

## Modern Geometry, Math 343, Kriloff

### List of Theorems - Fall, 2007

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This list will be continually updated throughout the term so please check back regularly. Theorems will be put in an order so that each theorem only depends on earlier results (of course such an order is not uniquely determined). This will involve inserting theorems in an order different than we encounter them in the course, so we will describe each theorem by a phrase or name instead of a number. When you use a theorem in a proof either refer to this name or describe the full result you are using.

**Postulate 1.** Given any two points we can construct the straight line through them. (Two points determine a unique line.)

**Postulate 2.** Given any line segment we can construct the line containing it.

**Postulate 3.** Given any point and length we can construct the circle with the point as its center and that length as its radius.

**Postulate 4.** All right angles are congruent.

**Postulate 5.** If two lines are cut by a transversal so that the interior angles on one side of the transversal sum to less than  $180^\circ$  then the two lines meet on that side of the transversal.

**Playfair's Postulate.** Given any line  $\ell$  and point  $P$  not on  $\ell$ , there is exactly one line containing  $P$  that is parallel to  $\ell$ .

We should really use Postulate 5 to prove this statement, but for now we will use it as a postulate. In fact Playfair's Postulate can also be used to prove Postulate 5 so only one or the other is needed on the list of postulates.

**Separation Postulate.** Any line divides the plane into two parts.

**Theorem (SAS).** *In  $\triangle ABC$  and  $\triangle DEF$  if  $AB = DE$ ,  $\angle BAC \cong \angle DEF$ , and  $AC = DF$ , then  $\triangle ABC \cong \triangle DEF$ . In other words if two sides and an included angle of one triangle are congruent to those of another triangle then the two triangles are congruent.*

**Theorem (Vertical Angles).** *If two lines intersect then the angles formed that share a vertex but not a common side are congruent.*

In this course we take these two results as familiar from high school geometry and will not prove them. In Chapter 2 we will work on proving other triangle congruence criteria (ASA, SSS, SAA) from this one. The ASA criterion will be inserted here since the proof we will see later will be independent of other results below (others will be inserted in appropriate spots). Thus the ASA result can be used for now.

**Theorem (Pons Asinorum and its converse).** *A triangle is isosceles if and only if the base angles are congruent.*

**Theorem (Exterior Angle Theorem - Weak Form).** *An exterior angle in a triangle has measure larger than the measures of each of the two remote interior angles.*

**Theorem** (Corollary to the Exterior Angle Theorem). *For a point not on a line there is a unique line through the point and perpendicular to the line.*

**Theorem** (Parallel Lines Theorem). *If two parallel lines are cut by a transversal then every pair of alternate interior angles are congruent and every pair of corresponding angles are congruent.*

**Theorem.** *If  $ABCD$  is a parallelogram then  $AB = DC$  and  $AD = BC$ . In other words, in a parallelogram opposite sides are equal.*

**Theorem** (Converse of the Parallel Lines Theorem). *If two lines are cut by a transversal and any single pair of alternate interior angles, or of corresponding angles, are congruent then the two lines are parallel.*

**Theorem** (Equal Sides Criterion for Parallelogram). *If in simple quadrilateral  $ABCD$  sides  $AB = DC$  and  $AD = BC$  then  $ABCD$  is a parallelogram.*

**Theorem** (Equal and Parallel Sides Criterion for Parallelogram). *If in simple quadrilateral  $ABCD$  sides  $AB = DC$  and  $AB \parallel DC$  then  $ABCD$  is a parallelogram.*

**Theorem** (Varignon's Theorem). *If the midpoints of an arbitrary quadrilateral are connected in clockwise (or counterclockwise) order then they form a parallelogram.*

**Theorem** (Exterior Angle Theorem - Strong Form). *An exterior angle in a triangle has measure equal to the sum of the measures of the two remote interior angles.*

**Theorem** (Triangle Angle-Sum Theorem). *The interior angles in a triangle have measures that sum to  $180^\circ$ .*

**Theorem** (Side/Angle Comparison). *In  $\triangle ABC$ ,  $m\angle A > m\angle B$  if and only if  $BC > AC$ .*

**Theorem** (Triangle Inequality). *The sum of the lengths of any two sides of a triangle is greater than the length of the third side.*

