Noncrystallographic Springer correspondences

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1. Reflection group data and Hecke algebra definition

2. Weights and tempered representations

3. Degree filtration and Kostka-Foulkes polynomials

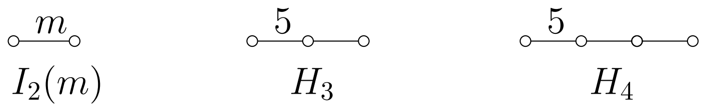
4. History and examples of Springer correspondence

5. Results and Techniques

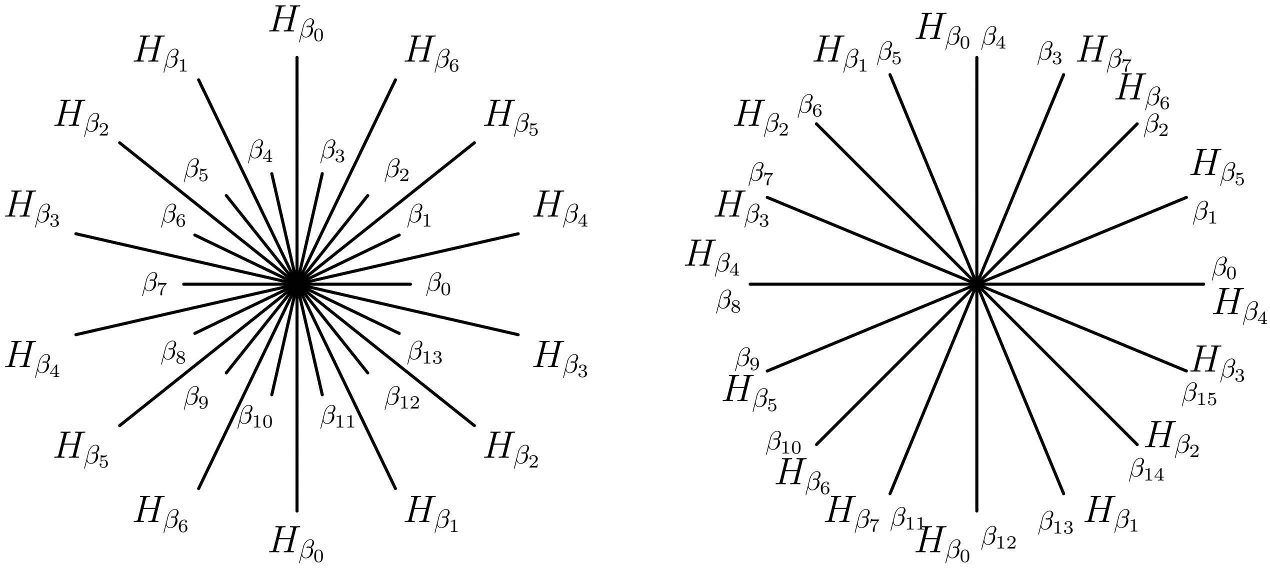
Reflection group data

W	finite reflection group
	noncrystallographic: *no known geometry*

Figure 1. Coxeter diagrams



$\mathfrak{h}_{\mathbb{R}}^*$	reflection representation of W
$\alpha \in \mathfrak{h}_{\mathbb{R}}^*$	roots in -1-eigenspace of reflections
$\alpha^\vee \in \mathfrak{h}_{\mathbb{R}}$	co-roots in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}^*, \mathbb{R})$ so that $\alpha^\vee(\alpha) = 2$
s_α	reflections: $\quad s_\alpha x = x - \langle x, \alpha^\vee \rangle \alpha$
H_α	reflecting hyperplanes,
	$H_\alpha = (+1 \text{ eigenspace of } s_\alpha) = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \alpha^\vee(x) = 0\}$
$\alpha_1, \ldots, \alpha_n$	simple roots
$\alpha_1^\vee, \ldots, \alpha_n^\vee$	simple co-roots
$s_1, \ldots, s_n$	corresponding simple reflections
$C^\vee = \sum_{i=1}^{n} \mathbb{R}_{<0} \alpha_i$	open negative cone with closure $\overline{C^\vee}$



$$\beta_k = \cos(\tfrac{k\pi}{n})\varepsilon_1 + \sin(\tfrac{k\pi}{n})\varepsilon_2, \qquad \varepsilon_1, \varepsilon_2 \text{ an orthonormal basis of } \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^2$$

FIGURE 2. Hyperplanes and roots for $I_2(7)$ and $I_2(8)$

Hecke algebra definition

Additional ingredients:

W acts on complexification $\mathfrak{h}^*_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}}$.

