Advanced Linear Algebra, Math 499/599, Kriloff
Homework 1 Problems - Spring, 2008

1.2.1 Verify that the set of complex numbers described in Example 4, $\{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$, is a subfield of $\mathbb{C}$.

1.2.4 Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} 2x_1 + (-1+i)x_2 \qquad\qquad + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad\qquad \begin{aligned} \left(1+\tfrac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

1.2.7 (Not required for 499 students) Prove that each subfield of the field of complex numbers contains every rational number.

1.3.4 Find a row-reduced matrix which is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

1.3.7 Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

1.3.8 Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix over the field F. Prove the following.

(1) If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.
(2) If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.
(3) If $ad - bc = 0$ and some entry of A is different from 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$, $x_2 = yx_2^0$.

1.4.7 Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 \qquad - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{aligned}$$

1.4.9 Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

1.4.10 Suppose R and R' are 2×3 row-reduced echelon matrixes and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

1.5.3 Find two different 2×2 matrices A such that $A^2 = 0$ but $A \neq 0$.

1.5.6 Let A be an $m \times n$ matrix and B an $n \times k$ matrix. Show that the columns of $C = AB$ are linear combinations of the columns of A. If $\alpha_1, \ldots, \alpha_n$ are the columns of A and $\gamma_1, \ldots, \gamma_k$ are the columns of C, then

$$\gamma_j = \sum_{r=1}^{n} B_{rj} \alpha_r.$$

1.5.8 Let

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices A and B such that $C = AB - BA$. Prove that such matrices can be found if and only if $C_{11} + C_{22} = 0$.

1.6.2 Let

$$A = \begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

1.6.7 Let A be an $n \times n$ (square) matrix. Prove the following two statements:

(1) If A is invertible and $AB = 0$ for some $n \times m$ matrix B, then $B = 0$.
(2) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

1.6.8 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove, using elementary row operations, that A is invertible if and only if $(ad - bc) \neq 0$.

1.6.10 Prove that if A is an $m \times n$ matrix, B is an $n \times m$ matrix, and $n < n$, then AB is not invertible.