negative, and so the surface is unstable also for the free boundary problem.

Now we consider the situation when a neck is located at $s = 0$. By standard ODE theory, there is a unique zero $\alpha \in [0, T]$ of the even Jacobi field w since the endpoints of this interval are consecutive zeroes of x' . It follows, by using w , that the first eigenvalue $\bar{\lambda}_1([0, \alpha])$ of the *partially* free boundary value problem (cf. (53)) is zero, where $s = 0$ is considered as the free boundary and $s = \alpha$ is considered as the fixed boundary.

Now we consider a second partially free boundary value problem (cf. (54)) on $[T, s_{2I}]$, where s_{2I} is the first inflection point above T . Consider $s = T$ as the fixed boundary and $s = s_{2I}$ as the free one and denote the first eigenvalue by $\underline{\lambda}_1([T, s_{2I}])$. By using the function x' , one sees that this eigenvalue is zero.

First consider the surface $X[0, s_{2I}]$. Since both the eigenvalues considered above are monotonically decreasing with respect to their fixed boundary components and since $\alpha < T$ holds, we can find $\epsilon \in (0, (T - \alpha)/2)$ so that $\bar{\lambda}_1([0, \alpha + \epsilon]) < 0$ and $\underline{\lambda}_1([T - \epsilon, s_{2I}]) < 0$ hold. A suitable linear combination of the eigenfunctions for these two eigenvalues will have integral zero and it can be extended to equal zero on $[\alpha + \epsilon, T - \epsilon]$ to give an admissible, energy decreasing, volume preserving variation.

Now consider $X[0, s_2]$ with $s_{2I} < s_2 < 2T$, where $s = 2T$ is the first neck above $s = 0$. In this case we can again take a linear combination of the eigenfunctions for $\bar{\lambda}_1([0, \alpha + \epsilon])$ and $\underline{\lambda}_1([T - \epsilon, s_{2I}])$ which has been extended to equal zero on $[\alpha + \epsilon, T - \epsilon]$ and continuously extended to be constant on $[s_{2I}, s_2]$. By considering the second variation formula in the form given by (38), one sees that because of the sign $z'' > 0$ on $[s_{2I}, s_2]$, the second variation will still be negative.

Finally, consider $X[0, s_2]$ with $2T < s_2$. Since the second Dirichlet eigenvalue of the Jacobi operator on $[0, 2T]$ is zero, it is negative on $[0, s_2]$. Therefore, $\tilde{\lambda}_2[0, s_2] < 0$, which implies that $X[0, s_2]$ is unstable. **q.e.d.**

In the rest of this section, we will assume that $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ is an anisotropic unduloid and $s = 0$ corresponds to a bulge.

Proof of Proposition 12.2. In view of Lemma 11.1, $0 < \alpha < s_I$ holds. Also note that $w(s) < 0$ holds for $\alpha < s < T$, where T is the half period of $x(s)$.

For s_0 with $w(s_0) \neq 0$, we set

$$R := \frac{w'(s_0)}{w(s_0)}. \quad (110)$$

Then, w is an eigenfunction belonging to the second eigenvalue $\lambda_2^R = 0$ of the eigenvalue problem

$$\tilde{L}[\varphi] = -\lambda x \varphi, \quad \varphi'(0) = 0, \quad (\varphi' - R\varphi)|_{s=s_0} = 0, \quad \varphi \in H^1([0, s_0]). \quad (111)$$

By using (66) and (67), we have, for small $\epsilon > 0$, at $s_0 := s_I + \epsilon$,

$$z'w' - z''w < 0, \quad B_1[w] < 0 \quad (112)$$

holds. Because of the first inequality of (112), if $w(s_0) < 0$, then $(w'/w) > (z''/z')$ holds at $s = s_0$. Therefore, from Lemma 13.3 (ii),

$$\tilde{\lambda}_2[0, s_0] > \lambda_2^R = 0$$

holds. On the other hand, from Lemma 7.3, $\tilde{\lambda}_1[0, s_0] < 0$ holds. Therefore, this is the case (II) of Lemma 7.2.