W acts on symmetric algebra $S(\mathfrak{h}^*_{\mathbb{C}})$ ($\cong$ polynomials on $\mathfrak{h}_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^*_{\mathbb{C}}, \mathbb{C})$).

Fix parameters $0 \neq c_\alpha \in \mathbb{C}$ such that

$$c_\alpha = c_{w\alpha} \quad \text{for } w \in W.$$

(If W acts irreducibly, 1 or 2 values depending on # orbits of W on roots.)

Group algebra is

$$\mathbb{C}W = \left\{ \sum_{w \in W} a_w t_w \mid a_w \in \mathbb{C} \right\} \qquad \text{with multiplication} \qquad t_w t_{w'} = t_{ww'}.$$

Graded Hecke algebra is $\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}^*_{\mathbb{C}})$

with multiplication as in $S(\mathfrak{h}^*_{\mathbb{C}})$ and in $\mathbb{C}W$ and

$$\begin{aligned} x t_{s_i} &= t_{s_i} s_i(x) + c_{\alpha_i} \langle x, \alpha_i^\vee \rangle \\ &= t_{s_i} s_i(x) + c_{\alpha_i} \frac{x - s_i(x)}{\alpha_i} \qquad \text{for } x \in \mathfrak{h}^*_{\mathbb{C}}. \end{aligned}$$

Center is $Z(\mathbb{H}) = S(\mathfrak{h}^*_{\mathbb{C}})^W = W$-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}$.

Weights and tempered representations

$Z(\mathbb{H})$ acts on simple $\mathbb{H}$-module M by scalars: for some $\gamma \in \mathfrak{h}_{\mathbb{C}}$,

$$pm = \gamma(p)m, \qquad \text{for all } m \in M,\ p \in S(\mathfrak{h}_{\mathbb{C}}^*)^W.$$

Since $\gamma(p) = w\gamma(p)$ for all $w \in W$, any $w\gamma \in W\gamma$ called <u>central character</u> of M.

For $\gamma \in \mathfrak{h}_{\mathbb{C}}$, <u>generalized γ-weight space</u> of finite dimensional $\mathbb{H}$-module M is

$$M_\gamma^{\text{gen}} = \{m \in M \mid \text{for all } x \in \mathfrak{h}_{\mathbb{C}}^*, (x - \gamma(x))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}.$$

As $S(\mathfrak{h}_{\mathbb{C}}^*)$-modules

$$M = \bigoplus_{\gamma \in \mathfrak{h}_{\mathbb{C}}} M_\gamma^{\text{gen}},$$

where γ called a <u>weight</u> of M if $M_\gamma^{\text{gen}} \neq 0$.

<u>Tempered</u> $\mathbb{H}$-module M has all weights in $\overline{C^\vee}$.

<u>Square integrable</u>=<u>cuspidal</u> $\mathbb{H}$-module M has all weights in $C^\vee$.

Degree filtration and Kostka-Foulkes polynomials

$M(\lambda)$ a simple tempered $\mathbb{H}$-module with *real* central character λ.

Assume there exists $v^- \in M(\lambda)$ such that $t_{s_i} v^- = -v^-$ for $1 \leq i \leq n$.

True for example for principal series modules, but not always.

Define filtration on $M(\lambda)$ by

$$M(\lambda)^k = \{pv^- \mid p \in S(\mathfrak{h}^*), \deg(p) \leq k\}.$$

For simple tempered $\mathbb{H}$-module $L(\mu)$, Kostka-Foulkes polynomial of $M(\lambda)$ is

$$K_{\lambda\mu}(t) = \sum_{k \geq 0} \left[M(\lambda)^k / M(\lambda)^{k-1} : L(\mu)\right] t^k,$$

where $[V : L(\mu)]$ is multiplicity of simple W-module $L(\mu)$ in V.

A Springer correspondence: matrix of $K_{\lambda\mu}(t)$:

(a) is square,
(b) has nonzero diagonal entries,
(c) is upper triangular.

Classical Springer correspondence

Fix usual data ($G \supseteq B \supseteq T$), G connected reductive algebraic group over $\mathbb{C}$.

