Deformations of Stationary Surfaces

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Abstract

We show that a space-like surface with zero mean curvature in a Lorentzian 4-manifold satisfying the null convergence condition is locally a stable minimum for the area functional when compared with nearby surfaces having isotropic (null) mean curvature vector.

By a *stationary surface* we will mean an immersed space-like surface in an n-dimensional Lorentzian manifold $\mathcal{L}^n$ with zero mean curvature. When $n = 3$ such surfaces are called *maximal* since small subdomains have the property that they maximize area among all nearby surfaces having the same boundary values. However, when the codimension is larger than one, stationary surfaces no longer have this property. This makes a variational approach to stationary surfaces difficult.

Let $\mathcal{L}^4$ be a Lorentzian 4-manifold (signature $(+++-)$) which satisfies the *null convergence condition* (NCC)

$$\mathrm{Ric}(Z, Z) \geq 0 \tag{1}$$

for all null vectors Z in $\mathcal{L}^4$, where Ric stands for the Ricci curvature tensor of $\mathcal{L}^4$. We remark that this condition is more general than the *timelike convergence condition* (TCC), that is, $\mathrm{Ric}(Z, Z) \geq 0$ for all timelike vectors Z in $\mathcal{L}^4$. This causality condition is of interest in general relativity and it means that, on average, gravity attracts.

Below we will show that small domains of a stationary surface in $\mathcal{L}^4$ locally *minimize* area among nearby competing surfaces which have the property that their mean curvature vectors are *isotropic* (null). It is interesting to note that surfaces with isotropic mean curvature play an important role in several diverse contexts. In relativity theory they are called *marginally-trapped* surfaces and are used in the study of space-time singularities [1]. They also play an important part in the conformal geometry of surfaces in both 3-dimensional conformally flat Riemannian and Lorentzian spaces [2, 3].

Theorem 1 *Let $\mathcal{L}^4$ be a Lorentzian 4-manifold satisfying the null convergence condition (1). Let Σ be a smooth surface and let $\chi : \Sigma \longrightarrow \mathcal{L}^4$ be a smooth, stationary immersion. Then each $p \in \Sigma$ has a neighborhood U in Σ such that $\chi|_U$ is a weak relative minimum for the area functional restricted to immersions $f : U \longrightarrow \mathcal{L}^4$ such that*
(i) f has isotropic mean curvature
(ii) $f \equiv \chi$ on some neighborhood of ∂U.

Remark By a *weak relative minimum* we mean a stable local minimum for the area functional restricted to variations through immersions satisfying (i) and (ii). The term refers to the fact that there exists a C^1 neighborhood (but *not* a priori a C^0 neighborhood) of χ in which any sufficiently smooth f satisfying (i) and (ii) has area greater than or equal to the area of χ. (See [4, p. 103] for a discussion of the Riemannian case).

Proof Let $\chi : \Sigma \longrightarrow \mathcal{L}^4$ be a smooth, stationary immersion. Denote the metric in $\mathcal{L}^4$ by $\langle\, , \,\rangle$. Our result is local so we may assume that there exists an adapted frame $\{e_1, e_2, a, b\}$ along Σ such that $\{e_1, e_2\}$ is an orthonormal frame for Σ and $\{a, b\}$ is a frame for $\perp \Sigma$ with

$$\langle a, a\rangle = 0 = \langle b, b\rangle \qquad \langle a, b\rangle = 1$$

at each point. We consider now a smooth one parameter family of immersions

$$\tilde{\chi} : I \times \Sigma \longrightarrow \mathcal{L}^4, \qquad I = (-c, c)$$

such that $\tilde{\chi}_0 = \chi$ and such that for all $t \in I$, $\tilde{\chi}_t \equiv \chi$ outside of some fixed compact set in Σ. Let $\xi := (\partial_t(\tilde{\chi})_{t=0})^{\perp}$ denote the normal part of the variation field.

