# Uniqueness theorems for stable anisotropic capillary surfaces

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**Abstract**

We consider capillary surfaces for certain rotationally invariant elliptic parametric functionals supported on two hydrophobically wetted horizontal plates separated by a fixed distance. It is shown that each such stable capillary surface is uniquely determined by the volume interior to the surface.

## 1 Introduction

When the temperature of a fluid is gradually lowered, it undergoes a process of crystallization in which its constituent atoms, molecules or ions will align themselves in a regular repeating pattern, It is rare that a single crystal will form and instead many crystals will form a polycrystal. This is the state in which, for example, most metals occur.

As the fluid cools, the usual isotropic surface energy (surface tension) will no longer be appropriate to model the shape of the interface of the fluid with its environment. Because of the internal structure of the material, the isotropic surface energy must be replaced by an anisotropic one; i.e. an energy that depends on the direction of the surface at each point. In this paper, we will treat a class of capillary problems for the simplest type of anisotropic surface energy: a constant coefficient, elliptic parametric functional.

Particularly, we consider a variational problem whose solution is a mathematical model of a drop of a cooled liquid trapped between two horizontal plates. The plates are hydrophobically wetted and are made of the same material. It is natural to consider the volume of the drop, the distance between the plates and the wetting constant $\omega$ which couples the energy of the fluid-plate interface to the free surface energy as "initial data" and then ask if the shape of the drop is uniquely determined. In our previous paper [5], we obtained a geometric characterization of such drops. In this paper, we show (Theorem 2.1) that under certain assumptions on the energy functional the uniqueness follows

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if the additional natural condition of stability is imposed. Not only do we have uniqueness but we are able to determine the shape of the drop (Theorem 2.3) to the extent that a parameterization can be easily obtained from our previous work [3].

We wish to emphasize that we have restricted ourselves to the cases of hydrophobic wetting and equal contact angle. This is not to suggest that the other cases are of lesser importance. At present we also ignore gravitational and other external forces. This paper should be considered as part of a program in which these more general problems will be considered.

Our assumptions imposed on the energy functional are satisfied by the usual area functional. In this important special case, solutions are constant mean curvature (CMC) surfaces which meet each of the supporting planes with constant angle. In the CMC case without wetting, the uniqueness and characterization of stable solutions follow from the results in Athanassenas[1], Vogel[6]. For hydrophobic wetting, they follow from the results in Vogel [7], and Finn and Vogel [8]. The lower bound for the volume of a stable spanning drop of height $h$ was shown by Finn and Vogel in [8] to be $h^3/\pi$, giving an affirmation of Carter's conjecture. We will generalize this result to anisotropic case with hydrophobic wetting (Theorem 2.4).

The paper is organized as follows. Section 2 contains precise statements of our main results. Sections 3 and 4 will be devoted to proofs of the results stated in Section 2. In Section 5, we will give a strict examination of the uniqueness for the case without wetting energy. Section 6 contains a summary of results concerning anisotropic Delaunay surfaces (rotationally symmetric surfaces with constant anisotropic mean curvature). These surfaces were introduced in detail in [3] and play a fundamental role in our stability and uniqueness analysis.

Finally we would like to convey our sincere thanks to the referees for calling the references [8], [9] and [10] to our attention and suggesting various improvements to our paper.

## 2 Statements of results

Let $F$ be a smooth, positive function on $S^2$. To an immersion $X : \Sigma \to \mathbf{R}^3$ from a two-dimensional oriented, connected, compact, smooth manifold $\Sigma$ (possibly with boundary $\partial\Sigma$) to the three-dimensional Euclidean space $\mathbf{R}^3$, we assign the free anisotropic energy

$$\mathcal{F}[X] := \int_\Sigma F(\nu)\, d\Sigma \,, \tag{1}$$

where $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \to S^2$ is the Gauss map of $X$, and $d\Sigma$ is the area form of the induced metric. We will assume that $F$ satisfies a "convexity condition" in the following sense: Denote by $DF$ and $D^2F$ the gradient and Hessian of $F$ on $S^2$. We assume that at each point in $S^2$ the matrix $D^2F + FI$ is positive definite. Such an energy functional $\mathcal{F}$ is then referred to as a constant coefficient elliptic parametric functional.

It is known that the energy $\mathcal{F}$ possesses a canonical critical point which minimizes $\mathcal{F}$ among closed surfaces enclosing a specific three dimensional volume ([2]), and it is known as the *Wulff shape* (for $\mathcal{F}$) which we will denote by $W$. $W$ is a uniformly convex smooth surface and given by the immersion $\chi : S^2 \to \mathbf{R}^3$ defined by $\chi(\nu) = DF(\nu) + F(\nu) \cdot \nu$. In the special case where $F \equiv 1$, $\mathcal{F}$ is the area functional and $W$ is the round sphere of radius 1 with center at the origin.

The property that $X$ is a critical point of $\mathcal{F}$ for all compactly supported volume-preserving variations is characterized by the property that the anisotropic mean curvature $\Lambda$ of $X$ is constant, where $\Lambda$ is given by

$$\Lambda := 2HF - \mathrm{div}_\Sigma DF = -\mathrm{trace}_\Sigma (D^2F + FI) \circ d\nu$$

(cf. [3]). Here $H$ is the mean curvature of $X$ and $I$ is the identity endomorphism field on $TS^2$. This definition is a generalization of the idea of constant mean curvature which arises from the area functional.

In this paper, we consider connected, compact surfaces $X$ with non-empty boundary embedded in a region $\Omega := \{z_0 \leq z \leq z_1\}$ whose interiors are included in the interior of $\Omega$, whose boundary components are restricted to lie on the two supporting (horizontal) planes $\Pi_i := \{z = z_i\}$, $i = 0, 1$, in $\Omega$, and which are constrained to enclose a fixed volume $V$. These considerations necessitate that the surface bounds a connected volume so that we preclude some physically important configurations like a "string of spheres". Also, for simplicity, we are assuming that each boundary component of the considered surface is homeomorphic to a circle. We will call such a surface an *anisotropic capillary surface* if it is in equilibrium for a functional

$$\mathcal{E}[X] := \mathcal{F}[X] + \omega_0 \mathcal{A}_0[X] + \omega_1 \mathcal{A}_1[X] \, . \tag{2}$$

Here $\mathcal{A}_i$ is the area in the plane $\Pi_i$ which is bounded by the boundary components of $X$ in $\Pi_i$ (physically, the area which is wetted by the material inside the surface) and the $\omega_i$'s are coupling constants. In practice the $\omega_i$'s are determined by the materials involved. Throughout this paper we use the term "capillary surface" to mean anisotropic capillary surface. We will use the adjective isotropic when it is needed to denote the special case when the free energy is the surface area.

For an embedding $X : (\Sigma, \partial\Sigma) \to (\Omega, \Pi_0 \cup \Pi_1)$ with outward pointing unit normal $\nu$, the contact angle of $X$ with $\Pi_i$ at $X(\zeta) \in \Pi_i$ ($\zeta \in \partial\Sigma$) is defined as the angle $\vartheta \in [0, \pi]$, between $\nu(\zeta)$ and $(-1)^i(0, 0, 1)$. The surface $X$ is a capillary surface if and only if the anisotropic mean curvature $\Lambda$ of $X$ is constant, and the contact angle $\vartheta$ of $X$ with each $\Pi_i$ is a constant $\vartheta(\omega_i)$. The precise value $\vartheta(\omega_i)$ will be given below.

A capillary surface is said to be stable if the second variation of the energy functional $\mathcal{E}$ is nonnegative for all volume-preserving variations satisfying the boundary condition.

A natural question to ask is whether one can uniquely determine the shape of the (stable) capillary surface from the 'initial data' $\mathcal{F}$, $V$, $h := z_1 - z_0$, $\omega_0$ and $\omega_1$. We will show that this is possible under certain conditions.

We will first impose conditions on the functional $\mathcal{F}$ which will be described via the corresponding Wulff shape $W$. It will be assumed that

(W1) $W$ is a uniformly convex surface of revolution with vertical rotation axis.

(W2) $W$ is symmetric with respect to reflection through the horizontal plane $z = 0$.

