Brief Calculus, Math 160, Kriloff, Fall 2006
Solution to 1.7 #16

16. If the quantity of a substance decreases by 4% in 10 hours, find its half-life.

Solution 1

Use $P = P_0 a^t$ to represent the percent of the substance left after t hours.
Then $P_0 = 100\%$ and $P = 96$ when $t = 10$ (or in decimal form, $P_0 = 1$ and $P = 0.96$ when $t = 10$).
So $0.96 = a^{10}$ and $a = \sqrt[10]{0.96} \approx 0.996$, which is also what we get by interpreting the information given as an hourly decay rate of 0.4% per hour and using $a = 1 + r$ with $r = -0.004$.

Now in $P = 100a^t$, set $P = 50$ and solve for t to find the half-life as $t = \frac{\ln(1/2)}{\ln a}$. If we use $a \approx 0.996$ then $t \approx 173$ hours, but if we use the more precise value of $a = \sqrt[10]{0.96}$ (or the ANS key on a TI calculator) then $t \approx 170$ hours.

(This is not a terribly large percentage error (3 is about 2% of 170), but maybe larger absolute error than you expect. The reason is because a is close to 1, $\ln a$ is close to 0 and dividing by a value near zero is very sensitive to changes in the number. This is the reason not to round early!)

Solution 2

Use $P = P_0 e^{kt}$ to represent the percent of the substance left after t hours.
Then $0.96 = e^{10k}$ and $k = \frac{\ln 0.96}{10} \approx -0.0041$. Note that this is slightly different from the hourly decay rate of -0.004.

Again set $P = 50$ and solve for t in $P = P_0 e^{kt}$ to find the half-life is $t = \frac{\ln(1/2)}{k}$. Once again, using the rounded value of $k = -0.004$ yields $t \approx 173$ hours but using the more precise value of $k = \frac{\ln 0.96}{10}$ yields $a \approx 170$ hours.

Solution 3

Use $P = P_0 \left(\frac{1}{2}\right)^{t/h}$, where h is the half-life, for the percent of the substance left after t hours.
Then $0.96 = \left(\frac{1}{2}\right)^{10/h}$ and $\ln 0.96 = \ln\left[\left(\frac{1}{2}\right)^{10/h}\right] = \frac{10}{h}\ln(1/2)$. So

$$\frac{10}{h} = \frac{\ln 0.96}{\ln(1/2)} \quad \text{or} \quad \frac{h}{10} = \frac{\ln(1/2)}{\ln 0.96} \quad \text{and} \quad h = 10\frac{\ln(1/2)}{\ln 0.96} \approx 170 \text{ hours.}$$

In all cases the half-life of the substance is around 170 hours.