Weyl group $W = N(T)/T$.

For $u \in G$ unipotent, component group (small finite, often trivial) is

$$A(u) = Z_G(u)/Z_G(u)^\circ.$$

Springer fiber is $\mathcal{B}_u = \{\text{Borel subgroups of } G \text{ containing } u\}$.

Springer (1976, 1978) discovered:

(a) $H^*(\mathcal{B}_u)$ is $(W, A(u))$-bimodule.

Can decompose $H^*(\mathcal{B}_u)$ according to the irreducible representations of $A(u)$

$$H^*(\mathcal{B}_u) \cong \bigoplus_{\rho \in A(u)^\wedge} H^*(\mathcal{B}_u)_\rho \otimes \rho, \qquad H^*(\mathcal{B}_u)_\rho \text{ is } W\text{-module.}$$

(b) $H^{\text{top}}(\mathcal{B}_u)_\rho$ is simple W-module, denoted $W^{(u,\rho)}$.

(c) Every simple W-module arises this way and *uniquely*, up to conjugation.

Thus: $\{\text{simple } W\text{-modules}\} \longleftrightarrow \{\text{conjugacy classes of pairs } (u, \rho)\}$.

Refinements of Springer correspondence

Borho, MacPherson (1981) explained (c) via intersection cohomology of unipotent orbits $\mathbb{O}_u$ with closure $\overline{\mathbb{O}_u}$.

$$K_{(u,\rho),(n,\tau)}(t) = 0 \quad \text{unless} \quad \mathbb{O}_n \subseteq \overline{\mathbb{O}_u}, \qquad \text{and} \qquad K_{(u,\rho),(u,\rho)} = 1.$$

In fact, $K_{(u,\rho),(n,\tau)}(t)$ are Poincaré polynomials for intersection cohomology sheaves on $\overline{\mathbb{O}_u}$:

$$K_{(u,\rho),(n,\tau)}(t) = \sum_{k\geq 0} \dim(H^k(IC(\overline{\mathbb{O}_u}, \chi^\rho)_n)_\tau))t^k.$$

Lusztig ($\sim$ 1990) showed pairs (u,ρ) index tempered simple modules of graded Hecke algebra.

Main Results (K, Ram)

Can extend Springer correspondence to noncrystallographic setting.

$I_2(m)$ for arbitrary parameters and H_3 substantially complete.

H_4 calculations in progress.

See Tables.

Example. $W = S_n$, type A_{n-1}. $M(\lambda) \cong \mathrm{Ind}_{S_\lambda}^{S_n}(\varepsilon)$

Two bases for the spherical Hecke algebra, Hall-Littlewood polynomials P_μ and Schur functions s_λ for partitions μ and λ, are related by

$$P_\mu = \sum_\lambda K_{\lambda\mu}(t) s_\lambda$$

and $K_{\lambda\mu}(1)$ are usual Kostka numbers.

Example. Unique simple $\mathbb{H}$-module $M(0)$ of central character $0 \in \mathfrak{h}_\mathbb{R}$ is

$$M(0) = \mathrm{Ind}_{S(\mathfrak{h}^*)}^{\mathbb{H}}(0) = \mathbb{C}\text{-span}\{t_w \otimes v_0 \mid w \in W\},$$

where $xv_0 = 0$ for all $x \in \mathfrak{h}_\mathbb{C}^*$.

As W-module, $M = M(0)$ is isomorphic to the regular representation.

Have antispherical vector

$$v^- = \sum_{w \in W} (-1)^{\ell(w)} t_w v_0,$$

and as W-module M^k/M^{k-1} is degree k homogeneous harmonic polynomials.

Thus the Kostka-Foulkes polynomials of $M(0)$ are the fake degrees of W.

Example. Define $\rho \in \mathfrak{h}_\mathbb{R}$ by $\rho(\alpha_i) = 1$, for $1 \le i \le n$.

Unique simple tempered $\mathbb{H}$-module $M(-\rho)$ with $-\rho$ as a weight.

$\dim M(-\rho) = 1$ and as W-module, $M(-\rho) \cong \varepsilon$, sign representation.

Thus the Kostka-Foulkes polynomials of $M(-\rho)$ are given by

$$K_{-\rho,\varepsilon} = 1, \qquad \text{and} \qquad K_{-\rho,\mu} = 0, \quad \text{for } \mu \in W^\wedge,\ \mu \neq \varepsilon.$$

Techniques

(a) Analyze weight space structure of $M(\lambda) = \bigoplus M(\lambda)^{\text{gen}}_{\gamma}$.