(W3) The generating curve of $W$ has non-decreasing curvature (with respect to the inward pointing normal) as a function of arc length on $\{z \geq 0\}$ as one moves in an upward direction.

In addition, it will be assumed that $\omega_0 = \omega_1 =: \omega \geq 0$ holds in (2). In the isotropic (liquid) case, the condition $\omega_i > 0$ is known as *hydrophobic wetting* since the material inside the surface will tend to avoid the supporting planes when minimizing energy. The case of the $\omega_i$'s being equal would occur (physically) if both supporting planes were made from the same material.

The Wulff shape $W$ can be represented as

$$(x_1, x_2, x_3) = (u(\sigma)\cos\theta, u(\sigma)\sin\theta, v(\sigma)),$$

where $\sigma$ is the arc length of the generating curve

$$\Gamma_W : (u(\sigma), v(\sigma))$$

of $W$. Denote by $\bar{\omega}$ the maximum height on $W$, that is $\bar{\omega} = \max_{\sigma} v(\sigma)$. At times we will also represent the generating curve of $W$ as a graph $(u(v), v)$, $-\bar{\omega} \leq v \leq \bar{\omega}$.

For $\omega \in (-\bar{\omega}, \bar{\omega})$, denote by $\vartheta(\omega)$ the contact angle between the region $W \cap \{x_3 \leq \omega\}$ of $W$ and the plane $\{x_3 = \omega\}$. Also we define $\vartheta(-\bar{\omega}) := 0$, $\vartheta(\bar{\omega}) := \pi$. Then, $\vartheta(\omega)$ is a continuous strictly-increasing function of $\omega$ on $[0, \pi]$ with $\vartheta(0) = \pi/2$. An embedding $X$ is a capillary surface for

$$\mathcal{E} := \mathcal{E}_\omega := \mathcal{F} + \omega\mathcal{A}_0 + \omega\mathcal{A}_1 \tag{3}$$

if and only if the anisotropic mean curvature of $X$ is constant, and the contact angle between $X$ and each bounding plane $\Pi_i$ is constant $\vartheta(\omega)$ along the boundary ([5, Propositions 3.1, 3.2]).

We will call an anisotropic capillary surface *spanning* if its intersection with both supporting planes is a circle of positive radius. We denote by $V_0(h, \omega)$ the infimum of the volumes of stable spanning anisotropic capillary surfaces having height $h$ and contact angle $\vartheta(\omega)$.

In keeping with the classical terminology, we will refer to a compact anisotropic capillary surface having non-empty boundary components only on the plane $z = z_0$, (respectively, $z = z_1$) as a *sessile drop*, (respectively, *pendent drop*). Such a surface is necessarily rotationally invariant, and therefore homothetic to a part of the Wulff shape ([5]).

If $|\omega| > \bar{\omega}$, then there is no capillary surface for the energy $\mathcal{E}_\omega$ ([5, Corollary 3.1]). For $0 \leq \omega \leq \bar{\omega}$, we will show the following uniqueness theorem.

**Theorem 2.1** *We assume (W1) through (W3) stated above.*

*[I] Assume* $0 \leq \omega < \bar{\omega}$*. Then,* $V_0(h, \omega) > 0$ *holds and,*

*(i) For volume* $V < V_0$*, any stable capillary surface for the energy* $\mathcal{E}_\omega$ *with volume* $V$ *and height* $h$ *is a sessile or pendent drop.*

*(ii) For volumes* $V \geq V_0$*, there exists a unique stable spanning capillary surface for the energy* $\mathcal{E}_\omega$ *with volume* $V$ *and height* $h$*.*

*[II] Assume* $\omega = \bar{\omega}$*. Then, any capillary surface for the energy* $\mathcal{E}_\omega$ *is tangent to the supporting planes* $\Pi_0 \cup \Pi_1$*.* $V_0(h, \omega) > 0$ *holds, and it coincides with the volume of the closed surface homothetic to the Wulff shape which is tangent to both of* $\Pi_0$ *and* $\Pi_1$*. And,*

*(i) For volume* $V \leq V_0$*, there is no stable capillary surface for the energy* $\mathcal{E}_\omega$ *with volume* $V$ *and height* $h$*.*

*(ii) For volumes* $V > V_0$*, there exists a unique stable capillary surface for the energy* $\mathcal{E}_\omega$ *with volume* $V$ *and height* $h$*. Moreover, this surface is spanning.*

Actually, we will later give analytic and geometric characterizations of each of the unique solutions for $V \geq V_0$ in Theorem 2.1. In order to do this, we first recall the classification of surfaces of revolution with constant anisotropic mean curvature (see §6). Such surfaces were studied in detail by the authors in [3] and are called *anisotropic Delaunay surfaces*. They are classified into six classes: horizontal plane, anisotropic catenoid, Wulff shape (up to translation and homothety), cylinder, anisotropic unduloid, and anisotropic nodoid. Each surface in each of these classes has similar properties to the corresponding Delaunay surface.

We let $\mu_i$, $i = 1, 2$ denote the principal curvatures of the Wulff shape $W$ with respect to the inward pointing normal. Here we let $\mu_1$ denote the curvature of the generating curve of $W$.

The following characterization of stable anisotropic capillary surfaces was obtained in our previous papers [4], [5].

**Theorem 2.2** *Let* $X$ *be a capillary surface with free boundary on two horizontal planes for the functional (3) with* $\omega \geq 0$ *and with the Wulff shape for the functional satisfying the conditions (W1) through (W3) stated above.*
*(i) If* $\omega = 0$*, then* $X$ *is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder which is perpendicular to* $\Pi_0 \cup \Pi_1$ *and whose height* $h$ *and radius* $R$ *satisfy*

$$\frac{\mu_1(0)}{\mu_2(0)}(1/R^2) \leq (\pi/h)^2 \, ,$$

*where* $\mu_i(0)$*,* $i = 1, 2$*, is the value of* $\mu_i$ *along the equator of* $W$*. (ii) If* $\omega > 0$ *holds, then* $X$ *is stable if and only if* $X$ *is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.*

Define

$$V_1 := V_1(h,\omega) := \pi h^3 \frac{\int_{-\omega}^{\omega} u^2\, dv}{\left(\int_{-\omega}^{\omega} dv\right)^3}\ .$$

$V_1$ is the volume of the capillary surface which is homothetic to the part of the Wulff shape with contact angle $\vartheta(\omega)$ on the plane $\Pi_i$, $i = 0, 1$.

$$V_2 := V_2(h,\omega) := \pi h^3 \frac{\int_{-\omega}^{\bar{\omega}} u^2\, dv}{\left(\int_{-\omega}^{\bar{\omega}} dv\right)^3}\ .$$

$V_2$ is the volume of the surface which is homothetic to the part of the Wulff shape which is tangent to the plane $\Pi_1$ and with contact angle $\vartheta(\omega)$ on the plane $\Pi_0$ .

**Theorem 2.3** *We assume (W1) through (W3) stated above.*

*(I) Assume* $0 < \omega < \bar{\omega}$*. Then,*

*(i) For volumes* $V_0 \leq V < V_1$*, there exists a unique stable spanning capillary surface with volume* $V$*, height* $h$ *and contact angle* $\vartheta(\omega)$*, and the surface is an anisotropic unduloid. For* $V = V_0$*, this surface has inflection points on the boundary, while, for* $V_0 < V < V_1$*, it does not have inflection points.*

*(ii) For* $V = V_1$*, there exists a unique stable capillary surface with volume* $V$*, height* $h$ *and contact angle* $\vartheta(\omega)$*, and the surface is homothetic to a part of the Wulff shape.*

*(iii) For* $V_1 < V$*, there exists a unique stable capillary surface with volume* $V$*, height* $h$ *and contact angle* $\vartheta(\omega)$*, and the surface is an anisotropic nodoid.*

*(II) Assume* $\omega = \bar{\omega}$*. Then, for* $V_0 < V$*, there exists a unique stable capillary surface with volume* $V$*, height* $h$ *and contact angle* $\vartheta(\omega)$*, and the surface is an anisotropic nodoid.*

Figure 2 shows the generating curves of examples of Theorem 2.3 (I) (i), (ii), and (iii) for the isotropic case, while Figure 3 shows examples for anisotropic case.

