Equilibria for anisotropic bending energies

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Abstract. We study a variational problem involving an anisotropic bending energy for surfaces. Surfaces with boundary and closed equilibria are discussed.

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The interface between immiscible materials may be regarded as a surface. The surface forms so as to minimize a free energy subject to whatever external forces, boundary conditions and constraints are imposed by the environment. For a homogeneous material in a disordered phase, the appropriate surface energy is isotropic; it is independent of the direction of the surface. However, for materials in an ordered phase the surface energy is anisotropic.

The classical bending energy has a long history which began with the Sophie Germain's investigation of the Chladni plates in the nineteenth century. More recently, the bending energy has been extensively used to study the morphology of vesicles, [1]. The aim of this paper is to discuss anisotropic versions of the bending energy of a smooth surface in the Euclidean space. A different version of an anisotropic bending energy is discussed in [2]. [3] Our current definition of anisotropic bending energy uses invariants of a tensor whose trace (3) has already been employed in classical works to characterize equilibria of anisotropic media [4], [5].

Our model for an anisotropic surface energy begins with a fixed smooth, closed, convex surface, called a Wulff shape, which we denote by W. We recall that for such a surface, the normal (Gauss) map, $N : W \to S^2$, defines a smooth bijection. If χ denotes the position vector of W, then the function $Q := \chi \cdot N$ is called the support function of W. Its value at $\omega \in W$ is the distance from the origin in $\mathbf{R}^3$ to the tangent plane to W at ω.

We let γ be the function defined on S^2 defined by, $\gamma = Q \circ N^{-1}$. For a smooth, oriented surface $X : \Sigma \to \mathbf{R}^3$ with normal field ν, we define its free energy to be

$$\mathscr{F}[\Sigma] := \int_\Sigma \gamma(\nu)\, d\Sigma\,. \qquad (1)$$

The fundamental fact relating the surface W to this free energy is known as Wulff's Theorem [6]: *Among all closed surfaces enclosing the same volume as W, the absolute minimum of* $\mathscr{F}$ *is attained at the surface W*. This means that W solves the isoperimetric

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problem for the anisotropic surface energy $\mathscr{F}$ and so W plays the role that the sphere plays for the area functional. We mention two other characterizations of the Wulff shape. One, [7], is that W is (again, up to homothey and certain rigid motions) the only stable equilibrium for the anisotropic free energy $\mathscr{F}$ among all closed surfaces containing a fixed volume. Another is a recent result, [8], that W is (again, up to homothey and certain rigid motions) , the only closed embedded equilibrium of $\mathscr{F}$. Other characterizations can be found in [9], [10].

If we impose the additional condition that W is invariant under the map $\chi \mapsto -\chi$, then W is the unit sphere for a norm $|\cdot|$ on three dimensional space and, in this case, the associated free energy can be written

$$\mathscr{F}[\Sigma] = \int_\Sigma |\nu|_* \, d\Sigma \, .$$

Here $|\cdot|_*$ denoted the dual norm defined by

$$|a|_* := \max_{|b|=1} a \cdot b \, .$$

It is usual in the literature to begin with the energy denisty γ in (1) defined on S^2 and use it to produce the Wulff shape W. This process is known as the Wulff construction. In the case where γ is smooth, W is parameterized by the Cahn-Hoffman field

$$\chi = D\gamma + \gamma\nu \, , \nu \in S^2 \, ,$$

where $D\gamma$ is the gradient of γ on the two dimensional sphere. For a surface, this field can be composed with the Gauss (normal) map to obtain a mapping of the surface into the Wulff shape. For a point p in the surface, the tangent plane to Σ at p and the tangent plane to W at $\chi(p)$ are parallel.

The type of surface energy in (1) was developed by Wulff to model the equilibrium shape of a small crystal and was used by Virga, [12], [13], to model the equilibrium shape of a small nematic liquid crystal submersed in an isotropic environment. In particular, the Rapini-Papouler functional

$$\gamma := 1 + \varepsilon\nu_3^2 \, , \quad \varepsilon \in (-1, 1)$$

is applied. Virga shows that as the size of the drop goes to zero, the director field tends to a constant. Thus, the bulk energy vanishes and anchoring energy is of the form $\gamma(\nu \cdot \vec{a})$ for a fixed vector $\vec{a}$.

