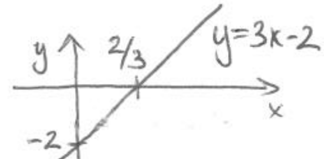
Final Exam – Math 143 – Fall 2005

1. Let f(x) = 3x-2, $g(x) = \dfrac{1}{1-x}$, $h(x) = \sqrt{x^2-1}$



a) Find the domain and range of f(x)

f(x) is a linear function with graph

All linear functions have domain $(-\infty,\infty)$ and range $(-\infty,\infty)$ (unless they are constant, $f(x)=c$, then range = $\{c\}$.).

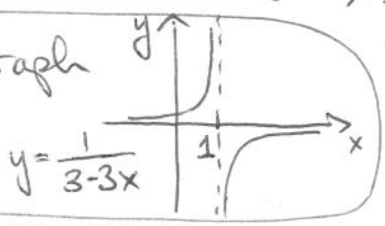
b) Find the domain of $(g \circ f)(x)$. Express your answer using intervals.

$$(g\circ f)(x) = g(3x-2) = \frac{1}{1-(3x-2)} = \frac{1}{3-3x}$$

The domain is all x so $3-3x \neq 0$, or all $x \neq 1$.

As intervals this is $(-\infty,1)\cup(1,\infty)$.

Or use the graph



Or use 3.5 p.183, domain of composite function

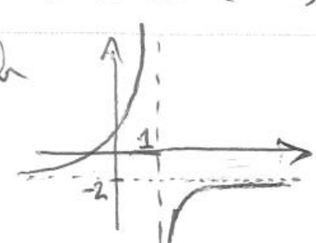
c) Find the domain of $(f \circ g)(x)$. Express your answer using intervals.

$$(f\circ g)(x) = f\left(\frac{1}{1-x}\right) = 3\cdot\frac{1}{1-x} - 2 = \frac{3}{1-x} - 2 = \frac{3}{1-x} - \frac{2(1-x)}{1-x} = \frac{3-2+2x}{1-x} = \frac{1+2x}{1-x}$$

The domain is all x so $1-x \neq 0$ or all $x \neq 1$.

As intervals this is $(-\infty,1)\cup(1,\infty)$.

Or use the graph



Or use p.183.

d) Find the domain and range of h(x). Express your answer using intervals.

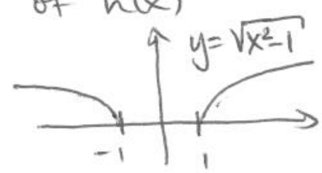
The domain is all x so $x^2-1 \geq 0$

Factor and build a sign chart.

$x^2-1 = (x-1)(x+1)$ or use the graph of $x^2-1=y$ to find domain $(-\infty,-1]\cup[1,\infty)$.

	-1		1	
$x-1$	$-$	$-$		$+$
$x+1$	$-$	$+$		$+$
x^2-1	$+$	$-$		$+$

Read range from graph of h(x) as $[0,\infty)$



2. Let $f(x) = \dfrac{2x-1}{x+3}$.

a) Find the domain of f(x)

Domain is all x so $x+3 \neq 0$, i.e. $\{x \mid x \neq -3\}$

As intervals, $(-\infty,-3) \cup (-3,\infty)$

b) Find the inverse function $x = f^{-1}(y)$

① Write as $y = \dfrac{2x-1}{x+3}$ $\xrightarrow{\text{OR}}$ ② Solve for x in terms of y (since asked for $x = f^{-1}(y)$)

② Swap x and y: $x = \dfrac{2y-1}{y+3}$

③ Solve for y: $x(y+3) = 2y-1$

$xy + 3x = 2y - 1$ $\quad$ $3x+1 = 2y - xy = (2-x)y$

$y = \dfrac{3x+1}{2-x} = f^{-1}(x)$

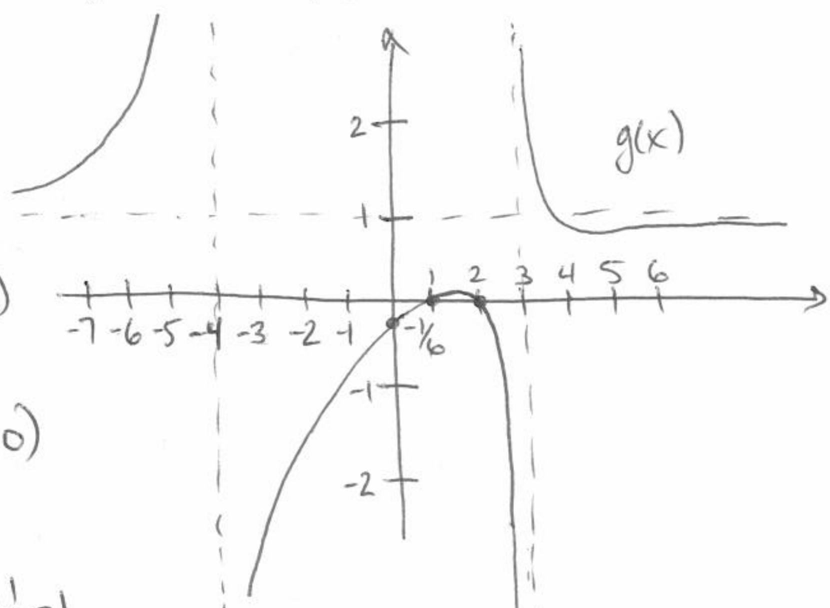
So $x = f^{-1}(y) = \dfrac{3y+1}{2-y}$

c) Find the range of f(x).