Controlled by Shi arrangement of central and affine hyperplanes.

Described combinatorially for $\gamma \in \overline{C}$ by

$P(\gamma) = \{\alpha \in R^+ \mid \gamma(\alpha) = 1\}$ and $Z(\gamma) = \{\alpha \in R \mid \gamma(\alpha) = 0\}$,

and local regions, defined for each $J \subseteq P(\gamma)$ by

$$\mathcal{F}^{(\gamma,J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset,\ R(w) \cap P(\gamma) = J\}.$$

$\mathcal{F}^{(\gamma,J)} \longleftrightarrow$ chambers at γ in $\mathfrak{h}_{\mathbb{R}} \longleftrightarrow$ weights $w\gamma$ of $M(\lambda)$

For any $J \subseteq P(\gamma)$,

$$\dim M(\lambda)^{\text{gen}}_{w\gamma} = \dim M(\lambda)^{\text{gen}}_{u\gamma}, \qquad \text{for all } u, w \in \mathcal{F}^{(\gamma,J)}.$$

(b) Find good central characters.

If $\mathbb{H}$-module is tempered and not cuspidal, then is induced from a tempered representation on a parabolic subalgebra.

If $\mathbb{H}$-module is cuspidal and has central character γ, apply combinatorics above at distinguished points where $\#P(\gamma) = \#Z(\gamma) + n$ (Heckman, Opdam).

Find greatest reducibility in principal series modules $\text{Ind}^{\mathbb{H}}_{S(\mathfrak{h}^*_{\mathbb{C}})}(\mathbb{C}v_{\gamma})$ at such characters, (intersections of hyperplanes).