Besides absolute minima, other stationary surfaces are also of interest. These are, of course, characterized by the vanishing of the first variation. For a deformation $X + \varepsilon\delta X + \mathscr{O}(\varepsilon^2)$, we obtain

$$\delta\mathscr{F} = -\int_\Sigma \Lambda\delta X \cdot \nu \, d\Sigma + \oint_{\partial\Sigma} (\delta X \times \frac{dX}{ds}) \cdot (D\gamma + \gamma\nu) \, ds \, . \tag{2}$$

If we disregard for the moment whatever boundary conditions may be imposed, we see that a necessary condition that the surface be in equilibrium for the free energy is that

$$\Lambda \equiv 0 \, .$$

Since the first variation of volume is given by

$$\delta V = \int_\Sigma \delta X \cdot \nu \, d\Sigma \,,$$

the equation $\Lambda \equiv$constant describes critical equilibria of the free energy subject to a volume constraint.

The quantity Λ will be called the anisotropic mean curvature. When $\gamma \equiv 1$ this quantity reduces to twice the usual mean curvature. The Cahn-Hoffman field X, defines a mapping of Σ into W. The tangent spaces to Σ at p and to W at $\chi(p)$ are parallel. Therefore, we can consider the differential $d\chi|_p$ as a linear map from $T_p\Sigma$ into itself. Although it is not self-adjoint, the differential is diagonalizable and we denote its eigenvalues by $\{\lambda_1, \lambda_2\}$. The eigenvalues are called the anisotropic principal curvatures. The local expression for Λ is

$$\lambda_1 + \lambda_2 = \Lambda \,. \tag{3}$$

Besides the trace, we can form the determinant:

$$\lambda_1 \lambda_2 = K_\Sigma / K_W \,,$$

where K_Σ is the Gaussian curvature of Σ and K_W is the Gaussian curvature of W at the point $\chi(p)$.

By analogy with the usual bending energy, we define an anisotropic bending energy by

$$E_{a,b,c} := \int_\Sigma a + \frac{b}{4}(\Lambda - \Lambda_0)^2 - cK_\Sigma / K_W \, d\Sigma \,, \tag{4}$$

where $a > 0, b$, and Λ_0 are constants. In (4) we have replaced the usual mean curvature and Gauss curvature by their anisotropic versions. Such energies have been applied, [14] to study surfactant laden liquid-liquid crystal interfaces.

We remark that for a closed , boundaryless surface, the integrand in the last term is a topological invariant:

$$\int_\Sigma K_\Sigma / K_W d\Sigma = \mathrm{degree}(\chi) \cdot \mathrm{Area}(W) \,,$$

which is therefore unchanged under smooth deformations. For surfaces with boundary, the integral of K_Σ / K_W is unchanged by deformations which keep the boundary fixed to first order. In either case, the functional $E_{a,b,c}$ is variationally equivalent to the anisotropic Willmore functional

$$E_{1,0,0} := \int_\Sigma \Lambda^2 \, d\Sigma \,. \tag{5}$$

For this functional, the Wulff shape is again characterized by an extremizing property, [11], [15] : *among all closed genus zero surfaces, the anisotropic Willmore functional attains its absolute minimum exactly when the surface is (up to homothey and certain rigid motions) the Wulff shape*. These rigid motions are the symmetries of W.