We will now impose a further restriction on our variation $\tilde{\chi}$, that for each t the mean curvature vector H_t of the immersion $\tilde{\chi}_t$ is isotropic, i.e.

$$\langle H_t, H_t\rangle \equiv 0.$$

Since $H_0 \equiv 0$, this implies that either

$$\langle \partial_t(H_t)_{t=0}, a\rangle \equiv 0 \qquad \text{or} \qquad \langle \partial_t(H_t)_{t=0}, b\rangle \equiv 0. \tag{2}$$

Recall that the *Jacobi operator* acting on sections $\xi \in \Gamma(\perp \Sigma)$ is given by

$$J[\xi] = \tilde{\Delta}^{\perp}\xi - \mathrm{Ric}^{\perp}\xi + \mathcal{B}\xi, \tag{3}$$

where

$$\tilde{\Delta}^{\perp}\xi := \sum_j (D^{\perp}_{e_j} D^{\perp}_{e_j} - D^{\perp}_{\nabla_{e_j} e_j})\xi$$

is the rough Laplacian in $\perp \Sigma$, $\mathcal{B}$ is the endomorphism of $\perp \Sigma$ defined by

$$\langle \mathcal{B}\xi, \eta\rangle := \langle D^T\xi, D^T\eta\rangle.$$

and

$$\mathrm{Ric}^{\perp}\xi := \sum_j (R(e_j,\xi)e_j)^{\perp}$$

where R is the curvature tensor of $\mathcal{L}^4$.

¿From the second variation formula

$$\delta^2_{\xi}\mathrm{Area}(\Sigma) = -\int_{\Sigma}\langle \xi, J[\xi]\rangle,$$

it is easy to deduce that

$$(\partial_t (H_t)_{t=0})^{\perp} = (1/2)J[\xi].$$

Thus the two conditions in (2) are equivalent to

$$\langle J[\xi], a\rangle \equiv 0 \qquad \text{or} \qquad \langle J[\xi], b\rangle \equiv 0. \tag{4}$$

Write $\xi = \sigma a + \tau b$. A computation gives

$$\begin{aligned} J[\xi] &= (\Delta\sigma + \sigma(\langle\tilde{\Delta}^{\perp}a, b\rangle + \langle(\mathcal{B} - \mathrm{Ric}^{\perp})a, b\rangle) + 2\langle D^{\perp}_{\nabla\sigma}a, b\rangle + \tau\langle(\mathcal{B} - \mathrm{Ric}^{\perp})b, b\rangle)a \\ &+ (\Delta\tau + \tau(\langle\tilde{\Delta}^{\perp}b, a\rangle + \langle(\mathcal{B} - \mathrm{Ric}^{\perp})a, b\rangle) + 2\langle D^{\perp}_{\nabla\tau}b, a\rangle + \sigma\langle(\mathcal{B} - \mathrm{Ric}^{\perp})a, a\rangle)b. \end{aligned}$$

Define operators acting on smooth functions by

$$L[u] := \Delta u + u[1/2(\langle\tilde{\Delta}^{\perp}a, b\rangle + \langle\tilde{\Delta}^{\perp}b, a\rangle) + \langle(\mathcal{B} - \mathrm{Ric}^{\perp})a, b\rangle] \tag{5}$$

$$A[u] := u/2[\langle\tilde{\Delta}^{\perp}a, b\rangle - \langle\tilde{\Delta}^{\perp}b, a\rangle] + 2\langle D^{\perp}_{\nabla u}a, b\rangle. \tag{6}$$

The operator L is clearly self-adjoint and we claim that A is skew-adjoint. To see this recall that for smooth compactly supported sections $\zeta, \eta \in \Gamma(\perp\Sigma)$ we have the Green's formula

$$0 = \int_{\Sigma}(\langle\tilde{\Delta}^{\perp}\zeta, \eta\rangle - \langle\tilde{\Delta}^{\perp}\eta, \zeta\rangle)$$

Applying this gives

$$\begin{aligned} 0 &= \int_{\Sigma}(\langle\tilde{\Delta}^{\perp}\sigma a, \sigma b\rangle - \langle\tilde{\Delta}^{\perp}\sigma b, \sigma a\rangle) \\ &= \int_{\Sigma}[(\sigma^2(\langle\tilde{\Delta}^{\perp}a, b\rangle - \langle\tilde{\Delta}^{\perp}b, a\rangle) + 4\sigma\langle D^{\perp}_{\nabla\sigma}a, b\rangle] \\ &= 2\int_{\Sigma}\sigma A[\sigma] \end{aligned}$$

which proves the claim.