**Theorem** (Viviani's Theorem). *The sum of the distances from a point  $P$  interior to (or on) an equilateral triangle to the sides of the triangle equals the height of the triangle.*

**Theorem** (Pasch's Theorem). *If  $A$ ,  $B$ , and  $C$  are distinct, noncollinear points and  $\ell$  is a line that intersects segment  $AB$ , then  $\ell$  also intersects either segment  $AC$  or segment  $BC$ .*

**Theorem** (Crossbar Theorem). *If  $\overrightarrow{AD}$  is between  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  (i.e.,  $D$  lies in the interior of  $\angle BAC$ ) then  $\overrightarrow{AD}$  intersects segment  $\overline{BC}$ .*

**Theorem** (Inscribed/Central Angle Theorem). *The measure of inscribed angle  $\angle PQR$  is half the measure of the central angle corresponding to arc  $PQR$ .*

**Theorem** (Right Triangle in Circle). *If a triangle is inscribed in a circle so that one side is a diameter of that circle, then the angle opposite that side is a right angle.*

**Theorem.** *A simple quadrilateral is cyclic if and only if opposite angles are supplementary.*

**Theorem.** *All angles in an equilateral triangle are congruent.*

**Theorem** (AA). *If two triangles have two pairs of angles of equal measure, then the triangles are similar.*

**Theorem** (ASA Congruence Criterion). *If two angles and the included side of one triangle are congruent respectively to two angles and the included side of another triangle, then the two triangles are congruent.*

**Theorem** (SSS Congruence Criterion). *If three sides of one triangle are congruent respectively to three sides of another triangle, then the two triangles are congruent.*

**Theorem** (SAA Congruence Criterion). *If two angles and a nonincluded side of one triangle are congruent respectively to two angles and a nonincluded side of another triangle, then the two triangles are congruent.*

**Theorem** (SSA Congruence Criterion for Right Triangles). *If the hypotenuse and a side of a right triangle are congruent respectively to the hypotenuse and a side of another right triangle, then the two triangles are congruent.*

**Theorem** (Parallel/Ratio Lemma). *Let  $U$  and  $V$  be points on sides  $\overline{AB}$  and  $\overline{AC}$  of  $\triangle ABC$  respectively. Then  $\overline{UV} \parallel \overline{BC}$  if and only if  $\frac{AU}{AB} = \frac{AV}{AC}$ .*

**Theorem** (SAS Similarity Criterion). *If two triangles have two pairs of corresponding sides with the same ratio and the included angles have equal measure, then the triangles are similar. In other words, in  $\triangle ABC$  and  $\triangle DEF$  if  $\frac{AB}{DE} = \frac{AC}{DF}$  and  $\angle A \cong \angle D$  then  $\triangle ABC \sim \triangle DEF$ .*

**Theorem** (SSS Similarity Criterion). *If the sides of one triangle are proportional to the corresponding sides of another triangle, then the triangles are similar. In other words, in  $\triangle ABC$  and  $\triangle DEF$  if  $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$  then  $\triangle ABC \sim \triangle DEF$ .*

**Theorem** (Proportionality Theorem). *If  $\triangle ABC \sim \triangle XYZ$ , then  $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$ .*

**Theorem** (Triangle Midpoint Corollary). *If points  $U$  and  $V$  are the midpoints of  $\overline{AB}$  and  $\overline{AC}$  in  $\triangle ABC$  respectively, then  $\overline{UV} \parallel \overline{BC}$  and  $UV = \frac{1}{2}BC$ .*

**Theorem** (Centroid Theorem). *The three medians of a triangle are concurrent. The point of concurrency, called the centroid is located twice as far from the vertex as it is from the midpoint of the opposite side.*

**Theorem** (Perpendicular Bisector Theorem). *A point is located on the perpendicular bisector of a segment if and only if the point is equidistant from the endpoints of the segment.*