For a fixed volume $V$, height $h$ and $\bar{\omega} \geq \omega > 0$, we let $U(V, h, \omega)$ (resp. $N(V, h, \omega)$) denote the stable anisotropic unduloid (resp. nodoid) with volume $V$, height $h := z_1 - z_0$, and contact angle $\vartheta(\omega)$ which we obtained in Theorem 2.3.

**Remark 2.1** In the theorems above “unique” means “unique up to horizontal translation”.

**Remark 2.2** Even in the isotropic case, there is no uniqueness without the stability assumption. Figure 4 shows the plots of the volumes of two families of capillary surfaces for the area functional. The top curve represents the volumes of stable capillary unduloids with height one and contact angle $\pi/4$ with two planes. The bottom curve shows the volumes of unstable capillary unduloids with the same height and contact angles. The generating curves of these surfaces have exactly one interior inflection point which makes them unstable by Theorem 2.2. This shows that volume does not uniquely determine the surface without the stability assumption.

Also, there is no positive lower bound for the volume without the assumption of stability. For any functional satisfying the conditions above, any vertical round cylinder is a capillary surface for the case $\omega = 0$. However the volume of the cylinder can be made arbitrarily small. Also, for $0 < \omega < \bar{\omega}$, there is an unstable unduloid with contact angle $\vartheta(\omega)$ and an arbitrarily small volume.

**Remark 2.3** For $V \geq V_2$, the capillary surface is unique. If $V_0 < V_2$, then, for $V_0 \leq V < V_2$, there exist exactly two stable capillary surfaces (up to translation) with volume $V$, height $h$ and contact angle $\vartheta(\omega)$. One of them is a sessile or pendent drop, while the other has two boundary components. In the isotropic case, these results follow from Chapter 6 of [10]. It would be interesting to know if $V_0 \leq V < V_2$ holds in general. This inequality is proved in the case $\omega = 0$ in §5.

The next result yields a numerical lower bound on the volume of a stable, spanning capillary surface.

**Theorem 2.4** *Assume that the Wulff shape satisfies the conditions (W1) through (W3). If* $0 < \omega \leq \bar{\omega}$*, then*

$$V_0(h, \omega) > \frac{h^3}{\pi}\left(\frac{2u(\omega)(u(0) - u(\omega))}{\omega^2 - (u(0) - u(\omega))^2}\right) > 0 \tag{4}$$

*holds. If* $\omega = 0$*, then*

$$V_0(h, 0) \geq \frac{h^3}{\pi}\left(\frac{\mu_1(0)}{\mu_2(0)}\right) \tag{5}$$

*holds, and this inequality is sharp in the sense that there is a stable cylinder which satisfies the equality in (5).*

**Remark 2.4** For CMC case, Theorem 2.4 implies that, if the contact angle $\vartheta = \vartheta(\omega)$ satisfies $\pi/2 \leq \vartheta \leq \pi$, then

$$V_0(h, \omega) \geq \frac{h^3}{\pi}$$

holds and the equality holds only for the most slender stable cylinder. This is exactly the result proved by Finn and Vogel in [8] for $0 < \vartheta \leq \pi$. Zhou [9] proved this for the general case where the contact angle on the lower and upper planes may be different.

Finally, we will show

**Theorem 2.5** *Assume* $0 \leq \omega \leq \bar{\omega}$ *and that the Wulff shape satisfies the conditions (W1) through (W3). For* $V \geq V_0$*, let* $\Sigma(V) = \Sigma(V, h, \omega)$ *denote the unique stable capillary surface with volume* $V$*, height* $h$ *and contact angles* $\vartheta(\omega)$ *with two boundary components. Here, we let* $\Sigma(V_0, h, \bar{\omega})$ *be the homothety of the Wulff shape with height* $h$ *which touches both of* $\Pi_0$ *and* $\Pi_1$*. Then the family of surfaces* $\Sigma(V)$*,* $V > V_0$*, foliate the open region of space which lies exterior to the surface* $\Sigma(V_0)$ *and lies between the planes* $z = z_i$*,* $i = 0, 1$*.*

# 3 Preliminary results

We introduce the auxiliary quantity

$$V^* := V^*(h, \omega) := \pi h^3 \left( \int_{-\omega}^{\omega} \frac{1}{\sqrt{u^2 - u^2(\omega)}} \, dv \right)^{-2}, \tag{6}$$

which will be used to obtain the lower bound for the volume in Theorem 2.4. The main result of this section is the following technical lemma.

**Lemma 3.1** *Assume* $0 < \omega \leq \bar{\omega}$ *and that the Wulff shape satisfies the conditions (W1) through (W3). Let* $\hat{R}$ *denote the radius of the circle through the points* $(u(\omega), \pm\omega), (u(0), 0)$*. Define*

$$A(\omega) = \left[ \frac{u(\omega) + (\hat{R} - u(0))}{u(\omega)} \right]^{1/2}. \tag{7}$$

*Then, there holds*

$$\int_{-\omega}^{\omega} \frac{dv}{\sqrt{u^2(v) - u^2(\omega)}} < \pi A(\omega), \tag{8}$$

*and consequently*

$$V^*(h, \omega) > \frac{h^3}{\pi} \left( \frac{2u(\omega)(u(0) - u(\omega))}{\omega^2 - (u(0) - u(\omega))^2} \right) > 0 \tag{9}$$

*holds.*

The generating curve $\Gamma_W^+$ of the Wulff shape is represented as

$$(u(\sigma), v(\sigma)), \qquad -L \leq \sigma \leq L,$$

$$u_0 := \max u = u(0) \geq u(\sigma) \geq 0 = u(-L) = u(L), \qquad \forall \sigma \in [-L, L],$$

$$\bar{\omega} := \max v = v(L) \geq v(\sigma) \geq -\bar{\omega} = v(-L), \qquad \forall \sigma \in [-L, L],$$

where $\sigma$ is the arc-length of $(u, v)$ and $2L$ is the length of $\Gamma_W^+$.

$$(u(\sigma), v(\sigma)), \qquad -2L \leq \sigma \leq 2L$$

gives a convex closed curve $\Gamma_W$ which is the section of $W$ by the $(x_1, x_3)$-plane. For simplicity, we set

$$\kappa := \mu_1,$$

which is the curvature of $(u(\sigma), v(\sigma))$ with respect to the inward pointing normal.

**Lemma 3.2**

$$f(\sigma) := u^2(\sigma) + v^2(\sigma)$$

*is a non-decreasing function of* $\sigma$ *in* $0 \leq \sigma \leq L$.

In order to prove Lemma 3.2, we need the following:

**Lemma 3.3** *Consider the eigenvalue problem*

$$\varphi'' + \kappa^2 \varphi = -\lambda \varphi \quad \text{in } 0 \leq \sigma \leq L, \qquad \varphi(0) = \varphi(L) = 0. \tag{10}$$

*Then, the first eigenvalue* $\lambda_1[0, L]$ *of the problem (10) is nonnegative.*

