Calculus I, Math 170, Kriloff
Final

Show **all** work completely on the pages below for full credit. **Simplify** when possible. Use **complete sentences** and **correct notation** throughout. When finished, **check** your work.

1. (14 points) Use **derivative rules** to find $f'(x)$. Answers need **not** be simplified.

   (a) $f(x) = 3\cos x \sin^2 x$

   $f'(x) = -3\sin^3 x + 3\cos x \cdot 2\sin x \cdot \cos x$
   $= -3\sin^3 x + 6\sin x \cos^2 x$

   (b) $f(x) = \dfrac{\sqrt[3]{x}}{ax^5 + b}$ (Let a and b be constant and give an answer in terms of a and b.)

   $$f'(x) = \frac{\frac{1}{3}x^{-2/3}(ax^5+b) - \sqrt[3]{x}(5ax^4)}{(ax^5+b)^2}$$

2. (6 points) The position of a particle at time t, measured in seconds, is given by the function $s(t) = t^4 - 4t^2 + 1$ where s is measured in centimeters. Find the acceleration of the particle at $t = 2$ seconds.

   $v(t) = s'(t) = 4t^3 - 8t$
   $a(t) = s''(t) = 12t^2 - 8$
   $a(2) = 12 \cdot 4 - 8 = 40$ cm/sec^2.

3. (7 points) Use the **definition** of the derivative to find $f'(a)$ if $f(x) = 3x^2$.

   $$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{3(a+h)^2 - 3a^2}{h}$$
   $$= \lim_{h\to 0} 3 \cdot \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} 3 \cdot \frac{2ah + h^2}{h}$$
   $$= \lim_{h\to 0} 3 \cdot \frac{h(2a+h)}{h} = \lim_{h\to 0} 3(2a+h) = 6a.$$

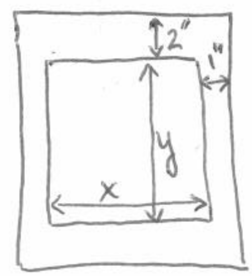
4. (11 points) Use **implicit differentiation** to find the equation of the tangent line to the curve $y^4 - x^3y = 11 + x$ at the point $(-2, 1)$.

$$4y^3 \cdot \frac{dy}{dx} - 3x^2y - x^3 \cdot \frac{dy}{dx} = 1$$

$$(4y^3 - x^3)\frac{dy}{dx} = 1 + 3x^2y$$

$$\frac{dy}{dx} = \frac{1+3x^2y}{4y^3-x^3} = \frac{1+3(-2)^2 \cdot 1}{4 \cdot 1^3 - (-2)^3} = \frac{1+12}{4+8} = \frac{13}{12}$$

5. Each page of a book will have $30\,\text{in}^2$ of print, and each page must have 2 inch margins at the top and bottom and 1 inch margins at each side. What is the minimum possible area of such a page?



(a) (2 points) Draw a diagram to illustrate the problem.

(b) (3 points) Assign variables for important quantities. Mark them on the diagram **and clearly** state below **in words** what each represents.

Let A_p = area of the page,
A_i = area of the print,
x = width of the print
y = height of the print.

We are given $xy = 30$ and are to minimize A_p.

(c) (3 points) Express the variable you are to minimize in terms of the other variables.

$$A_p = (x+2)(y+4)$$

(d) (8 points) Express the variable you are to minimize in terms of only **one** other variable, **simplify**, and write the **domain** then **stop**. Do **not** finish solving the problem.

Since $xy = 30$, $y = \frac{30}{x}$ and $A_p = (x+2)\left(\frac{30}{x} + 4\right)$

The domain is $(0, \infty)$

6. (9 points) The graph of the **derivative** $f'(x)$ of a function f is shown.

   (a) On what interval(s) is f decreasing?

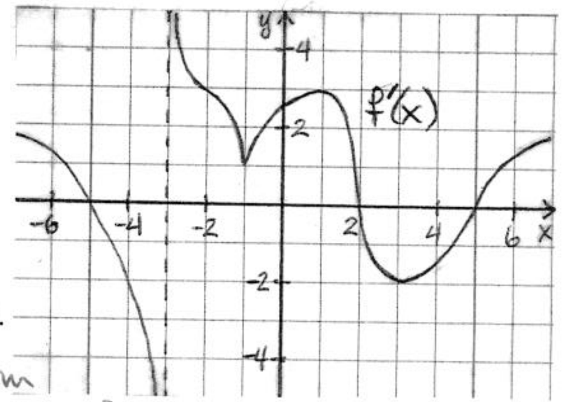
   $(-5,-3)$ and $(2,5)$

   (b) Give all x value(s) where f has a local minimum.