**Theorem** (Circumcenter Theorem). *The three perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrency, called the circumcenter, is equidistant from the three vertices of the triangle so there exists a circle with that center passing through the three vertices, called the circumcircle.*

**Theorem** (Angle Bisector Theorem). *A point is located on the bisector of an angle if and only if the point is equidistant from the sides of the angle.*

**Theorem** (Incenter Theorem). *The three bisectors of the angles of a triangle are concurrent. The point of concurrency, called the incenter, is equidistant from the three sides of the triangle so there exists a circle with that center passing through the feet of the three perpendiculars from the incenter to the sides, called the incircle.*

**Theorem** (Excenter Theorem). *The external angle bisectors at two vertices of a triangle and the internal angle bisector at the third vertex are concurrent. The three points of concurrency of the three different such triples of angle bisectors are called the excenters, and are equidistant from the three lines through the sides of the triangle.*

**Theorem** (Orthocenter Theorem). *The three altitudes from the vertices of a triangle to the lines containing the opposite sides are concurrent. The point of concurrency, called the orthocenter, is located inside the triangle if the triangle is acute, on the triangle if the triangle is right, and outside the triangle if the triangle is obtuse.*

**Theorem** (Euler Line Theorem). *The circumcenter,  $O$ , the orthocenter,  $H$ , and the centroid,  $G$ , are collinear with  $H - G - O$  and  $HG = 2GO$ .*

**Theorem** (Ceva's Theorem and its converse). *In  $\triangle ABC$ , the Cevians  $\overline{AX}$ ,  $\overline{BY}$ , and  $\overline{CZ}$  (with  $X$  on  $\overleftrightarrow{BC}$ ,  $Y$  on  $\overleftrightarrow{AC}$ , and  $Z$  on  $\overleftrightarrow{AB}$  and none of  $X, Y, Z$  the same as  $A, B,$  or  $C$ ) are concurrent (or parallel) if and only if  $\frac{AZ}{ZB} \frac{BX}{XC} \frac{CY}{YA} = 1$  and an odd number (one or three) of the Cevians are interior to  $\triangle ABC$ .*

**Theorem** (Menelaus' Theorem and its converse). *Given  $\triangle ABC$ , let  $X$  be on  $\overleftrightarrow{BC}$ ,  $Y$  be on  $\overleftrightarrow{AC}$ , and  $Z$  be on  $\overleftrightarrow{AB}$  and assume none of  $X, Y, Z$  are the same as  $A, B,$  or  $C$ . Then  $X, Y,$  and  $Z$  are collinear if and only if  $\frac{AZ}{ZB} \frac{BX}{XC} \frac{CY}{YA} = -1$  where a ratio is taken to be negative if the non-vertex does not lie on the side of the triangle (the minus sign is thus equivalent to an even number of the Cevians being interior to  $\triangle ABC$ ).*

**Theorem** (Collapsible Fence Theorem). *In  $\triangle ABC$ , draw line segments  $\overline{AQ}$  and  $\overline{BP}$ , where  $P$  and  $Q$  lie on sides  $\overline{AC}$  and  $\overline{BC}$  respectively. Now draw  $\overline{PY}$  parallel to  $\overline{AQ}$  and  $\overline{QX}$  parallel to  $\overline{BP}$ , where  $X$  and  $Y$  lie on  $\overline{AC}$  and  $\overline{BC}$ . Then  $\overline{XY} \parallel \overline{AB}$ .*

**Theorem** (Angles in Quadrilateral Theorem). *In convex quadrilateral  $ABCD$ , draw the diagonals  $\overline{AC}$  and  $\overline{BD}$ . Show if  $\angle ADB \cong \angle ACB$  then  $\angle ABD \cong \angle ACD$ .*

**Theorem.** *An exterior angle of a cyclic quadrilateral has the same measure as the opposite interior angle.*

**Theorem** (Law of Cosines). *If  $\triangle ABC$  has side lengths  $a, b, c$  opposite vertices  $A, B, C$  and  $m\angle C = \theta$ , then  $c^2 = a^2 + b^2 - 2ab \cos \theta$ .*