*Proof.* Set $\eta := v'$. Let $\phi$ be a function on $[0, L]$ which satisfies $\phi(0) = \phi(L) = 0$. Since

$$\eta > 0 \quad \text{in } 0 < \sigma < L, \quad \eta(L) = 0, \quad \eta'(L) \neq 0,$$

the function

$$\zeta := \phi/\eta$$

is well-defined on $0 \leq \sigma \leq L$. Elementary calculations show

$$\eta'' + \kappa^2 \eta = \kappa' u'.$$

Using this, we obtain

$$\phi'' + \kappa^2 \phi = \zeta'' \eta + 2\zeta' \eta' + \zeta \kappa' u'.$$

Therefore,

$$\begin{aligned} -\int_0^L \phi(\phi'' + \kappa^2 \phi)\, d\sigma &= -\int_o^L \zeta\eta(\zeta''\eta + 2\zeta'\eta' + \zeta\kappa' u')\, d\sigma \\ &= -\int_0^L \Big\{ (\zeta\zeta'\eta^2)' - (\zeta')^2\eta^2 + \kappa'\zeta^2\eta u' \Big\}\, d\sigma \\ &= -[\zeta'\eta\phi]_0^L + \int_0^L \Big\{ (\zeta')^2\eta^2 - \kappa'\zeta^2 u' v' \Big\}\, d\sigma \\ &= \int_0^L \Big\{ (\zeta')^2\eta^2 - \kappa'\zeta^2 u' v' \Big\}\, d\sigma \geq 0, \end{aligned}$$

which implies that $\lambda_1[0, L] \geq 0$. **q.e.d.**

*Proof of Lemma 3.2.* We will prove $f' \geq 0$. Note that $\kappa' = 0$ on some open interval $(\sigma_1, \sigma_2)$ is equivalent to $f =$ constant and so $f' = 0$ on $(\sigma_1, \sigma_2)$. Denote by $q$ the support function of the curve $(u, v)$. Then, $q = uv' - u'v$. By elementary calculations, we obtain

$$f''' + \kappa^2 f' = -2\kappa' q \leq 0.$$

Note that $f'(0) = f'(L) = 0$ holds. Now assume $f'(\sigma) < 0$ at some $\sigma \in (0, L)$. Then, there exist some $\sigma_1,\ \sigma_2 \in [0, L]$ such that $0 \leq \sigma_1 < \sigma_2 \leq L$ and

$$f'(\sigma) < 0, \quad \forall \sigma \in (\sigma_1, \sigma_2), \qquad f'(\sigma_1) = f'(\sigma_2) = 0$$

holds. We obtain

$$-\int_{\sigma_1}^{\sigma_2} f'(f''' + \kappa^2 f')\, d\sigma = 2\int_{\sigma_1}^{\sigma_2} \kappa' h f'\, d\sigma < 0. \tag{11}$$

Since the eigenvalues of the problem (10) have monotonicity with respect to the region, (11) implies that $\lambda_1[0, L] \leq \lambda_1[\sigma_1, \sigma_2] < 0$. This contradicts Lemma 3.3.
**q.e.d.**

We assume $0 < \omega \leq \bar{\omega}$. $\Gamma_W$ can be regarded as the graph $(u(v), v)$, $-\bar{\omega} \leq v \leq \bar{\omega}$, of a function $u(v)$ of $v$.

The line segment $\ell$ through the points $(u(\omega), \pm\omega)$ is the limit as $R \to \infty$ of a family of arcs $\alpha_R$ through $(u(\omega), \pm\omega)$ of circles $C_R$ of radius $R$ having centers $(-z_R, 0)$ on the real axis. Let $\Gamma$ denote the arc of $\Gamma_W$ with $u > 0$ and $-\omega < v < \omega$. It is clear that for $R >> 0$, $\alpha_R$ lies strictly between $\ell$ and $\Gamma$. From now on we will consider only these values of $R$. Thus, if $\alpha_R$ is given by $(U_R(v), v)$ with $-\omega < v < \omega$, then $0 \leq U_R(v) \leq u(v)$ holds. It is also clear that $0 \leq u(\omega) \leq U_R$ holds and so $U_R^2 - u^2(\omega) \geq 0$ holds. (See Figure 1.) It follows that

$$\int_{-\omega}^{\omega} \frac{1}{\sqrt{u^2 - u^2(\omega)}}\, dv < \int_{-\omega}^{\omega} \frac{1}{\sqrt{U_R^2 - u^2(\omega)}}\, dv\,. \tag{12}$$

We will try to obtain a lower bound of $U_R^2 - u^2(\omega)$.

The equation of $C_R$ is:

$$(U + z_R)^2 + v^2 = R^2\,, \tag{13}$$

and so

$$(u(\omega) + z_R)^2 + \omega^2 = R^2\,. \tag{14}$$

Subtracting these equations and performing elementary manipulations leads to

$$(U_R - u(\omega))(U_R + u(\omega) + 2z_R) = (\omega^2 - v^2)\,.$$

Letting $\hat{z}_R = \max(0, z_R)$, we have

$$U_R^2 - u^2(\omega) \geq \left[\frac{U_R + u(\omega)}{U_R + u(\omega) + 2\hat{z}_R}\right](\omega^2 - v^2)\,.$$

Since the function on the right is a non-decreasing function of $U_R(\geq u(\omega))$, we have

$$U_R^2 - u^2(\omega) \geq \left[\frac{2u(\omega)}{2u(\omega) + 2\hat{z}_R}\right](\omega^2 - v^2)\,. \tag{15}$$

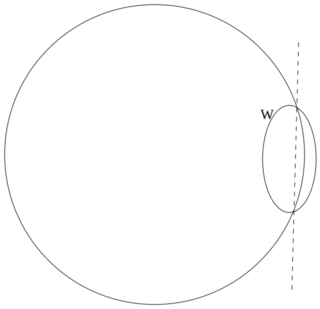

Figure 1:

In order to obtain a lower bound of $U_R^2 - u^2(\omega)$ from (15), we will need a lower bound on $\hat{z}_R$.

**Lemma 3.4** *We consider circles*

$$C_+(a,r) : (U-a)^2 + V^2 = r^2, \qquad U \geq a.$$

*If a circle* $C_+(a,r)$ *touches the right half*

$$\Gamma_W^+ := \{(u(\sigma), v(\sigma)) \mid u(\sigma) \geq 0\}$$

*of* $\Gamma_W$ *at a point* $(u_0, v_0)$ $(v_0 \neq 0)$ *from the left hand side, then* $\Gamma_W$ *is a circle.*

*Proof.* We denote by $\kappa(v) > 0$ the curvature of $\Gamma_W$ at $(u,v)$. Then $\kappa(v)$ is an even function and

$$\kappa'(v) \geq 0, \qquad 0 \leq \forall v \leq \bar{\omega}. \tag{16}$$

Set

$$\begin{aligned} \Gamma &:= \{(u,v) \in \Gamma_W^+ \mid -|v_0| \leq v \leq |v_0|\}, \\ C &:= \{(U,V) \in C_+(a,r) \mid -|v_0| \leq V \leq |v_0|\}. \end{aligned}$$

Because of symmetry, $C$ touches $\Gamma$ on the boundary from the left hand side.

About the curvatures of these two curves at the point $(u_0, v_0)$, it holds that

$$1/r \geq \kappa(v_0).$$

Therefore, by the assumption (W3),

$$1/r \geq \kappa(v), \qquad -v_0 \leq \forall v \leq v_0 \tag{17}$$

holds.

We now move $\Gamma$ in the negative direction of the $u$ axis so that it does not intersect $C$. Then we move $\Gamma$ toward the positive direction of the $u$ axis until it intersects $C$ for the first time and we denote by $\tilde{\Gamma}$ the translated curve at this time. $\tilde{\Gamma}$ is tangent to $C$ at an interior point $(\tilde{u}, \tilde{v})$. Because of the symmetry of

$\tilde{\Gamma}$ and $C$ with respect to the $v$ axis, we may assume that $0 < \tilde{v} \leq v_0$. Since $\tilde{\Gamma}$ lies in the negative side of $C$ with respect to $u$,

$$1/r \leq \kappa(\tilde{v}) \tag{18}$$

holds. It holds from (16), (17) and (18) that

$$1/r = \kappa(v), \qquad \forall v \in [-v_0, -\tilde{v}] \cup [\tilde{v}, v_0].$$

Therefore, $\tilde{\Gamma}$ is tangent to $C$ at the point $(u_0, v_0)$ and it lies on the negative side of $C$ with respect to $u$. On the other hand, since both of $\tilde{\Gamma}$ and $\Gamma$ are tangent to $C$ at point $(u_0, v_0)$, $\tilde{\Gamma}$ coincides with $\Gamma$. Recall $\Gamma$ lies to the positive side of $C$ with respect to $u$. Therefore, $\Gamma$ coincides with $C$. Again by the assumption (W3), $\Gamma_W$ is a circle. **q.e.d.**

**Lemma 3.5** *If we decrease the radius of the circle $C_R$, then the inequalities*

$$u(\omega) \leq U_R \leq u$$

*is satisfied until a value $R = \hat{R}$ is reached at which the curve $(U_{\hat{R}}(v), v)$ is tangent to the curve $(u, v)$ at the point $(u(0), 0)$. Moreover, $\hat{R} \geq u(0)$ holds. Here, the equality holds if and only if $\Gamma_W$ is a circle.*

*Proof.* If the Wulff shape $W$ is a sphere, then the statement clearly holds. Hence we assume that $W$ is not a sphere.