Set

$$\beta_{s_0} := -\frac{B_1[q]|_{s=s_0}}{B_1[w]|_{s=s_0}}, \quad \phi_{s_0} := q + \beta_{s_0}w.$$

Then,

$$B_1[\phi_{s_0}]|_{s=s_0} = 0$$

holds. On the other hand, $\tilde{L}[\phi_{s_0}] = \tilde{L}[q] = -\Lambda x$ holds (see (49)). Also, $B_1[q]|_{s=s_I} = 0$ holds, and so $\beta_{s_I} = 0$. Therefore,

$$\int_0^{s_I} \phi_{s_I} x \, ds = \int_0^{s_I} q x \, ds > 0$$

holds, where we used $q = xz' - x'z > 0$ on $0 \leq s \leq T$. Since $\int_0^{s_0} \phi_{s_0} x \, ds$ is continuous with respect to s_0 ,

$$\int_0^{s_0} \phi_{s_0} x \, ds > 0$$

holds for small $\epsilon > 0$. Therefore, in view of Lemma 7.2 (II-1), $X[0, s_0]$ is stable. **q.e.d.**

Proof of Proposition 12.3. We take ϵ_1, ϵ_2 small so that $\epsilon_1 < s_I < T + \epsilon_2$ holds. Let f be the function defined by (93). Then, since $f(s_I) = 0$, f is an eigenfunction belonging to the j th eigenvalue $\lambda_j = 0$ of the following eigenvalue problem for some $j \geq 2$:

$$\tilde{L}[\varphi] = -\lambda x \varphi, \quad \left(\varphi' - \frac{f'(\epsilon_1)}{f(\epsilon_1)} \varphi \right) \Big|_{s=\epsilon_1} = 0, \quad \left(\varphi' - \frac{f'(T+\epsilon_2)}{f(T+\epsilon_2)} \varphi \right) \Big|_{s=T+\epsilon_2} = 0, \quad \varphi \in H^1([\epsilon_1, T+\epsilon_2]).$$

Note that $f(0) = 1$ and $f(T) < 0$ hold. Lemma 5.3 in [11] implies that $f'(0) > 0$ and $f'(T) > 0$ hold. Therefore,

$$\frac{f'(0)}{f(0)} > 0 = \frac{z''(0)}{z'(0)}, \quad \frac{f'(T)}{f(T)} < 0 = \frac{z''(T)}{z'(T)}$$

holds. Hence, for $\epsilon_1, \epsilon_2 \in \mathbf{R}$ with sufficiently small $|\epsilon_1|, |\epsilon_2|$,

$$\frac{f'(\epsilon_1)}{f(\epsilon_1)} > \frac{z''(\epsilon_1)}{z'(\epsilon_1)}, \quad \frac{f'(T + \epsilon_2)}{f(T + \epsilon_2)} < \frac{z''(T + \epsilon_2)}{z'(T + \epsilon_2)}$$

holds. Therefore, from Lemma 13.3 (ii),

$$\tilde{\lambda}_2[\epsilon_1, T + \epsilon_2] < \hat{\lambda}_2 \leq \hat{\lambda}_j = 0$$

holds, which implies that $X[\epsilon_1, T + \epsilon_2]$ is unstable. **q.e.d.**

13 Appendix: Minimizing properties of eigenvalues

The theory in this section is probably well-known. It is included for the reader's convenience.

Let $\rho > 0$, Q be (sufficiently smooth) functions on the interval $[0, l]$. Set

$$J[\psi] = (\rho\psi')' + Q\psi \quad \text{in } [0, l].$$

Below, α, β, α_i , and β_i represent real numbers. Set, on $\partial[0, l]$,

$$B_{(\alpha, \beta)}[\psi] := \begin{cases} \rho\psi' - \alpha\psi & \text{at } s = 0, \\ \rho\psi' - \beta\psi, & \text{at } s = l, \end{cases} \quad B_{(0, *)}[\psi] := \begin{cases} \rho\psi' & \text{at } s = 0, \\ \psi & \text{at } s = l, \end{cases}$$

$$\begin{aligned} \mathcal{S} &:= \{\psi \in C^0([0, l]); \psi \text{ is piecewise } C^1 \text{ and piecewise } C^2 \text{ on } [0, l]\}, \\ \mathcal{S}_{(\alpha, \beta)} &:= \mathcal{S}, \\ \mathcal{S}_{(0, *)} &:= \{\psi \in \mathcal{S}; \psi(l) = 0\}. \end{aligned}$$

Set, for $(a, b) = (\alpha, \beta), (0, *)$,

$$I_{(a, b)}[\psi] := - \int_0^l \psi J[\psi] ds + [\psi B_{(a, b)}[\psi]]_0^l, \quad \psi \in \mathcal{S}_{(a, b)},$$

$$H[\psi] := \int_0^l \psi^2 ds, \quad H[\phi, \psi] := \int_0^l \phi\psi ds, \quad \phi, \psi \in \mathcal{S}.$$