   At $x=-3$ and $x=5$.

   (c) Give all x value(s) where f has an inflection point.

   At $x=-1,1,3$ (where f' changes from increasing to decreasing or vice versa).



7. (a) (6 points) Use a precise, grammatically correct sentence to explain the meaning of the mathematical sentence $f'(5) = -2$ in terms of the graph of f. Do **not** use the word "derivative" in your answer.

   The slope of the tangent line to $f(x)$ at $x=5$ is -2.

   (b) (3 points) Use a precise, grammatically correct sentence to explain what the mathematical sentence $f(3) = \lim_{x\to 3} f(x)$ tells us about f.

   The function f is continuous at $x=3$.

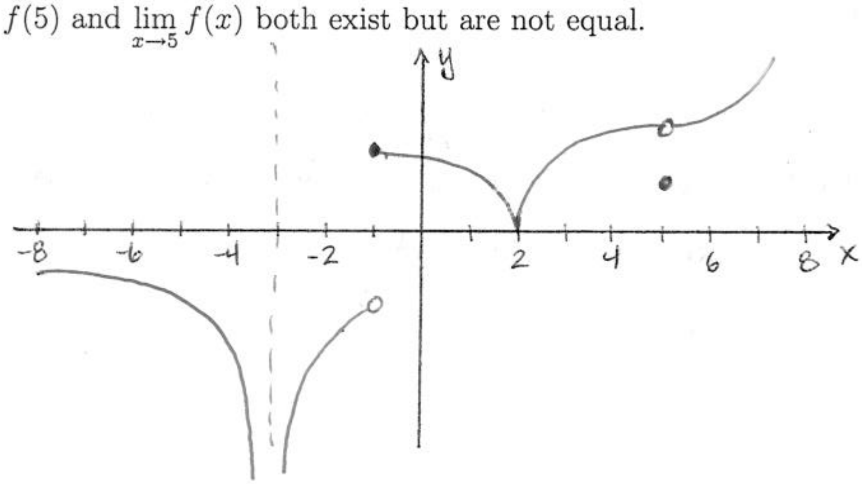
8. (8 points) Draw a careful graph of a function that has all of the following properties.

   (a) $\lim_{x\to -3} f(x) = -\infty$

   (b) f is defined but not continuous at $x=-1$

   (c) f is continuous but not differentiable at $x=2$

   (d) $f(5)$ and $\lim_{x\to 5} f(x)$ both exist but are not equal.



9. Find the exact limit or state that the limit does not exist. To receive full credit you must **show work** or **explain** and give an **exact** answer.

(a) (4 points) $\lim_{x\to -3} \frac{4x+12}{2x^2+5x-3}$ $= \lim_{x\to -3} \frac{4(x+3)}{(2x-1)(x+3)} = \lim_{x\to -3} \frac{4}{2x-1}$

$= \frac{4}{-6-1} = \frac{-4}{7}$.

(b) (7 points) $\lim_{x\to 16} \frac{x}{x-16}$ does not exist because

$\lim_{x\to 16^-} \frac{x}{x-16} = -\infty$ since x is positive and nonzero while $x-16$ is negative and approaching 0 as $x \to 16^-$

$\lim_{x\to 16^+} \frac{x}{x-16} = \infty$ since x is positive and nonzero while $x-16$ is positive and approaching 0 as $x \to 16^+$.

10. The speed of a runner increased steadily during the first 12 seconds of a race, as shown in the table.

Time t (seconds)	0	3	6	9	12
Velocity $v(t)$ (ft/sec)	0	9.1	14.3	17.6	19.4

(a) (6 points) Give an overestimate for the total distance the runner traveled during the first 12 seconds of the race.

$[v(3)+v(6)+v(9)+v(12)]\Delta t$

$= [9.1+14.3+17.6+19.4]\cdot 3$

$= 181.2$ ft.

(b) (4 points) Explain how you know your answer to part (a) is an overestimate.

Since $v(t)$ is increasing, the right hand endpoint estimate is an overestimate.