In order to discuss the Euler-Lagrange equation for the functional in (4), we write the deformation field

$$\delta X = \xi + \psi \nu \,,$$

where ξ is a tangent field to the surface. The induced infinitesimal change in Λ is,[15],

$$\delta \Lambda =: L[\psi] + \xi \cdot \nabla \Lambda \,. \tag{6}$$

Here, L is the self-adjoint linear operator defined by

$$L[\psi] = \mathrm{div} A \nabla \psi + \langle A \cdot d\nu, d\nu \rangle \psi \,,$$

and

$$A := D^2 \gamma + \gamma I \,,$$

were $D^2\gamma$ denotes the second derivative of γ on the two-sphere. Also, let $\mathscr{L} := L - H\Lambda$. A lengthy calculation gives

$$\begin{aligned}
\delta E_{a,b,c} \quad = \quad & \int_\Sigma \psi\left(\frac{b}{2}\mathscr{L}[\Lambda - \Lambda_0] - 2aH\right) d\Sigma \\
& + \oint_{\partial\Sigma} \left(a + \frac{b}{4}(\Lambda - \Lambda_0)^2 - cK_\Sigma / K_W\right) \xi \cdot n \, ds \\
& + \oint_{\partial\Sigma} \frac{b}{2}\left((\Lambda - \Lambda_0) A \nabla \psi \cdot n - \psi A \nabla \Lambda \cdot n\right) + c[A \nabla \psi, A \, d\nu(t), \nu] \, ds \,.
\end{aligned}$$

Here $[\cdot,\cdot,\cdot]$ is the vector triple product .

We will now assume that the free energy $\mathscr{F}$ is rotationally invariant about a vertical axis. Equivalently, we can assume that the Wulff shape W is axially symmetric.

In this case the operator A is diagonalized by the frame $\{e_1, e_2\}$ where e_1 is a unit vector pointing in the direction of the tangential component of the vertical vector E_3 and $e_2 = \nu \times e_1$. The tensor A then has the form

$$A = \frac{1}{\mu_1} e_1 \otimes e_1 + \frac{1}{\mu_2} e_2 \otimes e_2 \,.$$

$$\frac{1}{\mu_2} = \gamma(\nu_3) - \nu_3 \gamma'(\nu_3) \,, \qquad \frac{1}{\mu_1} = (1 - \nu_3^2)\gamma''(\nu_3) + \frac{1}{\mu_2} \,. \tag{7}$$

We also assume that the surface Σ has a free boundary component on a horizontal plane, which we assume to be the plane $x_3 = 0$, and that the rest of the surface lies above this plane. The surface may or may not have other boundary components. Along the boundary component in question, we have that the outward pointing normal to the boundary curve is given by $n = -e_1$ and the unit tangent to the curve is $t = \nu \times n = \pm e_2$.

$$\begin{aligned}
\delta E_{a,b,c} \quad = \quad & \int_\Sigma \psi\left(\frac{b}{2}\mathscr{L}[\Lambda - \Lambda_0] - 2aH\right) d\Sigma \\
& + \oint_{\partial\Sigma} \psi\left(\frac{-b\Lambda_n}{2\mu_1} - c\partial_t\left(\frac{\sigma_{12}}{\mu_1\mu_2}\right)\right) ds
\end{aligned}$$

$$+\oint_{\partial\Sigma}(a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W)\xi\cdot n\,ds$$
$$+\oint_{\partial\Sigma}\psi_n(\frac{b(\Lambda-\Lambda_0)}{2\mu_1}-\frac{ck_n}{\mu_1\mu_2})\,ds\,.$$

The corresponding conditions for equilibrium in the free boundary problem are

$$\frac{b}{2}\mathscr{L}[\Lambda-\Lambda_0]-2aH=0\,,\text{in}\,\Sigma\,, \tag{8}$$

$$\frac{b(\Lambda-\Lambda_0)}{2\mu_1}-\frac{ck_n}{\mu_1\mu_2}\equiv 0\,,\text{ on}\,\partial\Sigma\,, \tag{9}$$

$$(a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W)\nu\cdot E_3+(\frac{b\Lambda_n}{2\mu_1}+c\partial_t(\frac{\sigma_{12}}{\mu_1\mu_2}))n\cdot E_3\equiv 0\,,\text{ on}\,\partial\Sigma\,. \tag{10}$$

The Euler-Lagrange equation corresponding to the anisotropic Willmore functional (5) is

$$\text{div}A\nabla\Lambda+(\langle Ad\nu,d\nu\rangle-\Lambda H)\Lambda=0\,. \tag{11}$$