Now, assume that the circle $(U_R(v), v)$ is tangent to the curve $(u, v)$ at a point $(u(v_0), v_0)$ $(0 < v_0 \leq \omega)$ and

$$u(\omega) \leq U_R(v) \leq u(v), \qquad -\omega \leq \forall v \leq \omega$$

holds. Then, by Lemma 3.4, $W$ must be a sphere, which is a contradiction. Therefore, we have proved the first statement.

Next, we prove $\hat{R} \geq u(0)$. We consider circles $C(r)$ with center at the origin. If $r > 0$ is small, then $C(r)$ is contained in the domain bounded by $\Gamma_W$. If we increase $r$, then, at a certain $r_1$, $C(r)$ touches $\Gamma_W$ for the first time from the inside of $\Gamma_W$. Because of Lemma 3.4, $C(r)$ touches $\Gamma_W$ at $(\pm u(0), v(0))$. This implies $\hat{R} \geq u(0)$.

If $\hat{R} = u(0)$, then $C(u(0)) = C_{\hat{R}}$ and this circle touches $\Gamma_W$ at $(u(\omega), \pm\omega)$. Therefore, by Lemma 3.4, $\Gamma_W$ is a circle. **q.e.d.**

*Proof of Lemma 3.1.* Lemma 3.5 supplies a lower bound for $z_R$ which we denote by $\hat{z}(\omega)$, that is, if $\hat{R}$ is the radius of the circle passing the three points $(u(\omega), \pm\omega), (u(0), 0)$, then

$$\hat{z}(\omega) = \hat{R} - u(0).$$

Therefore,

$$A(\omega) := \left[\frac{u(\omega) + \hat{R} - u(0)}{u(\omega)}\right]^{1/2} = \left[\frac{u(\omega) + \hat{z}(\omega)}{u(\omega)}\right]^{1/2}. \tag{19}$$

We obtain from (12) and (15),

$$\int_{-\omega}^{\omega} \frac{1}{\sqrt{u^2 - u^2(\omega)}}\, dv < A(\omega) \int_{-\omega}^{\omega} \frac{1}{\sqrt{\omega^2 - v^2}}\, dv = A(\omega)\pi\,. \tag{20}$$

This implies (8). Since the points $(u(\omega), \pm\omega), (u(0), 0)$ lie on the circle given by (13) with center $(-\hat{z}(\omega), 0)$ and radius $\hat{R}$, we have

$$(u(\omega) + \hat{z}(\omega))^2 + \omega^2 = \hat{R}^2 = (u(0) + \hat{z}(\omega))^2\,.$$

This leads to

$$\hat{z}(\omega) = \frac{u^2(\omega) + \omega^2 - u^2(0)}{2(u(0) - u(\omega))} > 0\,. \tag{21}$$

The last inequality is because the numerator above is nonnegative by Lemma 3.2. By substituting (21) into (19), we obtain

$$A(\omega) = \left[\frac{\omega^2 - (u(0) - u(\omega))^2}{2u(\omega)(u(0) - u(\omega))}\right]^{1/2}\,. \tag{22}$$

The first inequality in (9) follows from (6), (8), and (22). **q.e.d.**

# 4 Proofs of Theorems 2.1, 2.3, 2.4 and 2.5

First, we give a lemma.

**Lemma 4.1** *$V_1 > V_2$ holds.*

*Proof.*

$$V_1 = \pi h^3 \frac{\int_{-\omega}^{\omega} u^2\, dv}{(2\omega)^3}\,, \tag{23}$$

$$V_2 = \pi h^3 \frac{\int_{-\omega}^{\bar{\omega}} u^2\, dv}{(\bar{\omega} + \omega)^3} = \pi h^3 \frac{1}{2\omega(\bar{\omega} + \omega)^2} \int_{\frac{-2\omega^2}{\bar{\omega}+\omega}}^{\frac{2\omega\bar{\omega}}{\bar{\omega}+\omega}} u^2(v(\eta))\, d\eta, \quad \eta := \frac{2\omega}{\bar{\omega} + \omega} v\,. \tag{24}$$

Since

$$v(\eta) = \frac{\bar{\omega} + \omega}{2\omega}\eta, \qquad \frac{\bar{\omega} + \omega}{2\omega} > 1$$

holds,

$$u(v(\eta)) < u(\eta)$$

holds. Hence,

$$\int_{\frac{-2\omega^2}{\bar{\omega}+\omega}}^{\omega} u^2(v(\eta))\, d\eta < \int_{\frac{-2\omega^2}{\bar{\omega}+\omega}}^{\omega} u^2(v)\, dv \tag{25}$$

holds. Set

$$A := \int_{\omega}^{\frac{2\omega\bar{\omega}}{\bar{\omega}+\omega}} u^2(v(\eta))\, d\eta, \qquad B := \int_{-\omega}^{\frac{-2\omega^2}{\bar{\omega}+\omega}} u^2(v)\, dv.$$

We will show $A < B$. By the symmetry of $u(v)$ with respect to $v$, we have

$$B = \int_{\frac{2\omega^2}{\bar{\omega}+\omega}}^{\omega} u^2(v)\, dv.$$

Set

$$\xi(\eta) := \eta - \frac{\omega(\bar{\omega}-\omega)}{\bar{\omega}+\omega}.$$

Then,

$$v(\eta) = \frac{\bar{\omega}+\omega}{2\omega}\xi + \frac{\bar{\omega}-\omega}{2} > \xi.$$

Therefore, we have

$$A = \int_{\frac{2\omega^2}{\bar{\omega}+\omega}}^{\omega} u^2(v(\eta(\xi)))\, d\xi < \int_{\frac{2\omega^2}{\bar{\omega}+\omega}}^{\omega} u^2(\xi)\, d\xi = B. \tag{26}$$

(23) – (26) implies that $V_2 < V_1$ holds. **q.e.d.**

*Proof of Theorems 2.1, 2.3 and 2.4.* First note that, for $V > V_2$, there is no sessile or pendent drop. Especially, by Lemma 4.1, for $V \geq V_1$, there is no sessile or pendent drop.

Assume $0 < \omega \leq \bar{\omega}$. Let $X(s,\theta) = (x(s)e^{i\theta}, z(s))$ be a stable spanning capillary surface. Then, by Theorem 2.2 (ii), $X$ is a convex part of an anisotropic Delaunay surface. From the representation formula (Theorem 6.2) for anisotropic Delaunay surfaces and Remark 6.1, it follows that for stable capillary surfaces, the height $h$ and volume $V$ are given as follows: First note that

$$dz = \frac{z_s}{x_s}dx = \frac{v_\sigma}{u_\sigma}dx = x_u dv$$

holds. Therefore,

$$h = \int_{v=-\omega}^{v=\omega} dz = \int_{-\omega}^{\omega} x_u\, dv = \frac{1}{-\Lambda}\int_{-\omega}^{\omega} 1 + \frac{u}{\sqrt{u^2+\Lambda c}}\, dv, \tag{27}$$

$$V = \pi \int_{v=-\omega}^{v=\omega} x^2\, dz = \pi(-\Lambda)^{-3}\int_{-\omega}^{\omega} \frac{(u+\sqrt{u^2+\Lambda c})^3}{\sqrt{u^2+\Lambda c}}\, dv,$$

here $\Lambda \leq 0$ is the anisotropic mean curvature of $X$ and $c$ is the flux parameter for $X$.