**Theorem** (Law of Sines). *Given  $\triangle ABC$  with circumradius  $R$ , let  $a, b, c$  be the lengths of the sides opposite vertices  $A, B, C$  respectively. Then*

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)} = 2R.$$

**Theorem** (Shape on Sides of Triangle Theorem). *On  $\triangle ABC$  with largest angle  $\angle C$ , construct three equilateral triangles whose side lengths are the same as one side of  $\triangle ABC$ , or three circles whose diameters are the same as one side of  $\triangle ABC$ . Label the area of the figure opposite vertex  $A$  as  $A_a$ , opposite vertex  $B$  as  $A_b$ , and opposite vertex  $C$  as  $A_c$ . Then*

- if  $m\angle C < 90^\circ$  then  $A_a + A_b < A_c$ ,*
- if  $m\angle C = 90^\circ$  then  $A_a + A_b = A_c$ , and*
- if  $m\angle C > 90^\circ$  then  $A_a + A_b > A_c$ .*

**Theorem (I=O Theorem).** *If the incenter and the circumcenter of a triangle are the same then the triangle is equilateral.*

**Theorem (Gergonne Point Theorem).** *The Cevians joining the vertices of a triangle with the points of tangency of the incircle are concurrent (at what is called the Gergonne point of the triangle).*

**Theorem (Tangent to Circle Theorem).** *A line tangent to a circle is perpendicular to the radius of the circle constructed from the center of the circle to the point of intersection with the tangent line.*

**Theorem (Inscribed Angle Theorem).** *An inscribed angle in a circle is equal in degrees to one half its subtended arc. Equivalently, the arc subtended by an inscribed angle is measured by twice the angle.*

**Theorem (Exterior Secants Corollary).** *The angle between two secants drawn to a circle from an exterior point is equal in degrees to half the difference of the two subtended arcs.*

**Theorem (Interior Chords Corollary).** *The angle between two chords that intersect in the interior of a circle is equal in degrees to half the sum of the two subtended arcs.*

**Theorem (Converse of Right Triangle in Circle Theorem).** *In a right triangle the side opposite the right angle is a diameter of the circumcircle of the triangle.*

**Theorem (Chord-Tangent Theorem).** *The angle between a chord and the tangent at one of its endpoints is equal in degrees to half the subtended arc.*

**Theorem (Secant-Tangent Corollary).** *The angle between a secant and a tangent meeting at a point outside the circle is equal in degrees to half the difference of the subtended arcs.*

**Theorem (Two-Tangents Theorem).** *Let  $P$  be a point exterior to a circle centered at point  $O$  and draw the two tangents to the circle from  $P$ . Let  $S$  and  $T$  be the two points of tangency. Then  $\overline{OP}$  bisects  $\angle SPT$  and  $PS = PT$ .*

**Theorem (Perpendicular Bisectors Meet Circumcircle Theorem).** *The perpendicular bisectors of the sides of a triangle meet the circumcircle of that triangle at points that also lie on the bisector of either an interior or exterior angle at a vertex of the triangle.*

**Theorem (Altitudes Correspond to Angle Bisectors Theorem).** *In acute angled  $\triangle ABC$ , extend the altitudes from  $A$ ,  $B$ , and  $C$  to meet the circumcircle at points  $X$ ,  $Y$ , and  $Z$  respectively. Then  $\overline{AX}$ ,  $\overline{BY}$ , and  $\overline{CZ}$  bisect the three angles of  $\triangle XYZ$ .*

**Theorem (Distance to Incenter Theorem).** *Extend the bisector of one of the angles of a triangle to meet the circumcircle at point  $P$ . Then the distance from  $P$  to each of the other two vertices of the triangles is equal to the distance from  $P$  to the incenter of the triangle.*

**Theorem (I=H Theorem).** *If the incenter and the orthocenter of a triangle are the same then the triangle is equilateral.*

**Theorem (I=G Theorem).** *If the incenter and the centroid of a triangle are the same then the triangle is equilateral.*

**Theorem (Circumcenter/Orthocenter Theorem).** *Show that the circumcenter of  $\triangle ABC$  is the orthocenter of the medial triangle of  $\triangle ABC$ .*

**Theorem.** Let  $P$  be a point exterior to a circle centered at point  $O$  and draw the two tangents to the circle from  $P$ . Let  $S$  and  $T$  be the two points of tangency. Prove  $\widehat{ST} \cong 180^\circ - m\angle SPT$ .