Making a one parameter family of vertical translations will clearly not affect the value of the energy, although this variation does not preserve the boundary constraint. The variational field is just the constant vertical unit vector E_3 and we obtain,

$$0=\oint_{\partial\Sigma}-(\frac{b\Lambda_n}{2\mu_1}+c\partial_t(\frac{\sigma_{12}}{\mu_1\mu_2}))\nu\cdot E_3+(a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W))n\cdot E_3\,ds\,. \tag{12}$$

Note that in (10) we have $a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W\geq 0$ and $E_3\cdot n\leq 0$ since the surface is contained the upper half-space. It then follows from (10) that

$$(\frac{b\Lambda_n}{2\mu_1}+c\partial_t(\frac{\sigma_{12}}{\mu_1\mu_2}))\nu\cdot E_3\geq 0$$

must hold. But this means that both the terms in the integrand of (12) are non positive, so they both must vanish,

$$(\frac{b\Lambda_n}{2\mu_1}+c\partial_t(\frac{\sigma_{12}}{\mu_1\mu_2}))\nu\cdot E_3\equiv 0\,, \tag{13}$$

and

$$(a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W)n\cdot E_3\equiv 0\,, \tag{14}$$

We now consider some special cases. First consider the case $a=0=c$, $b=1$. Then it is clear that any surface with constant anisotropic mean curvature $\Lambda\equiv\Lambda_0$ is an extremal. On the other hand, for an equilibrium surface, we obtain from (8), (9) and (10), $\mathscr{L}[\Lambda-\Lambda_0]=0$ in Σ and $\Lambda-\Lambda_0\equiv 0\equiv[\partial_n(\Lambda-\Lambda_0)]n\cdot E_3$ on $\partial\Sigma$. From (14), we obtain $[\partial_n(\Lambda-\Lambda_0)]\nu\cdot E_3\equiv 0$ on the boundary. Since $(n\cdot E_3)^2+(\nu\cdot E_3)^2\equiv 1$ on the boundary, it follows that $\partial_n(\Lambda-\Lambda_0)$ vanishes on the boundary. However, $\mathscr{L}$ is a second order linear elliptic differential operator and any solution which vanishes identically to first order along the boundary must vanish identically. We can conclude that the surfaces with $\Lambda\equiv\Lambda_0$ are the *only* equilibria.

Let $\Omega:=a+\frac{b}{4}(\Lambda-\Lambda_0)^2-cK_\Sigma/K_W$ with ,

$$c>0\,,\quad 1>\varepsilon>0\,,\qquad b=c(1-\varepsilon^2)^{-1}\,,\qquad a=\frac{c(\varepsilon^{-2}-1)}{4(1-\varepsilon^2)}\Lambda_0^2\,.$$

Note that $0\le(\lambda_1-\lambda_2)^2$ implies that $0\le(\Lambda^2/4)-K_\Sigma/K_W$. Thus

$$\Omega\ge a+\frac{b}{4}((1-\varepsilon^2)\Lambda^2+(1-\varepsilon^{-2})\Lambda)^2)-cK_\Sigma/K_W=c(\frac{\Lambda^2}{4}-K_\Sigma/K_W)\ge 0..$$

Since equality holds if and only if $\Lambda\equiv\varepsilon^{-2}\Lambda_0$ and $\frac{\Lambda^2}{4}-K_\Sigma/K_W\equiv 0$, we find that the functional attains its absolute minimum exactly when the surface is part of a rescaled Wulff shape.

ANISOTROPIC WILLMORE SURFACES OF REVOLUTION.

In this section we will first repeat the derivation of the axially symmetric anisotropic Willmore surfaces which was derived in [16]. The derivation in the isotropic case can be found in [17]. By taking a slightly different approach we are able to produce closed examples with toroidal topology.

