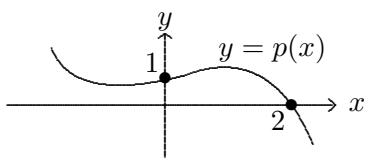


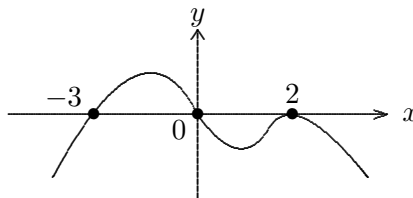
- 12.1 Complex numbers—the complex plane, real and imaginary parts, the arithmetic of complex numbers, complex conjugates.
- 12.2 Long division of polynomials—the algorithm and what it tells you about polynomials and rational functions. The relationship between the dividend, divisor, quotient, and remainder. Large scale behavior of rational functions, slant asymptotes.
- 12.3 Roots of polynomials and the factor and remainder theorems. The multiplicity of a root.
- 12.4 The fundamental theorem of algebra. Recovering polynomials from their roots.
- 12.6 The conjugate roots theorem.

1. Find a real quadratic function that has a root at  $2i - 3$ .
2. Determine a possible formula for a real polynomial with the following graph, assuming that you also know that  $-1 + i$  is a root of the polynomial.



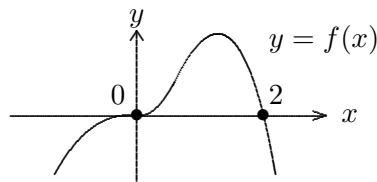
3. Let  $u = 2 + 3i$ ,  $v = -2 + i$ , and  $w = 3 - i$ . Compute
  - a.  $2u - v$
  - b.  $u \cdot w$
  - c.  $v^3$
  - d.  $v/w$
  - e.  $\overline{u \cdot v}$
4. Compute the quotient and remainder when  $2x^3 + 3$  is divided by  $x^2 + 2x$  and determine all asymptotes (horizontal, vertical, or slant) for the rational function  $f(x) = \frac{2x^3+3}{x^2+2x}$ .
5. When a polynomial  $p(x)$  is divided by  $2x^2 + 3$  the quotient is  $3x - 5$  and the remainder is  $x + 4$ . What is  $p(x)$ ?
6. MULTIPLE CHOICE: The graph  $y = \frac{5x^2-x^3+1}{x^2+3x-1}$  has which line as a slant asymptote?
  - a.  $y = \frac{2}{3}x - 1$
  - b.  $y = 2 - x$
  - c.  $y = 5 + x$
  - d.  $y = 8 - x$
  - e.  $y = 9 - 25x$
7. MULTIPLE CHOICE: Which polynomial has a graph most similar to the given graph?

- a.  $y = 2x(x - 2)^2(x + 3)$
- b.  $y = 3x^2(2 - x)(x + 3)$
- c.  $y = -x(x + 2)^2(x - 3)$
- d.  $y = -2(x - 2)^2(x + 3)^2$
- e.  $y = -2x(2 - x)^2(x + 3)$



8. One root of  $p(x) = 6x^3 - x^2 - 21x + 10$  is  $x = -2$ . Verify this and use this fact to help you factor  $p(x)$  completely.
9. Let  $p(x)$  be a nonconstant polynomial with real coefficients. What does the factor theorem tell us about  $p(x)$ ? What does the fundamental theorem of algebra tell us about  $p(x)$ ? To what extent can  $p(x)$  be factored if the factors are required to have real coefficients? How is the conjugate roots theorem relevant?
10. Design a polynomial with real coefficients that has  $2 + i$  as a root of multiplicity 2.
11. TRUE/FALSE: Assuming that  $f$  is a polynomial and that the graph shows all relevant features, which of the following statements are necessarily true? possibly true? or necessarily false?

- a. The degree of  $f(x)$  is 4.
- b. The degree of  $f(x)$  is even.
- c.  $(x + 2)$  is a factor of  $f(x)$ .
- d.  $x^3$  is a factor of  $f(x)$ .