Note that in general we have for any $\zeta, \eta \in \Gamma(\perp\Sigma)$

$$\langle\mathrm{Ric}^{\perp}\zeta, \eta\rangle = -\mathrm{Ric}(\zeta, \eta) + \langle R(a, \zeta)\eta, b\rangle + \langle R(b, \zeta)\eta, a\rangle,$$

where Ric stands for the Ricci curvature tensor of $\mathcal{L}^4$. Therefore, J can be expressed

$$J[\xi] = [(L+A)[\sigma] + \tau\langle \mathcal{C}b, b\rangle]a + [(L-A)[\tau] + \sigma\langle \mathcal{C}a, a\rangle]b, \tag{7}$$

where $\mathcal{C}$ is the endomorphism of $\perp \Sigma$ given by

$$\langle \mathcal{C}\zeta, \eta\rangle := \langle \mathcal{B}\zeta, \eta\rangle + \mathrm{Ric}(\zeta, \eta).$$

Observe that under NCC it follows that

$$\langle \mathcal{C}\eta, \eta\rangle = \langle \mathcal{B}\eta, \eta\rangle + \mathrm{Ric}(\eta, \eta) \geq \langle \mathcal{B}\eta, \eta\rangle \geq 0,$$

for any isotropic $\eta \in \Gamma(\perp \Sigma)$.

Thus the two conditions in (4) are equivalent to

$$(L-A)[\tau] + \sigma\langle \mathcal{C}a, a\rangle \equiv 0, \tag{8}$$

or

$$(L+A)[\sigma] + \tau\langle \mathcal{C}b, b\rangle \equiv 0. \tag{9}$$

We assume first that (8) holds.

Case 1: If $\langle \mathcal{C}a, a\rangle > 0$ at a point $p \in \Sigma$, then on some neighborhood V of p we may change the frame $\{a, b\}$ according to $a \longrightarrow \langle \mathcal{C}a, a\rangle^{-1/2}a$, $b \longrightarrow \langle \mathcal{C}a, a\rangle^{1/2}b$, to obtain a new frame with $\langle \mathcal{C}a, a\rangle \equiv 1$. We assume this has been done and that ξ vanishes near ∂V. Then (8) can be expressed as

$$\sigma = -(L-A)[\tau]. \tag{10}$$

Therefore

$$\begin{aligned}
\delta^2_\xi \mathrm{Area}(\Sigma) &= -\int_\Sigma \langle \xi, J[\xi]\rangle \\
&= -\int_\Sigma \tau\langle J[\xi], b\rangle \\
&= -\int_\Sigma \tau((L+A)[\sigma] + \tau\langle \mathcal{C}b, b\rangle) \\
&= \int_\Sigma \tau((L+A)(L-A)[\tau] - \tau\langle \mathcal{C}b, b\rangle) \\
&= \int_\Sigma \tau\mathcal{F}[\tau],
\end{aligned}$$

where

$$\mathcal{F} := (L+A)(L-A) - \langle \mathcal{C}b, b\rangle. \tag{11}$$

Note that the operator $\mathcal{F}$ is a self-adjoint elliptic fourth order operator whose principal part is the biharmonic operator. By general theory, p has a neighborhood U_1 on which the first eigenvalue λ_1 of

$$\mathcal{F}[u] = \lambda u, \qquad u|_{\partial U} = 0, \quad \partial_n u = 0 \tag{12}$$