We will now restrict to the case where $a=0=\Lambda_0$. As noted above, the variational problem is then equivalent to $E_{(1,0,0)}$. We assume that the Wulff shape W is a surface of revolution (with vertical axis) and look for another surface of revolution Σ which satisfies the Euler-Lagrange equation (11). For this we will use the boundary terms in the first variation formula. We take the variation field δX to be a symmetry of the Lagrangian, Then if Σ is a part of a surface of revolution satisfying (11), we have

$$0=\delta E_{1,0,0}=\oint_{\partial\Sigma}2\Lambda(A\nabla\eta)\cdot n-2\eta(A\nabla\Lambda)\cdot n\rangle+\Lambda^2\xi\cdot n\,ds. \tag{15}$$

Assuming that the boundary consists of two horizontal circles.. We take first, $\delta X=E_3=E_3^T+\nu_3\nu$ and then take $\delta X=X=X^T+q\nu$ where q is the support function. These variations fields correspond to vertical translation and homothety which are symmetries of Ξ. For both these variational fields, the integrands in (15) are constant on each of the boundary circles. The integrals are just a constant times the arc length $2\pi x$ and the results are the two equations given below:

$$x_s\frac{2\Lambda_s}{\mu_1}+z_s\Lambda(\frac{k_2}{\mu_2}-\frac{k_1}{\mu_1})=\frac{C_1}{x}\,, \tag{16}$$

$$-q\frac{2\Lambda_s}{\mu_1}+X\cdot X_s\Lambda(\frac{k_2}{\mu_2}-\frac{k_1}{\mu_1})=\frac{C_2}{x}\,. \tag{17}$$

Computing the determinant $x_sX\cdot X_s+qz_s=x$ and inverting this system, yields

$$\frac{2\Lambda_s}{\mu_1}=\frac{1}{x^2}(-C_2z_S+C_1X\cdot X_s) \tag{18}$$

$$\Lambda(\frac{k_2}{\mu_2}-\frac{k_1}{\mu_1})=\frac{1}{x^2}(C_2x_s+C_1q) \tag{19}$$

if we make a vertical translation of the generating curve $(x,z)\longrightarrow(x,z+c)$ changes the support function q according to $q\longrightarrow q-cx_s=:q_1$. Thus choosing $cc_1=-c_2$ changes (19) to the simpler

$$\Lambda(\lambda_2-\lambda_1)=\frac{cq}{x^2}\,, \tag{20}$$

where we have renamed q_1 as q. In the isotropic case $F\equiv 1$, this equation occurs in [17].
Remarks. (i) In the case of a surface of revolution , the equation (11) becomes,

$$\frac{1}{x}(\frac{x\Lambda_s}{\mu_1})_s+(\frac{k_1^2}{\mu_1}+\frac{k_2^2}{\mu_2}-\Lambda H)\Lambda=0\,. \tag{21}$$

It can be shown that, except for the case when the surface is a round cylinder, equations (20) implies equation (21) . For this one needs the "Codazzi equation "

$$x^2\Lambda_s=(x^2(\lambda_1-\lambda_2))_s\,.$$

(ii) An anisotropic Willmore surface of revolution with zero or one boundary component is, up to rescaling, the Wulf shape W or a flat disc. To see this, note that in either of these cases we can let one of the boundary circles shrink to a point and we obtain in (18) and (19) that the C_i's are both zero. Thus from (20) $\Lambda(\lambda_2-\lambda_1)=0$ holds. If $\Lambda\equiv 0$, is is easy to see that the surface must be a flat disc since the Gaussian curvature is non positive everywhere. If $\lambda_1\equiv\lambda_2$, then the surface is known, [18], to be homothetic to a part of W.

In order to make these conclusions, it is essential that the surface be sufficiently regular, i.e. of class C^4. For example, one can invert the catenoid to get a closed surface but this surface fails to have the required differentiability.