We consider the scale invariant quantity,

$$a := -\Lambda c.$$

Then,

$$h = \frac{1}{-\Lambda}\int_{-\omega}^{\omega} 1 + \frac{u}{\sqrt{u^2-a}}\, dv, \tag{28}$$

$$V = \pi(-\Lambda)^{-3}\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv, \tag{29}$$

and we obtain

$$V = \pi h^3\Big(\int_{-\omega}^{\omega} 1+\frac{u}{\sqrt{u^2-a}}\,dv\Big)^{-3}\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv\,. \tag{30}$$

By applying Hölder's inequality for the measure $dv/\sqrt{u^2-1}$ to the formula for $h$, we obtain

$$h \le \frac{1}{-\Lambda}\Big(\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv\Big)^{1/3}\Big(\int_{-\omega}^{\omega}\frac{1}{\sqrt{u^2-a}}\,dv\Big)^{2/3}.$$

It then follows that

$$V/h^3 \ge \pi\Big(\int_{-\omega}^{\omega}\frac{1}{\sqrt{u^2-a}}\,dv\Big)^{-2} \ge \pi\Big(\int_{-\omega}^{\omega}\frac{1}{\sqrt{u^2-u^2(\omega)}}\,dv\Big)^{-2}. \tag{31}$$

If $V_0(h,\omega)$ is defined as the infimum of the volume of all stable capillary surfaces with the given height and $\omega$ having *non empty* boundary components on both planes, (31) shows that $V_0(h,\omega) \ge V^*(h,\omega)$ holds. The first half of Theorem 2.4 then follows from this inequality and Lemma 3.1. The second half of Theorem 2.4 follows easily from Theorem 2.2 (i).

Define

$$\Gamma(a,\omega) := \pi\Big(\int_{-\omega}^{\omega} 1+\frac{u}{\sqrt{u^2-a}}\,dv\Big)^{-3}\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv\,. \tag{32}$$

It follows from (28) and (29) that a necessary and sufficient condition that there exists a spanning, stable capillary surface with prescribed $h$ and $V$ is that $V/h^3 = \Gamma(a,\omega)$ for some real value $a$.

Note that

$$a = -\Lambda c \le u^2(\omega) \quad \text{for } 0<\omega\le\bar\omega,$$
$$0 = u(\bar\omega) < u(\omega) < u(0) \quad \text{for } 0<\omega<\bar\omega$$

hold. Also note (see Section 6) that for $a > 0$ the capillary surfaces are anisotropic unduloids, for $a = 0$ they are part of the Wulff shape (up to translation and homothety), while for $a < 0$ they are anisotropic nodoids. The result will then follow by showing that with the height $h$ fixed, the volume is a strictly decreasing function of $a$ for $a \le u^2(\omega)$.

First assume $a < u^2(\omega)$. Differentiating (30) with respect to $a$, we have

$$\begin{aligned}
&-2(\pi h^3)^{-1}V_a\Big(\int_{-\omega}^{\omega} 1+\frac{u}{\sqrt{u^2-a}}\,dv\Big)^4\\
&= 3\int_{-\omega}^{\omega}\frac{u}{(u^2-a)^{3/2}}\,dv\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv\\
&-\int_{-\omega}^{\omega}\frac{u+\sqrt{u^2-a}}{\sqrt{u^2-a}}\,dv\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^2(u-2\sqrt{u^2-a})}{(u^2-a)^{3/2}}\,dv.
\end{aligned} \tag{33}$$

We will show that the right hand side of (33) is positive. If

$$u-2\sqrt{u^2-a}\le 0$$

holds for all $u$ for $-\omega \le v \le \omega$, then it is done. Assume now that

$$u-2\sqrt{u^2-a}>0 \tag{34}$$

holds for some $u$. In particular, $a$ must be positive. Note that

$$\begin{aligned}
&-2(\pi h^3)^{-1}V_a\left(\int_{-\omega}^{\omega}1+\frac{u}{\sqrt{u^2-a}}\,dv\right)^4\\
&=\frac{1}{a}\left(3\int_{-\omega}^{\omega}\frac{au}{(u^2-a)^{3/2}}\,dv\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^3}{\sqrt{u^2-a}}\,dv\right.\\
&\left.-\int_{-\omega}^{\omega}\frac{a(u+\sqrt{u^2-a})}{\sqrt{u^2-a}}\,dv\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^2(u-2\sqrt{u^2-a})}{(u^2-a)^{3/2}}\,dv\right).
\end{aligned}\tag{35}$$

We will prove that

$$(u+\sqrt{u^2-a})^3>a(u+\sqrt{u^2-a}), \tag{36}$$

$$au>(u+\sqrt{u^2-a})^2(u-2\sqrt{u^2-a}) \tag{37}$$

holds for any $u$ satisfying (34). Because $a<u^2$ holds, (36) clearly holds. (34) is equivalent to

$$a<u^2<\frac{4}{3}a. \tag{38}$$

Set

$$f(u):=au-(u+\sqrt{u^2-a})^2(u-2\sqrt{u^2-a}).$$

Then,

$$f(u)=2(u+\sqrt{u^2-a})(u^2-a)>0$$

holds. This proves (37). Combining (35) with (36) and (37) gives

$$V_a<0.$$

This implies that $V$ is a strictly decreasing function of $a$.

It remains to show that the monotonicity extends to the point $a=u^2(\omega)$. This will follow if we can show that, again with the height fixed, $V$ has an extension to $u^2(\omega)$ which is continuous from below.

Both integrals in (32) are of the form

$$\int_{-\omega}^{\omega}\frac{(u+\sqrt{u^2-a})^p}{\sqrt{u^2-a}}\,dv$$

with $p=1,3$. Also, both integrands are bounded by constant$\cdot(\sqrt{u^2-a})^{-1}\le$ constant$\cdot(\sqrt{u^2-u^2(\omega)})^{-1}$. Therefore, the continuity of $V$ from below follows by the Dominated Convergence Theorem since it follows from Theorem 2.4 that the integral

$$I := \int_{-\omega}^{\omega} \frac{1}{\sqrt{u^2 - u^2(\omega)}}\, dv$$

is convergent. **q.e.d.**

*Proof of Theorem 2.5.* In the case where $\omega = 0$, because of Theorem 2.2 (i), the statement clearly holds. So, we will assume $0 < \omega \leq \bar{\omega}$. For convenience we will assume that the height is 1. We may assume $z_0 = -1/2$ and $z_1 = 1/2$. For $-\omega \leq v \leq \omega$, we write the generating curve of the Wulff shape as $(u(v), v)$. By using Theorem 6.2 and (28), we can express the coordinates of each capillary surface as $v \mapsto (x(a,v), z(a,v))$, with

$$x(a,v) = \frac{u(v) + \sqrt{u^2(v) - a}}{\int_{-\omega}^{\omega} 1 + u(v)/\sqrt{u^2(v) - a}\, dv}\,.$$

Then for $a < u^2(\omega)$,

$$\begin{aligned}(\partial_a x)(a,v) \quad &= \quad -(1/2)\left(\sqrt{u^2(v)-a}\int_{-\omega}^{\omega} 1 + u(v)/\sqrt{u^2(v)-a}\, dv\right)^{-1} \\ &\qquad -(1/2)(u(v) + \sqrt{u^2(v)-a})\left(\int_{-\omega}^{\omega} u(v)(u^2(v)-a)^{-3/2}\, dv\right) \times \\ &\qquad \left(\int_{-\omega}^{\omega} (1 + u(v)/\sqrt{u^2(v)-a})\, dv\right)^{-2} \\ &< \quad 0\,.\end{aligned}$$

Thus, for $v$ fixed, $x(a,v)$ is strictly decreasing as a function of $a$ for $a < u^2(\omega)$. Note that the generating curve of each capillary surface can also be represented as a graph $x = \underline{x}(a,z)$, $-1/2 \leq z \leq 1/2$.

We now assume that two generating curves $(x(a,v), z(a,v))$, $(x(b,v), z(b,v))$, $a < b < u^2(\omega)$, intersect. By the above, it is clear that $x(b,0) < x(a,0)$ and equivalently $\underline{x}(b,0) < \underline{x}(a,0)$. Similarly, $x(b,\pm\omega) < x(a,\pm\omega)$ holds, and so equivalently $\underline{x}(b,\pm 1/2) < \underline{x}(a,\pm 1/2)$.

Note that these two curves cannot have any non-transversal intersections for $-1/2 < z < 1/2$. If they did, then at the point of intersection, the values of $v$ (which depends only on the tangent at each point) for both curves must agree, contradicting the fact that $x(A,v)$ is strictly decreasing in $A$ for $A < u^2(\omega)$.