is positive. Hence $\delta^2_\xi \text{Area}(\Sigma) \geq 0$ holds for all normal sections ξ vanishing near ∂U_1 and satisfying $\langle J[\xi], a\rangle \equiv 0$.
Case 2: If $\langle \mathcal{C}a, a\rangle = 0$ at a point $p \in \Sigma$, then on some neighborhood V of p we have $0 \leq \langle \mathcal{C}a, a\rangle \leq 1$. Using (8) we can write

$$\sigma = \sigma(1 - \langle \mathcal{C}a, a\rangle) + \sigma\langle \mathcal{C}a, a\rangle = \sigma(1 - \langle \mathcal{C}a, a\rangle) - (L - A)[\tau].$$

Therefore

$$\begin{aligned}
\delta^2_\xi \text{Area}(\Sigma) &= -\int_\Sigma \langle \xi, J[\xi]\rangle \\
&= -\int_\Sigma \tau\langle J[\xi], b\rangle \\
&= -\int_\Sigma \tau((L + A)[\sigma] + \tau\langle \mathcal{C}b, b\rangle) \\
&= \int_\Sigma \tau((L + A)(L - A)[\tau] - \tau\langle \mathcal{C}b, b\rangle) - \int_\Sigma \tau(L + A)[\sigma(1 - \langle \mathcal{C}a, a\rangle)] \\
&= \int_\Sigma \tau\mathcal{F}[\tau] - \int_\Sigma (L - A)[\tau]\sigma(1 - \langle \mathcal{C}a, a\rangle) \\
&= \int_\Sigma \tau\mathcal{F}[\tau] + \int_\Sigma \sigma^2\langle \mathcal{C}a, a\rangle(1 - \langle \mathcal{C}a, a\rangle),
\end{aligned}$$

since $(L - A)$ is the adjoint of $(L + A)$, where $\mathcal{F}$ is given by (11). We can find for p a neighborhood U_1 on which the first eigenvalue λ_1 of (12) is positive and $0 \leq \langle \mathcal{C}a, a\rangle \leq 1$. Hence $\delta^2_\xi \text{Area}(\Sigma) \geq 0$ holds for all normal sections ξ vanishing near ∂U_1 and satisfying $\langle J[\xi], a\rangle \equiv 0$.

We then proceed analogously using (9) (that is, the second condition in (4)) to find a neighborhood U_2 of p for which $\delta^2_\xi \text{Area}(\Sigma) \geq 0$ holds for all normal sections ξ vanishing near ∂U_2 and satisfying $\langle J[\xi], b\rangle \equiv 0$. Then $U = U_1 \cap U_2$ has the desired property. **q.e.d.**

Note that by equation (10) there are, in case 1, many variations satisfying (4) since given an arbitrary smooth compactly supported function τ we can simply define σ using (10).

We will now discuss examples which are well known in the context of conformal geometry. Let $\mathbf{S}^4_1$ denote the deSitter space equipped with the usual metric

$$dS^2 = -dt^2 + ch^2 t d\sigma^2$$

where $d\sigma^2$ is the metric on the three-sphere $\mathbf{S}^3$. Identify $\mathbf{S}^3$ with the slice $t = 0$ and let $\Sigma \subset \mathbf{S}^3$ be an equatorial 2-sphere. Also let $T \subset \mathbf{S}^3$ be the *Clifford torus* which is the lift of an equator in the 2-sphere via the Hopf fibration. Then both Σ and T are minimal in $\mathbf{S}^3$ and are hence stationary in $\mathbf{S}^4_1$. It can be shown that both Σ and T are *globally* weak relative minima in the sense described above. Specifically *any* deformations of these surfaces through surfaces having isotropic mean curvature will initially increase area. Note that both Σ and T (in fact all closed minimal surfaces) are unstable in the three-sphere. It was shown by the second author in [3] that there are compact stationary tori in $\mathbf{S}_1^4$ which are unstable with respect to variations having isotropic mean curvature.

References

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