To construct axially symmetric solutions, we write

$$\lambda_1=k_1/\mu_1=\frac{z''}{x'\mu_1},\qquad \lambda_2=k_2/\mu_2=-\frac{-z'}{x\mu_2}\,.$$

Making the substitutions $x'=\cos\theta$, $z'=\sin\theta$, leads to the following system:

$$x' = \cos\theta \tag{22}$$

$$z' = \sin\theta \tag{23}$$

$$\theta' = \pm\frac{\mu_1}{x}\cdot\{\frac{\sin^2\theta}{\mu_2^2}-c(x\sin\theta-z\cos\theta)\}^{1/2} \tag{24}$$

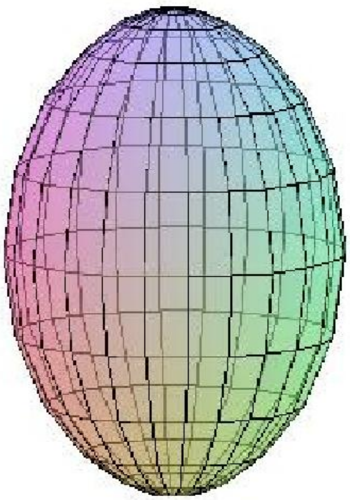

FIGURE 1. Wulff shape for $\gamma = 1 + 0.5\nu_3^2$

We will produce examples of closed anisotropic Willmore surfaces which are axially symmetric. Our examples will be obtained for the anisotropic surface energy derived from the Rapini-Papouler funcitionals

$$\gamma := 1 + \varepsilon \nu_3^2 \,, \quad \varepsilon \in (-1, 1)$$

For this range of ε the functional possesses the desired smoothness and convexity assumptions.

The principal curvatures of the corresponding Wulff shape are given by

$$1/\mu_2 = \gamma(\nu_3) - \nu_3 \gamma'(\nu_3) = 1 - \varepsilon \nu_3^2 \,, \tag{25}$$

$$1/\mu_1 = (1 - \nu_3^2)\gamma''(\nu_3) + 1/\mu_2 = 2\varepsilon(1 - \nu_3^2) + 1 - \varepsilon \nu_3^2 \,. \tag{26}$$

Some examples of solutions for the energy whose Wulff shape is given in Figure (1) are given in Figures (2) and (3).

ANOTHER POINT OF VIEW

Consider an oriented surface Σ embedded in the Euclidean space $\mathbf{R}^3$ and let ν be its normal map. The energy of this map is given by

$$E[\nu] = \int_\Sigma g^{ij} \nu_{,i} \nu_{,j} \sqrt{g} d^2 x = \int_\Sigma 4H^2 - 2K \, d\Sigma \,.$$

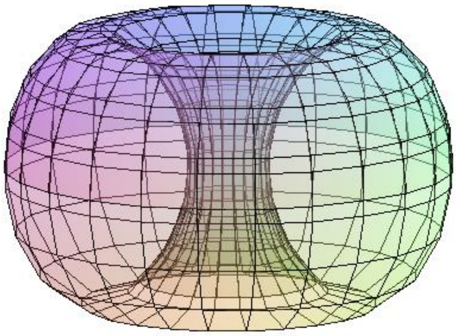

FIGURE 2. Toroidal anisotropic Willmore surface for the Wulff shape in Figure (1)

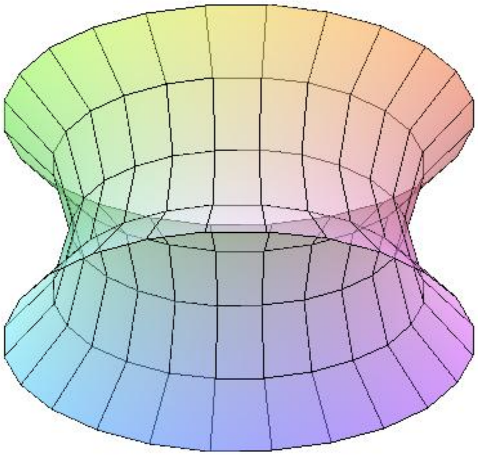

FIGURE 3. A catenoid-like anisotropic Willmore surface for the Wulff shape in Figure (1)

Note that for a fixed surface, the value of the energy differs from the four times the Willmore functional by a fixed topological constant. However the usual variational problem for the energy is not equivalent to the variational problem for the Willmore functional, When we vary the energy, we usually only vary the map. If we want to use E to study the Willmore functional, we need to vary both the map and the underlying metric g_{ij} in such a manner as would arise through a deformation of the surface.

