Calculus I, Math 170, Kriloff
Exam 1

A correct answer with little or no supporting work will be worth very little, so show **all** work. **Simplify** and use **complete sentences** and **correct notation** throughout. When finished, **check** your work.

1. (8 points) Find the equation of the tangent line to $f(x) = \sqrt[3]{x^2} - \dfrac{8}{\sqrt[6]{x}}$ at $x = 1$.

Rewrite $f(x) = x^{2/3} - 8x^{-1/6}$

Then $f'(x) = \frac{2}{3}x^{-1/3} - 8(-\frac{1}{6})x^{-7/6}$

$= \frac{2}{3}x^{-1/3} + \frac{4}{3}x^{-7/6}$ so $f'(1) = \frac{2}{3} + \frac{4}{3} = 2$

Since $f(1) = 1 - 8 = -7$, the equation of the tangent line to $f(x)$ at $x=1$ is $y - (-7) = 2(x-1)$

or $y = 2x - 9$

2. (16 points) Use derivative formulas to find y' in terms of x or f.

(a) $y = \dfrac{(x-2)(3x+5)}{x+\sqrt{7}} = \dfrac{3x^2 - x - 10}{x + \sqrt{7}}$

So by the Quotient Rule,

$y' = \dfrac{(6x-1)(x+\sqrt{7}) - (3x^2 - x - 10)\cdot 1}{(x+\sqrt{7})^2} = \dfrac{6x^2 + 6\sqrt{7}x - x - \sqrt{7} - 3x^2 + x + 10}{(x+\sqrt{7})^2}$

$= \dfrac{3x^2 + 6\sqrt{7}x + 10 - \sqrt{7}}{(x+\sqrt{7})^2}$

(b) $y = [f(x)]^2 = f(x)f(x)$

$y' = f'(x)f(x) + f(x)f'(x)$

$= 2f(x)f'(x)$

3. (4 points) Ohm's Law relating resistance R, voltage V, and current I, is $R = V/I$. Assuming resistance is constant, find the rate of change of voltage with respect to current.

Solve $R = V/I$ for V: $V = RI$.

So $\dfrac{dV}{dI} = R$.

4. (8 points) Find $f'(x)$ using the definition of the derivative if $f(x) = \dfrac{4}{3-2x}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{4}{3-2(x+h)} - \frac{4}{3-2x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{4(3-2x) - 4(3-2(x+h))}{(3-2x)(3-2(x+h))}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{12-8x-12+8x+8h}{(3-2x)(3-2(x+h))}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{8h}{(3-2x)(3-2x-2h)}$$

$$= \lim_{h\to 0} \frac{8}{(3-2x)(2-2x-2h)} = \frac{8}{(3-2x)^2}$$

5. (18 points) Find the following limits. Explain your reasoning clearly.

(a) $\displaystyle\lim_{t\to -6} \frac{2t+12}{t^2+3t-18} = \lim_{t\to -6} \frac{2(t+6)}{(t+6)(t-3)} = \lim_{t\to -6} \frac{2}{t-3} = \frac{2}{-6-3} = \frac{-2}{9}$

(b) $\displaystyle\lim_{x\to 5^-} \frac{2x}{x-5} = -\infty$ because $\displaystyle\lim_{x\to 5^-} 2x = 10$ and $\displaystyle\lim_{x\to 5^-} x-5 = 0$ so the values of $\frac{2x}{x-5}$ become infinite, and $10>0$ while $x-5<0$ for $x<5$ so the ratio is negative.

(c) $\displaystyle\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 2+x, & \text{if } x<1; \\ 2/x, & \text{if } x \geq 1 \end{cases}$

$\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2+x = 2+1 = 3$ and

$\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2/x = 2/1 = 2$.

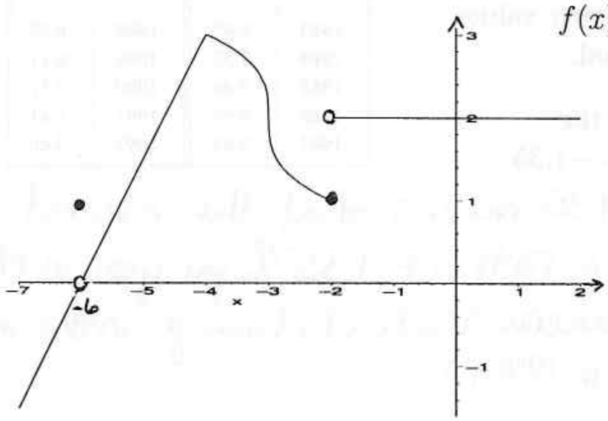
Since these are not equal, $\displaystyle\lim_{x\to 1} f(x)$ does not exist.