Then, it holds that

$$I_{(\alpha, \beta)}[\psi] = \int_0^l (\rho(\psi')^2 - Q\psi^2) ds - \beta(\psi(l))^2 + \alpha(\psi(0))^2, \quad (113)$$

$$I_{(0, *)}[\psi] = \int_0^l (\rho(\psi')^2 - Q\psi^2) ds. \quad (114)$$

We consider eigenvalue problem

$$J[\psi] = -\lambda\psi \quad \text{in } [0, l], \quad B_{(a,b)}[\psi] = 0 \quad \text{on } \partial[0, l], \quad \psi \in \mathcal{S}_{(a,b)}. \quad (115)$$

Denote by $\lambda_1^{(a,b)} < \lambda_2^{(a,b)} < \dots$ the eigenvalues of (115). Let $e_i^{(a,b)}$ be an eigenfunction of (115) belonging to $\lambda_i^{(a,b)}$ which satisfies

$$H[e_i^{(a,b)}] = 1, \quad H[e_i^{(a,b)}, e_k^{(a,b)}] = 0 \quad (i \neq k).$$

Then, the following Lemma holds (cf. [4]).

Lemma 13.1

$$\lambda_i^{(a,b)} = I_{(a,b)}[e_i^{(a,b)}] = \min\{I_{(a,b)}[\psi]; \psi \in \mathcal{S}_{(a,b)}, H[\psi] = 1, H[\psi, e_k^{(a,b)}] = 0 \ (k = 1, \dots, i-1)\}$$

holds. Moreover, $I_{(a,b)}[\psi] = \lambda_i^{(a,b)}$ holds if and only if ψ satisfies (115) for $\lambda = \lambda_i^{(a,b)}$.

Also, the following so-called min-max principle holds. Denote by $\mathcal{V}_i^{(a,b)}$ the set of all i -dimensional subspaces of $\mathcal{S}_{(a,b)}$. Then,

Lemma 13.2

$$\lambda_i^{(a,b)} = \min_{V \in \mathcal{V}_i^{(a,b)}} \max_{\psi \in V - \{0\}} (I_{(a,b)}[\psi] / H[\psi]). \quad (116)$$

By using the above lemmas, we obtain the following:

Lemma 13.3 *For each $i \in \mathbf{N}$, the following inequalities hold.*

- (i) *If $\beta \geq 0$ holds, then $\lambda_i^{(0,\beta)} < \lambda_i^{(0,*)}$ holds.*
- (ii) *If $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$ holds, then $\lambda_i^{(\alpha_1,\beta_1)} \geq \lambda_i^{(\alpha_2,\beta_2)}$ holds for all $i \in \mathbf{N}$. Here, $\lambda_i^{(\alpha_1,\beta_1)} = \lambda_i^{(\alpha_2,\beta_2)}$ holds for some i if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ holds.*
- (iii) *Set $\lambda_1^{(0,*)}[0, l] := \lambda_1^{(0,*)}$. If $0 < l_1 < l_2$, then $\lambda_1^{(0,*)}[0, l_1] > \lambda_1^{(0,*)}[0, l_2]$ holds.*

Proof. (i) Because of (113) and (114), for $\psi \in \mathcal{S}_{(0,*)}$,

$$I_{(0,\beta)}[\psi] = I_{(0,*)}[\psi]$$

holds. Therefore, by Lemma 13.2, $\lambda_i^{(0,\beta)} \leq \lambda_i^{(0,*)}$ holds. Moreover, by Lemma 13.1, $\lambda_i^{(0,\beta)} < \lambda_i^{(0,*)}$ holds.

- (ii) is proved also by using (113), Lemma 13.2, and Lemma 13.1.
- (iii) holds from Lemma 13.1. **q.e.d.**

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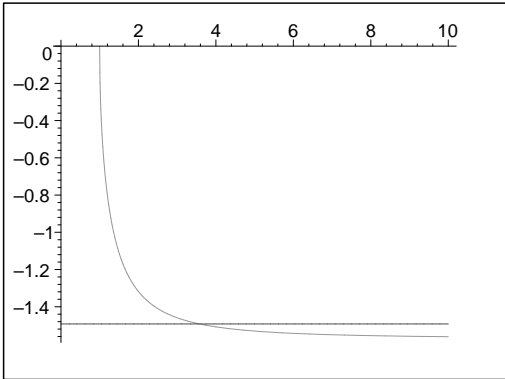


Figure 1: The generating curve of the anisotropic catenoid \mathcal{C} of Example 10.1. The Wulff shape is given in Figure 4. The horizontal line indicates the depth to which \mathcal{C} is stable for the free boundary problem.