11. (23 points) Evaluate the following integrals or state that they do not exist. To receive full credit you must **show work** and give an **exact** answer.

(a) $\int_0^4 (\sqrt{x}+1)^2\,dx$ $= \int_0^4 (x^{1/2}+1)^2\,dx = \int_0^4 (x + 2x^{1/2} + 1)\,dx$

$= \frac{x^2}{2} + 2\cdot\frac{2}{3}x^{3/2} + x \Big|_0^4 = \frac{16}{2} + \frac{4}{3}\cdot 8 + 4 = 8 + \frac{32}{3} + 4 = \frac{68}{3}$

(b) $\int \frac{x^3}{(2+x^4)^5}\,dx$ $= \frac{1}{4}\int \frac{du}{u^5} = \frac{1}{4}\int u^{-5}\,du = \frac{1}{4}\cdot\frac{u^{-4}}{-4} + C$

$u = 2 + x^4$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$= -\frac{1}{16}u^{-4} + C$

$= \frac{-1}{16(2+x^4)^4} + C$

(c) $\int_0^{\pi/b} a\sin(bx)\,dx$ (Let a and b be constant and give an answer in terms of a and b.)

$= a\left(-\frac{\cos bx}{b}\right)\Big|_0^{\pi/b} = -\frac{a}{b}(\cos\pi - \cos 0) = -\frac{a}{b}(-1-1) = \frac{2a}{b}$

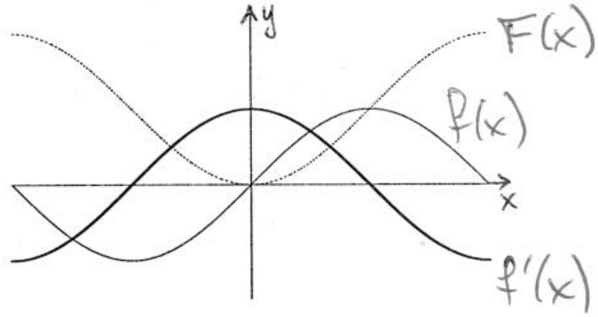
12. (6 points) Suppose the rate of change in the height of a plant in cm per week during its first 8 weeks of growth is modeled by $h'(t) = 6t^2 + 4t$ and the height after 5 weeks is 350 cm. Find the formula for the height of the plant after t weeks.

$h(t) = \frac{6t^3}{3} + \frac{4t^2}{2} + C = 2t^3 + 2t^2 + C$

$h(5) = 2\cdot 5^3 + 2\cdot 5^2 + C = 250 + 50 + C = 300 + C = 350$ so $C = 50$.

$h(t) = 2t^3 + 2t^2 + 50$

13. (6 points) Label each curve as one of $f(x)$, $f'(x)$, or $F(x) = \int_0^x f(t)dt$. No explanation required.

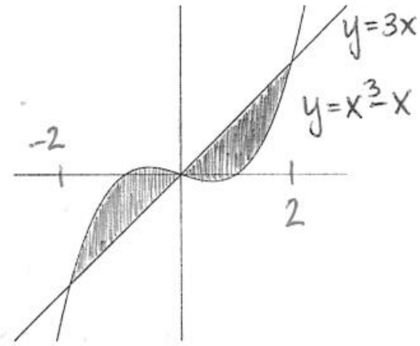


14. (8 points) Find the exact **area** of the shaded region bounded by $y = x^3 - x$ and $y = 3x$.

$$\int_{-2}^{2} |x^3 - x - 3x|\,dx = \int_{-2}^{2} |x^3 - 4x|\,dx$$

by symmetry $= 2\int_0^2 (4x - x^3)\,dx = 2\left(\frac{4x^2}{2} - \frac{x^4}{4}\right)\Big|_0^2$

$$= 2\cdot\left(\frac{16}{2} - \frac{16}{4}\right) = 2\cdot\frac{16}{4} = 8.$$



$x^3 - x = 3x$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0 \quad x = 0, \pm 2$

15. (6 points) Consider the region bounded by $y = x(x-3)^2$ and the x-axis (see the graph). Set up but **do not** evaluate an integral to calculate the volume of the solid formed by revolving this region about the y-axis.

Use cylindrical shells

$$\int_0^3 2\pi x \cdot x(x-3)^2\,dx$$