It follows from the inequalities given above, that the two curves have at least two transversal intersections at heights $0 < z = \zeta_1 < \zeta_2 < z_1$. We will assume that the $\zeta_1$ is the height of the 'first' such intersection and that $\zeta_2$ is the next such intersection.

Since $\underline{x}(b,0) < \underline{x}(a,0)$, holds, we must have

$$\partial_z \underline{x}(a,\zeta_1) \leq \partial_z \underline{x}(b,\zeta_1) < 0\,,$$

$$\partial_z \underline{x}(b,\zeta_2) \le \partial_z \underline{x}(a,\zeta_2) < 0\,.$$

By the Intermediate Value Theorem, $\partial_z \underline{x}(a,\zeta^*) = \partial_z \underline{x}(b,\zeta^*)$ must hold for some $\zeta^* \in [\zeta_1,\zeta_2]$. Note that for $z \in (\zeta_1,\zeta_2)$, $\underline{x}(a,z) < \underline{x}(b,z)$ holds. A contradiction is reached because at the points $(\underline{x}(a,\zeta^*),\zeta^*)$ and $(\underline{x}(b,\zeta^*),\zeta^*)$, the tangent vectors agree and hence the values of $v$ at both points agree. Thus $\underline{x}(a,\zeta^*) > \underline{x}(b,\zeta^*)$ must hold by monotonicity of $x(A,v)$ with respect to $A$. This shows that distinct generating curves do not intersect.

By (28), it follows that $-\Lambda \to 2\omega$ as $a \to -\infty$. It then follows from the formula for $x(a,v)$, that

$$x(a,0) \to \infty, \quad x(a,\pm\omega) \to \infty, \quad \text{as} \quad a \to -\infty\,.$$

It then follows that since the deformation of the generating curves depends continuously on $a$, the family of surfaces $\Sigma(V)$ fill out the region exterior to $\Sigma(V_0)$. **q.e.d.**

# 5 Deflating a cylinder

Theorem 2.1 (I) asserts that for each $\omega \in [0,\bar{\omega})$ there is a least volume, stable capillary surface having two boundary components on the planes $z = z_i$, $i = 0, 1$. One may ask what will occur if the volume of this surface is decreased. It is expected that one or both boundary components will detach from the supporting planes and that the surface of the drop forms into one or more sessile or pendent drops or rescaled Wulff shapes.

In order for the drop to remain, it must be the case that the sessile drop with contact angle $\vartheta(\omega)$ or the entire rescaled Wulff shape with height $z_1 - z_0$ has volume at least as large as $V_0$. We consider here only the case $\omega = 0$. Since, for the fixed height, the entire rescaled Wulff shape contains one fourth the volume of half of the Wulff shape, we consider the first possibility.

We assume for convenience that $z_1 - z_0 = 1$. Theorem 2.2 (i) implies that the radius $R$ of the least volume stable capillary cylinder satisfies

$$R = \big(\mu_1(0)/\mu_2(0)\big)^{1/2}\pi^{-1}.$$

Therefore, the minimum volume is

$$V_0(\omega = 0) = \frac{\mu_1(0)}{\pi\mu_2(0)}\,.$$

Representing, the generating curve of the Wulff shape as $u = u(v)$, we have $\mu_1(0) = -u_{vv}(0)$, $\mu_2(0) = 1/u(0)$, and hence

$$V_0(\omega = 0) = \frac{u(0)|u_{vv}(0)|}{\pi}\,. \tag{39}$$

**Proposition 5.1** *Assume (W1) – (W3). Then, there holds*

$$V_2(\omega = 0) > V_0(\omega = 0)\ .$$

*Proof.* Let $u = u(v)$ be the generating curve of $W$. We claim that

$$u(t\bar{\omega}) \geq (1-t)u(0) - (u_{vv}(0)/2)\bar{\omega}^2 t(1-t) \tag{40}$$

holds. Assume this for now.

Using the inequality $(a+b)^2 \geq 4ab$, for all $a, b \geq 0$, we obtain

$$u^2(t\bar{\omega}) \geq 2u(0)|u_{vv}(0)|\bar{\omega}^2 t(1-t)^2\ .$$

Therefore,

$$\begin{aligned} \int_0^{\bar{\omega}} u^2\, dv &= \bar{\omega} \int_0^1 u^2(t\bar{\omega})\, dt \\ &\geq 2u(0)|u_{vv}(0)|\bar{\omega}^3 \int_0^1 t(1-t)^2\, dt \\ &= (1/6)u(0)|u_{vv}(0)|\bar{\omega}^3\ . \end{aligned}$$

It follows that

$$\begin{aligned} V_2(\omega = 0) &= (\pi/\bar{\omega}^3) \int_0^{\bar{\omega}} u^2\, dv \geq (\pi/6)u(0)|u_{vv}(0)| \\ &> (1/\pi)u(0)|u_{vv}(0)| = V_0(\omega = 0)\ . \end{aligned}$$

We now show (40). Let $H(t) = u(t\bar{\omega}) - (1-t)u(0) + (u_{vv}(0)/2)\bar{\omega}^2 t(1-t)$. Note $H(0) = H(1) = 0$. If (40) doesn't hold, then $H$ attains a negative minimum at some point in $(0,1)$ where $H'' \geq 0$ holds. A simple calculation shows that $H''(t) = \bar{\omega}^2(u_{vv}(t\bar{\omega}) - u_{vv}(0))$ which is negative since $-u_{vv} > -u_{vv}(1+u_v^2)^{-3/2} \geq -u_{vv}(0)$ holds on $(0, \bar{\omega})$ by the assumption (W3) on the curvature of the Wulff shape. **q.e.d.**

# 6 Appendix: Anisotropic Delaunay surfaces

We summarize important results about surfaces of revolution with constant anisotropic mean curvature for a rotationally symmetric energy functional (anisotropic Delaunay surfaces). Such surfaces were studied in detail by the authors in [3] (see also [4] and [5]).

Let

$$\chi(\sigma, \theta) = (u(\sigma)e^{i\theta}, v(\sigma))$$

be a parametrization of the Wulff shape $W$, where $(u(\sigma), v(\sigma))$ is the arc length parametrization of the generating curve. We have identified $\mathbf{R}^3$ with $\mathbf{C} \times \mathbf{R}$

in the formula above. We may extend $(u(\sigma), v(\sigma))$ so that it is defined for all real number $\sigma$. In this case, $(u(\sigma), v(\sigma))$ represents the section of $W$ by $(x_1, x_3)$-plane.

Consider an anisotropic Delaunay surface $\Sigma$ parameterized by

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)) \,,$$

where $(x(s), z(s))$ is the arc length parameterization of the generating curve, and $x(s) \geq 0$ holds for all $s$. The Gauss map of the surface $X$ is given by

$$\nu = (z'(s)e^{i\theta}, -x'(s)).$$

We choose the orientation of the generating curve so that $\nu$ points "outward" from the surface. There is a natural map from the surface to the Wulff shape $W$ defined by the requirement that the oriented tangent planes to both surfaces agree at corresponding points. Thus, at corresponding points the outward pointing unit normals must agree and we have

$$x' = u_\sigma \,, \qquad z' = v_\sigma. \tag{41}$$

In [3], we showed that the profile curve $(x, z)$ satisfies the equation

$$2{\mu_2}^{-1} x z' + \Lambda x^2 = c \,, \tag{42}$$

where $\Lambda$ is the anisotropic mean curvature and $c$ is a real constant called the flux parameter. Also, $-\mu_2$ is the principal curvature of the Wulff shape in the $\theta$ direction. Since $W$ is a surface of revolution, we have $\mu_2 = \mu_2(\nu_3) = \mu_2(-u_\sigma) = \mu_2(-x')$ by (41). Computing the principal curvature $-\mu_2 = -v_\sigma/u$, (42) can be expressed as

$$2ux + \Lambda x^2 = c \,. \tag{43}$$

The orientation of an anisotropic Delaunay surface may be chosen so that $\Lambda \leq 0$ holds and then the anisotropic Delaunay surfaces fall into six cases as follows:

- (I-1) $\Lambda = 0$ and $c = 0$: *horizontal plane.*
- (I-2) $\Lambda = 0$ and $c \neq 0$: *anisotropic catenoid.*
- (II-1) $\Lambda < 0$ and $c = 0$: *Wulff shape (up to vertical translation and homothety).*
- (II-2) $\Lambda < 0$ and $c = ((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1}$: *cylinder of radius* $(\mu_2|_{\nu_3=0}|\Lambda|)^{-1}$.
- (II-3) $\Lambda < 0$ and $((\mu_2|_{\nu_3=0})^2|\Lambda|)^{-1} > c > 0$: *anisotropic unduloid.*
- (II-4) $\Lambda < 0$ and $c < 0$: *anisotropic nodoid.*

Any surface in each case above is complete, and it has similar properties to the corresponding CMC surface in the sense of the following Lemma.