**Proposition.** If  $F$  and  $G$  are isometries of  $\mathbb{R}^2$ , prove that  $F \circ G$  preserves distances.

**Proposition.** (1) If  $F = T_{\mathbf{a}}$  is a translation, describe  $F^{-1}$  with proof.

(2) If  $F = M_\ell$  is a reflection, describe  $F^{-1}$  with proof.

(3) If  $F = R_{P,\theta}$  is a rotation, describe  $F^{-1}$  with proof.

(4) If  $F = G_{\mathbf{a}}$  is a glide reflection, describe  $F^{-1}$  with proof.

You may use for these that the inverse of a bijective function is unique.

**Proposition (I).** Let  $A, B, C \in \mathbb{R}^2$  and  $d_1 = AB, d_2 = BC, d_3 = AC$ .

(1)  $B$  lies on  $AC$  with  $A - B - C$  if and only if  $d_1 + d_2 = d_3$ .

(2)  $A, B,$  and  $C$  are collinear if and only if one of  $d_1, d_2,$  and  $d_3$  is the sum of the other two.

(3) Let  $F$  be an isometry.  $A, B,$  and  $C$  are collinear if and only if  $F(A), F(B),$  and  $F(C)$  are collinear.

**Proposition (II).** Let  $F$  and  $G$  be isometries of  $\mathbb{R}^2$ . If  $A, B, C \in \mathbb{R}^2$  are not collinear, and  $F(A) = G(A), F(B) = G(B),$  and  $F(C) = G(C)$ , then  $F = G$ .

**Proposition (III).** Let  $A, B \in \mathbb{R}^2$  and  $A', B' \in \mathbb{R}^2$ . There are at most two isometries  $F$  such that  $F(A) = A'$  and  $F(B) = B'$ .

**Theorem (Classification Theorem).** Every isometry of  $\mathbb{R}^2$  is either the identity, a translation, a rotation, a reflection, or a glide reflection.

**Theorem (Nine-point circle).** Given any triangle, the following nine points all lie on a common circle: the midpoints of the sides, pedal points, and Euler points (midpoints of the segments joining a vertex of the triangle to its orthocenter). Each of the line segments joining an Euler point to the midpoint of the opposite side is a diameter of this circle. The radius of this circle is half the circumradius of the triangle.

**Theorem.** Suppose distinct points  $A$  and  $B$  are each fixed points of an isometry  $F$ . Prove, without using the Classification Theorem, that every point on the line  $\overleftrightarrow{AB}$  must also be fixed by  $F$  (this shows any isometry that has more than one fixed point must have infinitely many).

**Proposition.** A parallelogram has perpendicular diagonals if and only if it is a rhombus.

**Proposition.** Let  $U$  and  $V$  be points on sides  $\overline{AB}$  and  $\overline{AC}$ , respectively, of  $\triangle ABC$  and suppose that  $\overline{UV} \parallel \overline{BC}$ . Show that  $\overline{UC}, \overline{VB},$  and the median from  $A$  are concurrent.

**Proposition.** Draw the two medians of a triangle. This subdivides the interior of the triangle into four pieces: three triangles and a quadrilateral. Show that two of the three small triangles have equal area and that the area of the third is equal to that of the quadrilateral.

**Theorem (Incenter/Orthocenter Theorem).** Extend the bisectors of  $\angle A, \angle B,$  and  $\angle C$  of  $\triangle ABC$  to meet the circumcircle of  $\triangle ABC$  at points  $X, Y,$  and  $Z$ . Show that  $I$  is the orthocenter of  $\triangle XYZ$ .