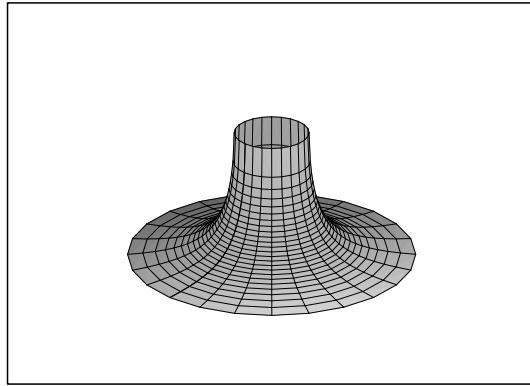


Figure 2: The corresponding end

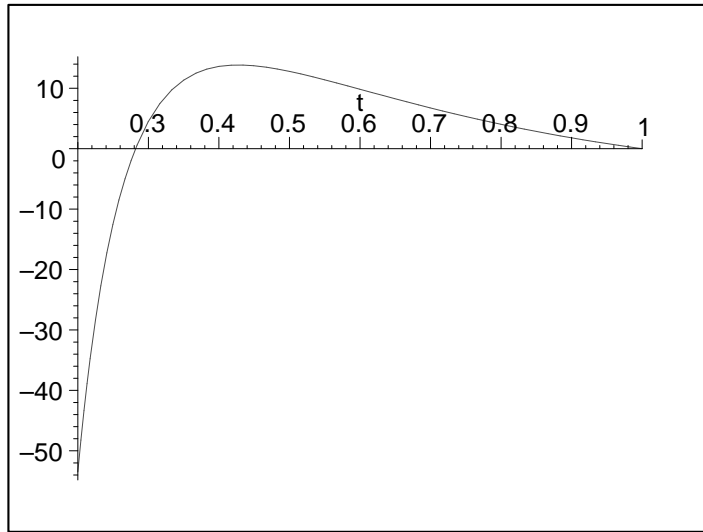


Figure 3: Plot of Γ as a function of t .

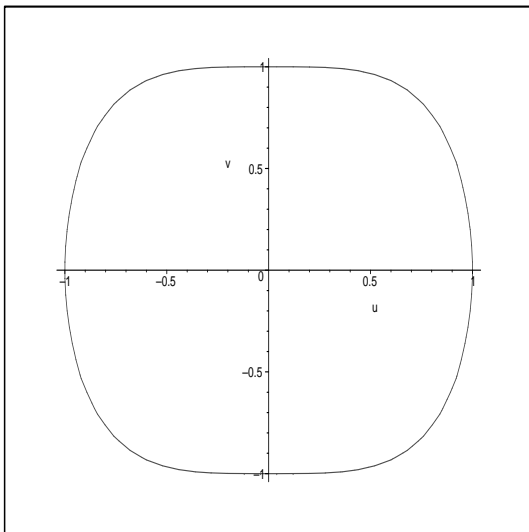


Figure 4: The generating curve of the Wulff shape W given by $u^4 + v^2 = 1$.

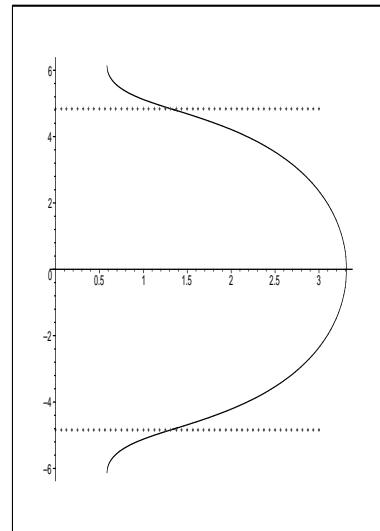


Figure 5: The generating curve of an anisotropic unduloid \mathcal{U} for the functional with Wulff shape W in Figure 4, with $\Lambda = -1/2$ and $c = 1$. The part of the curve between the dotted lines generates the largest symmetric stable region in \mathcal{U} .

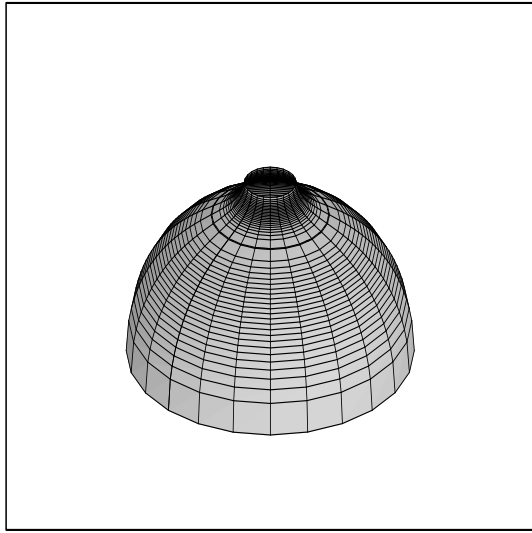


Figure 6: A part of the anisotropic unduloid generated by the curve in Figure 5.