## ANSWERS

1. If  $x = 2i - 3$  then  $(x + 3)^2 = (2i)^2 = -4$  so that  $x^2 + 6x + 13 = 0$ .
2. We require a polynomial with roots  $-1 + i$ ,  $2$ , and, by the conjugate roots theorem,  $-1 - i$ . Let  $p(x) = a(x + 1 - i)(x + 1 + i)(x - 2) = a(x^2 + 2x + 2)(x - 2)$ . The  $y$ -intercept gives  $a = -\frac{1}{4}$  and thus one possible solution is  $p(x) = -\frac{1}{4}x^3 + \frac{1}{2}x + 1$ .
3. a.  $6 + 5i$       b.  $9 + 7i$       c.  $-2 + 11i$       d.  $-\frac{7}{10} + \frac{1}{10}i$       e.  $-7 + 4i$
4. The quotient is  $2x - 4$  with a remainder of  $8x + 3$ . It follows that  $f(x) = (2x - 4) + \frac{8x + 3}{x^2 + 2x}$  so that  $y = 2x - 4$  is a slant asymptote. There are also vertical asymptotes at the roots of  $x^2 + 2x$ , i.e., the lines  $x = -2$  and  $x = 0$ .
5.  $p(x) = (2x^2 + 3)(3x - 5) + (x + 4) = 6x^3 - 10x^2 + 10x - 11$ .
6. By long division, the quotient is  $-x + 8$  with a remainder of  $-25x + 9$  so that the slant asymptote is (d)  $y = 8 - x$ .
7. We require simple roots at  $x = -3$  and  $x = 0$  and a double root at  $x = 2$ . Both (a) and (e) have these roots. However, the large scale behavior shows that the degree must be even and the leading coefficient must be negative. Thus, the correct answer is (e)  $y = -2x(2 - x)^2(x + 3)$ .
8.  $p(-2) = -48 - 4 + 42 + 10 = 0$  so  $-2$  is a root and  $(x + 2)$  is a factor of  $p(x)$ . By long division,  $p(x) = (x + 2)(6x^2 - 13x + 5)$ . By the quadratic formula, the other two roots are  $\frac{5}{3}$  and  $\frac{1}{2}$  so that  $p(x) = (x + 2)(3x - 5)(2x - 1)$ .
9. Let  $p(x)$  be a nonconstant polynomial with real coefficients. The factor theorem tells us that  $r$  is a root if and only if  $x - r$  is a factor, i.e.,  $p(r) = 0$  if and only if  $p(x) = (x - r) \cdot q(x)$  for some polynomial  $q(x)$ . The fundamental theorem of algebra tells us that  $p(x)$  must have at least one root within the complex number system. If  $r = a + bi$  is a nonreal root, then by the conjugate roots theorem  $a - bi$  is also a root, and thus,  $(x - a - bi)(x - a + bi) = x^2 - 2ax + (a^2 + b^2)$  must be a factor. It follows that  $p(x)$  can be factored as a product of real linear factors and irreducible quadratic factors, where an irreducible quadratic factor is a real quadratic with nonreal (conjugate) roots.
10. If  $2 + i$  is a root of multiplicity 2 of a real polynomial, then by the conjugate roots theorem,  $2 - i$  must also be a root of multiplicity 2. Thus one possibility is  $p(x) = (x - (2 + i))^2(x - (2 - i))^2 = (x^2 - 4x + 5)^2 = x^4 - 8x^3 + 26x^2 - 40x + 25$ .
11. a. The graph shows a simple root at  $x = 2$  and a 3-fold (possibly 5-fold, 7-fold, ...) root at  $x = 0$ . We also cannot see complex roots, if any. So although the graph is consistent with the graph of a polynomial of degree 4, the degree could be greater than 4, i.e., this part is possibly true.
- b. The large scale behavior shows that the degree of the polynomial is even and that the leading coefficient is negative, so this is necessarily true.
- c. This is definitely false:  $f(x)$  has a root at  $x = 2$ , not at  $x = -2$ .
- d. This is necessarily true: the behavior near  $x = 0$  indicates a root of odd multiplicity greater than or equal to 3. A triple root at  $x = 0$  corresponds to a factor of  $x^3$ .