The range of $f(x)$ is the same as the domain of $f^{-1}(y)$, which is all y so $2-y \neq 0$, i.e. $\{y \mid y \neq 2\}$, or $(-\infty,2) \cup (2,\infty)$.

3. Graph the function $g(x) = \dfrac{x^2-3x+2}{x^2+x-12}$. Clearly label all intercepts, horizontal asymptotes and vertical asymptotes.

$g(x) = \dfrac{(x-1)(x-2)}{(x+4)(x-3)}$



y-intercept: $g(0) = \dfrac{2}{-12} = \dfrac{-1}{6}$

x-intercepts: 1, 2 (so $g(x)=0$)

vertical asymptotes:

$x=-4$, $x=3$ (so $x^2+x-12=0$)

horizontal asymptote:

$$\frac{x^2-3x+2}{x^2+x-12} \cdot \frac{1/x^2}{1/x^2} = \frac{1-3/x+2/x^2}{1+1/x-12/x^2} \approx \frac{1}{1} = 1$$

for $|x|$ large, so $y=1$ is the H.A.

4. a) Solve for x in the equation $\ln(x^2 - 9) - \ln(x+3) + \ln(2) = 1$

$$\ln \frac{x^2-9}{x+3} + \ln(2) = 1$$

$$\ln\left(\frac{x^2-9}{x+3}\right) + \ln 2 = \ln\left(\frac{(x-3)(x+3)}{x+3}\right) + \ln 2$$

So, $\ln(x-3) + \ln 2 = 1$

$$\ln(2x-6) = 1$$

$$2x-6 = e^1 = e$$

$$2x = 6+e$$

$$x = \frac{6+e}{2}$$

CHECK: $x = (6+e)/2 \approx 4.359$

$\ln(Ans^2-9) - \ln(Ans+3) + \ln 2 = 1$ ✓

or $\ln(4.359^2-9) - \ln(4.359+3) + \ln 2 \approx .9999$ ✓

b) Find all x satisfying the inequality $\ln|x-2| \leq 1$

Raise e to both sides

$$e^{\ln|x-2|} \leq e^1$$

$$|x-2| \leq e$$

$$-e \leq x-2 \leq e$$

$$2-e \leq x \leq e+2$$

CHECK: $2-e \approx -.718$

$2+e \approx 4.718$

pick e.g. $x=0$ $\ln|0-2| = \ln 2 \approx .69 < 1$ ✓

$x=1$ $\ln|1-2| = \ln 1 = 0 < 1$ ✓

c) Completely expand $\log_{10}\left(\dfrac{5x^2y}{z^2-1}\right)$ as a sum of logarithms.

$$\log_{10}\left(\frac{5x^2y}{z^2-1}\right) = \log_{10}(5x^2y) - \log_{10}(z^2-1)$$

$$= \log_{10} 5 + \log_{10} x^2 + \log_{10} y - \log_{10}(z^2-1)$$

$$= \log_{10} 5 + 2\log_{10} x + \log_{10} y - \log_{10}(z^2-1)$$

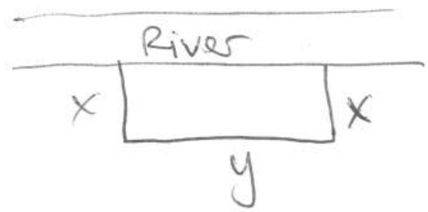
d) If $\log_3 y = 22$, find $\log_9 y$

Since $\log_3 y = 22$, $3^{22} = y$ and since $\log_9 y = x$ means $9^x = y$, we rewrite the left hand side as $3^{2\cdot 11} = (3^2)^{11} = 9^{11}$.

Thus $\log_9 y = 11$.

5. Two hundred feet of fencing is available for a rectangular field alongside a river serving as one side of the rectangle (so that only three sides require fencing). Find the largest possible area that can be enclosed and find the dimensions of the field of maximum area.

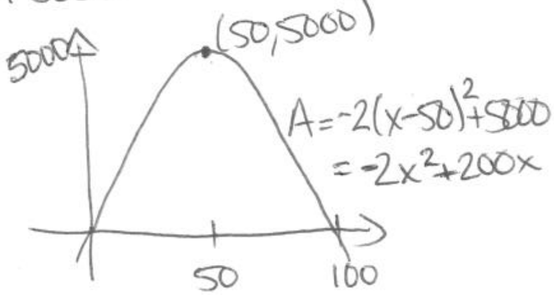
Given: perimeter $P = 200$ ft.



Find: largest possible area A and dimensions yielding the largest A.