Analogously, we consider the functional defined by the energy of the map $\chi : \Sigma \to W$. Introduce the function $\Gamma := \text{trace} \nabla \chi \circ J$, where J denotes rotation by 90^o in the tangent space in the direction such that $J = \nu \times \cdot$.

We have

$$E[\chi] = \int_\Sigma |d\chi|^2 \, d\Sigma = \int_\Sigma \Lambda^2 + \Gamma^2 - 2K_\Sigma / K_W \, d\Sigma \, .$$

Because $\Lambda^2 - 4K_\Sigma/K_W = (\lambda_1 - \lambda_2)^2 \geq 0$ holds, we have the inequality

$$E[\chi] \geq 2\int_\Sigma \frac{K_\Sigma}{K_W} d\Sigma = 2\text{degree}(\chi)\cdot \text{Area}(W)\,,$$

for any closed, boundaryless surface Σ. In particular, this implies that the Wulff shape is again, up to homothety and those rigid motions that are symmetries of the energy, the unique absolute minimizer of the energy E among all genus zero surfaces.

It is well known that, in fact, the energy's dependence on the metric g is unchanged by the conformal rescalings $g \rightarrow e^u g$. When computing variations of this energy, we will vary both the map χ and the metric as follows. Consider a variation of the surface $\delta X = \xi + \psi\nu$, where ξ is a tangent field to the surface. The corresponding variations of the map χ and the induced metric are

$$\delta\chi = A\delta\nu = A(-\nabla\psi + d\nu\xi)\,,$$

$$\delta g = L_\xi g - 2\psi II\,.$$

Here L denotes the Lie derivative and II denotes the second fundamental form. If we regard the energy E as depending on both the map χ and the metric, then the first variation is

$$\delta E = \frac{\partial E}{\partial\chi}[\delta\chi] + \frac{\delta E}{\delta g}[\delta g]\,, \tag{27}$$

where

$$\begin{aligned}\frac{\partial E}{\partial\chi}[\delta\chi] : &= 2\int_\Sigma \nabla\chi\cdot\nabla\delta\chi\, d\Sigma &(28)\\ &= -2\int_\Sigma \tau[\chi]\cdot\delta\chi\, d\Sigma + 2\oint_{\partial\Sigma}\delta\chi\cdot\nabla_n\chi\, ds\,, &(29)\end{aligned}$$

and

$$\frac{\partial E}{\delta g}[\delta g] := \int_\Sigma \Psi[\chi]\cdot\delta g\, d\Sigma\,. \tag{30}$$

The field $\tau[\chi]$ is called the tension field of the map χ and is given by the tangential part of the Lapalcian $\nabla^2\chi$. A calculation gives

$$\tau[\chi] = -\nabla\Lambda - J\nabla\Gamma\,. \tag{31}$$

The field $\Psi[\chi]$ is called the stress energy tensor and it is given by $\Psi[\chi] := (1/2)|d\chi|^2 g - d\chi(\cdot)\cdot d\chi(\cdot)$. This is the trace-free part of minus the pull-back of the metric on the Wulff shape W. If we restrict to a compactly supported, normal variation $\delta X = \psi\nu$, we obtain for the first variation

$$\begin{aligned}\delta E[\chi] &= \int_\Sigma -2\tau\cdot\delta\chi + \langle\Psi[\chi],\delta g\rangle\, d\Sigma\\ &= \int_\Sigma -2(\nabla\Lambda + J\nabla\Gamma)\cdot A\nabla\psi - 2\psi\langle\Psi[\chi], II\rangle\, d\Sigma\\ &= 2\int_\Sigma \psi\Big(\text{div}[A(\nabla\Lambda + J\nabla\Gamma)] - \langle\Psi[\chi], II\rangle\Big)\, d\Sigma\end{aligned}$$

The Euler-Lagrange equation is thus

$$\mathrm{div}[A(\nabla\Lambda + J\nabla\Gamma)] - \langle \Psi[\chi], II\rangle = 0\,. \tag{32}$$

In the case where $\Gamma \equiv 0$, the Euler-Lagrange equation agrees with the one given in (8) with $a = 0 = \Lambda_0$. The condition $\Gamma \equiv 0$ will always hold, for example, when both W and Σ are assumed to be axially symmetric.