**Theorem 6.1 ([3], [4], [5])** *(i) The generating curve* $C : (x(s), z(s))$ *of an anisotropic catenoid is a graph over the whole* $z$*-axis, and* $z'(s) \neq 0$ *for all* $s$*.* $C$ *is perpendicular to the horizontal line at a unique point.*

*(ii) Let* $(x(s), z(s))$*,* $(x \geq 0)$*, be the generating curve of an anisotropic unduloid or an anisotropic nodoid. Then, there is a unique local maximum* $B$ *and a unique local minimum* $N > 0$ *of* $x$*, which we will call a bulge and a neck respectively.*

*(iii) The generating curve* $C : (x(s), z(s))$ *of an anisotropic unduloid is a graph over the* $z$*-axis, and* $z'(s) > 0$ *for all* $s$*.* $C$ *is a periodic curve with respect to the vertical translation, and the region from a neck to the next neck (and/or a bulge to the next bulge) gives one period. Therefore,* $C$ *has a unique inflection point* $(x, z)$ *between each neck and the next bulge, which satisfies* $x = \sqrt{c/(-\Lambda)}$*.*

*(iv) The curvature of the generating curve* $C$ *of an anisotropic nodoid has a definite sign.* $C$ *is a non-embedding periodic curve with respect to the vertical translation. The region from a neck to the next neck (and/or a bulge to the next bulge) gives one period.*

In the previous sections, we needed a representation formula for the profile curves which is summarized in the following result from [3].

**Theorem 6.2 ([3])** *Let* $W$ *be the Wulff shape of a rotationally symmetric anisotropic surface energy* $\mathcal{F}$*. Let*

$$\sigma \mapsto (u(\sigma), v(\sigma)), \quad \sigma \in (-\infty, \infty),$$

*be the profile curve of* $W$*, where* $\sigma$ *is the arc length. Then*

$$\mu_2^{-1} v_\sigma - u = 0$$

*holds. Let* $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ *be a surface with constant anisotropic mean curvature* $\Lambda \leq 0$*, and let the Gauss map of* $X$ *coincide with that of* $W$ *at* $s = s(\sigma)$*. Then* $X$ *is given as follows.*

*(i) When* $X$ *is an anisotropic catenoid,*

$$x = c/(2u)$$

*for some nonzero constant* $c$*.*

*(ii) When* $X$ *is an anisotropic unduloid,*

$$x = \frac{u \pm \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

*for some constants* $c > 0$ *and* $\Lambda < 0$*, where* $x = x(u(\sigma))$ *is defined in* $\{\sigma | u \geq \sqrt{-\Lambda c}\}$*.*

*(iii) When* $X$ *is an anisotropic nodoid,*

$$x = \frac{u + \sqrt{u^2 + \Lambda c}}{-\Lambda}$$

*for some constants* $c < 0$ *and* $\Lambda < 0$*, where* $x = x(u(\sigma))$ *is defined in* $\{-\infty < \sigma < \infty\}$*.*

*In all cases above,* $z$ *is given by*

$$z = \int^{u} v_u x_u \, du. \tag{44}$$

*Conversely, for a Wulff shape* $W$ *defined as above, define* $x$ *and* $z$ *as in (i) – (iii) and (44). Then* $X(s, \theta) = (x(s)e^{i\theta}, z(s))$ *is an anisotropic Delaunay surface which satisfies*

$$2\mu_2^{-1} z_s x + \Lambda x^2 = c,$$

*where* $s$ *is the arc length of* $(x, z)$*, and* $\Lambda$ *is supposed to be zero for Case (i). Moreover,* $X$ *has the same regularity as that of* $W$*.*

**Remark 6.1** In (ii) in Theorem 6.2, $x = (-\Lambda)^{-1}(u + \sqrt{u^2 + \Lambda c})$ gives the part of the anisotropic unduloid whose Gaussian curvature is positive (i.e. the convex part), while $x = (-\Lambda)^{-1}(u - \sqrt{u^2 + \Lambda c})$ gives the part of the anisotropic unduloid whose Gaussian curvature is negative.

**Remark 6.2** In (iii) in Theorem 6.2, $u > 0$ corresponds to the part of the anisotropic nodoid whose Gaussian curvature is positive (i.e. the convex part), while $u < 0$ gives the part of the anisotropic nodoid whose Gaussian curvature is negative.

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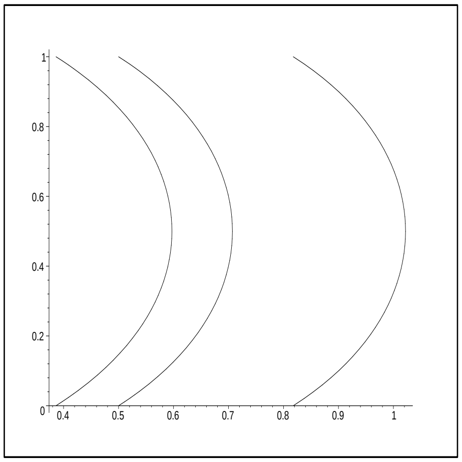

Figure 2: The innermost curve generates an (isotropic) unduloid, the middle curve is a sphere and the outer curve is a nodoid. The height is 1 and the contact angle is $\pi/4$. The values of $a$ are $0.25, 0$ and $-1$.

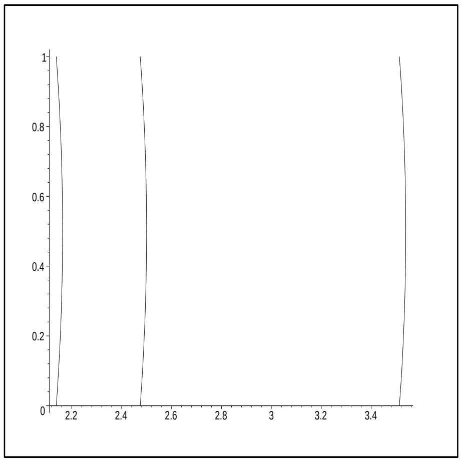

Figure 3: The innermost curve generates an anisotropic unduloid, the middle curve is a part of the Wulff shape $u^2 + v^4 = 1$ and the outer curve is an anisotropic nodoid. The height is 1 and the contact angle is $\omega = \pi/4$. The values of $a$ are $0.25, 0$ and $-1$.

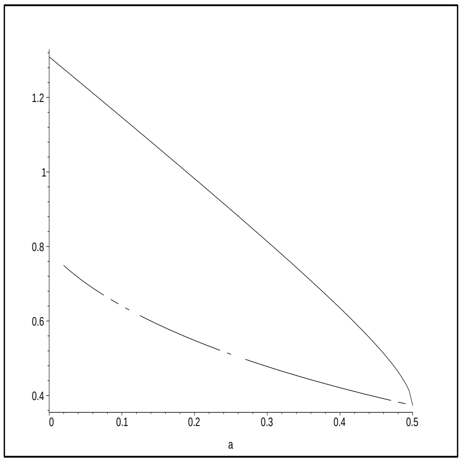

Figure 4: Plot of the volumes of stable (upper) and unstable (lower) unduloids as a function of $a$. The height is 1 and the contact angle is $\pi/4$. The generating curves of the unstable unduloids have exactly one inflection point.