$P = 2x + y = 200$ so $y = 200 - 2x$

$$A = xy = x(200-2x) = -2x^2 + 200x$$
$$= -2(x^2 - 100x + 2500) + 5000$$
$$= -2(x-50)^2 + 5000$$



So the largest area is 5000 ft^2 and occurs when $x = 50$ ft, in which case $y = 200 - 2x = 200 - 100 = 100$ ft.

6. The population of a town is assumed to grow exponentially. In 1900, the population was 2000 and in 1950 the population was 6000.

a) Find an expression for the population P(t) at any time t.

Let t be the number of years after 1900. Then

$6000 = 2000 \cdot e^{k \cdot 50}$

$3 = e^{50k}$

$\ln 3 = 50k$

$k = \frac{\ln 3}{50} \approx .022$

So $P(t) = 2000e^{kt}$ where $k = \frac{\ln 3}{50}$

or $P(t) = 2000e^{.022t}$

b) Find the year when the population reaches 12000.

$12000 = 2000e^{kt}$

$6 = e^{kt}$

$\ln 6 = kt$

$t = \frac{\ln 6}{k} = \frac{50 \cdot \ln 6}{\ln 3} \approx 81.5$ years after 1900, so in the middle of 1981 the population hits 12,000.

7. The initial mass of a sample on an isotope is 50 grams. After 100 years, the sample has a mass of 40 grams.

a) Find an expression for the mass of the sample after t years.

$N(t) = N_0 e^{kt}$

$40 = 50 \cdot e^{k \cdot 100}$

$e^{k \cdot 100} = \frac{40}{50} = \frac{4}{5}$

$100k = \ln(4/5)$

$k = \ln(4/5)/100 \approx -.0022$ So $N(t) = 50e^{-.0022t}$. or, $N(t) = 50e^{kt}$ where $k = \frac{\ln(4/5)}{100}$

b) What is the value of the half-life of this isotope.

$k = \frac{\ln(1/2)}{\text{half-life}}$ so $\text{half-life} = \frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{\frac{\ln(4/5)}{100}} = \frac{100\ln(1/2)}{\ln(4/5)}$

≈ 310.6 years

or set $25 = 50e^{kt}$ and solve for t.

8. You are told that the complex number 1-2i is a root of the polynomial $3x^3 - 7x^2 + 17x - 5$. Find all the other roots.

$1+2i$ is also a root, so $x-(1-2i)$ and $x-(1+2i)$ are factors.

$(x-(1-2i))(x-(1+2i)) = x^2 - 2x + 5$ ($1-2i+1+2i = 2$ and $(1-2i)(1+2i) = 1^2+2^2 = 5$)

$$
\begin{array}{r}
3x - 1 \\
x^2-2x+5 \,\overline{)\, 3x^3 - 7x^2 + 17x - 5} \\
-(3x^3 - 6x^2 + 15x) \\
\hline
-x^2 + 2x - 5 \\
-(-x^2 + 2x - 5) \\
\hline
0
\end{array}
$$

So the remaining root is $\frac{1}{3}$ (so $3x-1=0$)

The roots are $1 \pm 2i$ and $\frac{1}{3}$, all of multiplicity 1.

9. Graph the function $y = f(x) = 1 - e^{-(x-2)}$. Clearly label all intercepts and asymptotes.

Shift $y=e^x$ left 2 units, reflect in the y-axis (or reflect in the y-axis then shift right 2 units), then reflect in the x-axis and shift up 1.

This moves the horizontal asymptote of $y=0$ to $y=1$.

y-intercept: $f(0)=1-e^{-(0-2)}=1-e^2\approx-6.389$

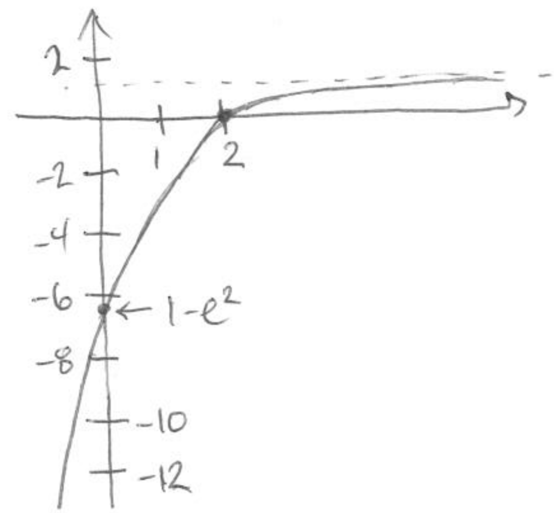
x-intercept: $0=1-e^{-(x-2)}$

$1=e^{-(x-2)}$

$0=\ln 1=-(x-2)=-x+2$

$-x=\ln(1)-2=0-2=-2$

$x=2$



10. Graph the polynomial $P(x) = (x^2 - x - 2)(x-3)$. Clearly label all intercepts.

y-intercept: $P(0)=(-2)(-3)=6$

$P(x)=(x-2)(x+1)(x-3)$

x-intercepts: $x=3,2,-1$