For surfaces with non-empty boundary, we have for an arbitrary variation:

$$\begin{aligned}\delta E[\chi] &= 2\int_\Sigma \psi\Big(\mathrm{div}[A(\nabla\Lambda + J\nabla\Gamma)] - \langle\Psi[\chi], II\rangle\Big)\,d\Sigma \\ &\quad - \oint_{\partial\Sigma} 2A\nabla\psi\cdot\nabla_n\chi + 2\psi A(\nabla\Lambda + J\nabla\Gamma)\cdot n - |d\chi|^2\xi\cdot n\,ds\,.\end{aligned}$$

Let H_W denote the mean curvature of the Wulff spape at the point $\chi(\nu)$. A calculation gives that

$$-\langle\Psi[\chi], II\rangle = H(\Gamma^2 + \Lambda^2) - \frac{2H_W\Lambda K_\Sigma}{K_W}\,,$$

so that (32) can be written,

$$\mathrm{div}[A(\nabla\Lambda + J\nabla\Gamma)] + H(\Gamma^2 + \Lambda^2) - \frac{2H_W\Lambda K_\Sigma}{K_W} = 0\,, \tag{33}$$

Two interesting subcases of the Euler-Lagrange equation are

$$\Gamma \equiv 0\,, \qquad \mathrm{div}\,[A\nabla\Lambda] + (H\Lambda - \frac{2H_W K_\Sigma}{K_W})\Lambda = 0\,. \tag{34}$$

$$\Lambda \equiv 0 \qquad \mathrm{div}\,[AJ\nabla\Gamma] + (H\Gamma)\Gamma = 0\,. \tag{35}$$

For an axially symmetric Wulff shape, examples of solutions of (34) are furnished by the anisotropic Willmore surfaces considered above. To provide examples of solutions of (35), we consider the helicoids

$$f := (r\cos\theta, r\sin\theta, \alpha r)\,, \quad \alpha \neq 0\,.$$

It was already shown, [16], that for any rotationally invariant functional, these surfaces have zero anisotropic mean curvature. The induced metric is $dr^2 + (r^2+\alpha^2)d\theta^2$. The normal field for this surface is given by $\nu = (r^2+\alpha^2)^{-1/2}(-\alpha\sin\theta, \alpha\cos\theta, -r)$ and so $\nu_r = -\alpha(r^2+\alpha^2)^{-3/2}f_\theta$. Therefore $\sigma_{r\theta} = \alpha(r^2+\alpha^2)^{-1}$. A calculation shows that

$$\Gamma = -\sigma_{12}(1/\mu_1 - 1/\mu_2)\,.$$

Since μ_j, $j = 1,2$ are functions of $\nu_3 = -r(r^2+\alpha^2)^{-1/2}$, they are independent of θ. Thus Γ is a function of r. This gives $\nabla\Gamma =: \phi(r)f_r$, $A\nabla\Gamma =: \phi_1(r)f_r$, $AJ\nabla\Gamma =: \phi_2(r)f_\theta$. It follows that $\mathrm{div}AJ(\nabla\Gamma) = 0$. Since the helicoid is a minimal surface, $H \equiv 0$, equation (35) follows.

CONCLUSIONS

We have considered two formulations of anisotropic bending energies for surfaces in three dimensional space. The first is obtained by replacing the usual mean and Gaussian curvature by their anisotropic analogues. For this version we have obtained strong restrictions on solutions of certain free boundary problems. In the case of one axially symmetric energy based on the Rapini-Papouler functional, we have produced graphics of a solution with toroidal topology.

The second version of the anisotropic bending energy is given by the energy of an anisotropic Gauss map which takes values in the Wulff shape. Under deformations, the induced metric tensor is coupled to the map energy in a particular way. This energy reduces to the previously defined case when both the surface and the Wulff shape are axially symmetric. In general, the two formulations are, however, distinct.

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