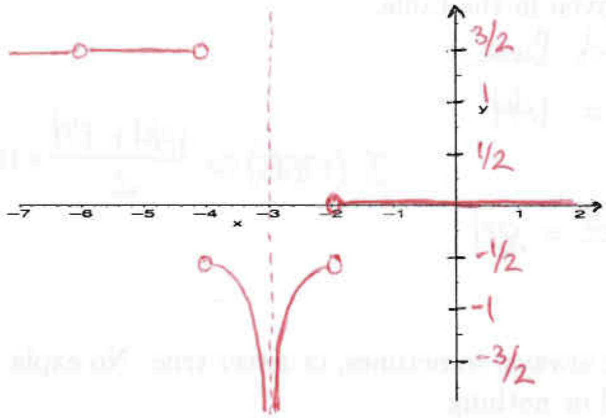
6. (4 points) Give a complete sentence stating a precise definition of continuity.

A function f is continuous at $x=a$ if $f(a)$ exists, $\displaystyle\lim_{x\to a} f(x)$ exists, and $\displaystyle\lim_{x\to a} f(x) = f(a)$.

7. (22 points) Below is the graph of f.

$$f(x)=\begin{cases} \frac{3}{2}x+9 & \text{if } x<-6 \text{ or } -6<x<-4 \\ 1 & \text{if } x=-6 \\ ? & \text{if } -4\le x\le -2 \\ 2 & \text{if } -2<x \end{cases}$$





(a) Suppose that on the interval $-4\le x\le -2$, $f(x)$ is given by translating the graph of $g(x)=-x^{1/3}$ left 3 units and up 2. Find the formula to replace ? in $f(x)$.

This means ? stands for $g(x+3)+2=-(x+3)^{1/3}+2$.

(b) Carefully sketch the graph of $f'(x)$ on the axes above. Mark the y-axis scale.

(c) Fill in the table with all values a so that f is not differentiable at $x=a$ and give a **thorough reason** for each.

At	the function f is not differentiable because
$x=-6$ and $x=-2$	f is not continuous at $x=-6$ and $x=-2$.
$x=-4$	the one-sided limits of this difference quotient are not equal: $\lim_{x\to -4^-}\frac{f(x)-f(-4)}{x+4}=-\frac{1}{2}\neq 0=\lim_{x\to -4^+}\frac{f(x)-f(-4)}{x+4}$.
$x=-3$	f has a vertical tangent line at $x=-3$.

8. (14 points) The interest rate on U.S. treasury bills is a function of time. The table gives midyear values of this function $I(t)$ over a nine-year period.

t	$I(t)$	t	$I(t)$
1983	8.62	1988	6.67
1984	9.57	1989	8.11
1985	7.49	1990	7.51
1986	5.97	1991	5.41
1987	5.83	1992	3.46

(a) Give a complete sentence describing the meaning of the statement $I'(1990) = -1.35$.

The statement $I'(1990) = -1.35$ means that the interest rate was decreasing at a rate of 1.35% per year in 1990. (Alternatively, the (instantaneous) rate of change in the interest rate was -1.35% per year in 1990.)

(b) Estimate $I'(1988)$ using the values given in the table.

Average the slopes of secant lines:

$$\frac{I(1989)-I(1988)}{1989-1988} = \frac{8.11-6.67}{1} = 1.44$$

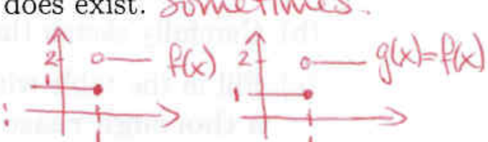
$$\frac{I(1988)-I(1987)}{1988-1987} = \frac{6.67-5.83}{1} = .84$$

$$I'(1988) \approx \frac{1.44+.84}{2} = 1.14\%/\text{yr}.$$

9. (9 points) State whether the following are always, sometimes, or never true. No explanation required. Each part is 3 points, all or nothing.

(a) If $f'(r)$ exists, then $\lim_{x \to r} f(x) = f(r)$. Always.
(This says if f is differentiable at r, then f is continuous at $x=r$, which is Theorem 4 in Section 3.2.)

(b) $\lim_{x\to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, but $\lim_{x\to a}[f(x)+g(x)]$ does exist. Sometimes.
(We saw an example in class where this was true, here's one that's not:



(c) If $f(a) = 4$, $f(b) = -7$ and f is not continuous on $[a, b]$, then there is no value c between a and b so that $f(c) = 0$. Sometimes.
(This is like #40 in Section 2.5.)

Optional bonus question. (5 points) Give a grammatically correct sentence stating the complete precise definition of the statement $\lim_{x\to 0} \frac{\sin x}{x} = 1$. (This is proven in Section 3.5.)

The statement $\lim_{x \to 0} \frac{\sin x}{x} = 1$ means given any $\varepsilon > 0$ there exists $\delta > 0$ so that if $0 < |x| < \delta$, then $\left|\frac{\sin x}{x} - 1\right| < \varepsilon$.