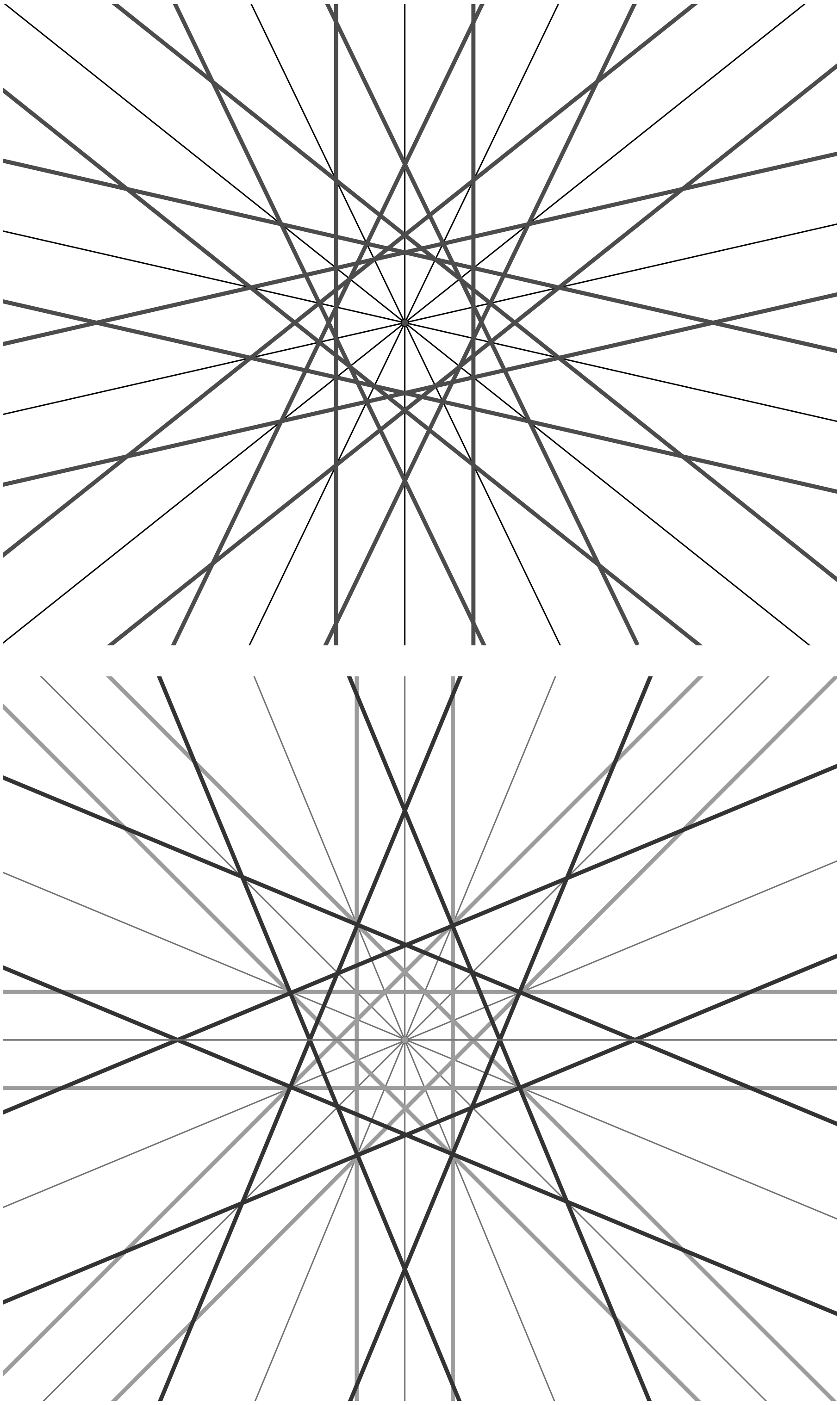


FIGURE 3. Hyperplanes for $I_2(7)$ and $I_2(8)$.

Table 1. Character table of $WI_2(m)$

	1	s_1	s_2	$(s_1s_2)^k,\ 1 \leq k \leq \lfloor \frac{m}{2} \rfloor$	
χ_1^+	1	1	1	1	
χ_1^{+-}	1	1	-1	$(-1)^k$	
χ_1^{-+}	1	-1	1	$(-1)^k$	
χ_2^λ	2	0	0	$2\cos\left(\frac{2\pi\lambda k}{m}\right),$	$1 \leq \lambda \leq \lfloor \frac{m-1}{2} \rfloor$
χ_1^-	1	-1	-1	1	

TABLE 2. Tempered representations of $\mathbb{H}I_2(m)$

Label	Tempered Representation	Dimension	Central Character	$Z(\gamma)$ $P(\gamma)$
M^0	$\mathrm{Ind}_{S(\mathfrak{h}^*)}^{\mathbb{H}}(0)$	$2m$	$(0,0)$	$Z(\gamma)=R^+$ $P(\gamma)=\emptyset$
M^{+-}	$\mathrm{Ind}_{\mathbb{H}_1}^{\mathbb{H}}(St\otimes 0)$	m	$\beta_{\frac{m}{2}}$	$Z(\gamma)=\{\alpha_1\}$ $P(\gamma)=\{\beta_{\frac{m}{2}}\}$
M^{-+}	$\mathrm{Ind}_{\mathbb{H}_2}^{\mathbb{H}}(0\otimes St)$	m	$\beta_{\frac{m}{2}-1}$	$Z(\gamma)=\{\alpha_2\}$ $P(\gamma)=\{\beta_{\frac{m}{2}-1}\}$
$M^{(m-1)/2}$	$\mathrm{Ind}_{\mathbb{H}_1}^{\mathbb{H}}(St\otimes 0)$	m	$\beta_{\frac{m-1}{2}}$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\beta_{\frac{m-1}{2}}\}$
$M^{\lambda}, 1\le\lambda\le\lfloor\frac{m}{2}\rfloor$	cuspidal	$2\lambda-1$	$\frac{1}{\sin\left(\frac{2\pi\lambda}{m}\right)}\beta^*_{\frac{m-1}{2}}$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\beta_{k-1},\beta_{m-k}\}$

$\beta_k=\cos(\frac{k\pi}{n})\varepsilon_1+\sin(\frac{k\pi}{n})\varepsilon_2,\qquad \varepsilon_1,\varepsilon_2$ an orthonormal basis of $\mathfrak{h}^*_{\mathbb{R}}=\mathbb{R}^2$

$*$ This is correct for m odd, but not for m even.

(K - 1995; K, Ram - ERT, 2002)

Table 3. Decomposition matrix for $I_2(m)$

dim.		χ_1^+	χ_1^{+-}	χ_1^{-+}	$\chi_2^{\lfloor (m-1)/2 \rfloor}$	$\cdots$	χ_2^2	χ_2^1	χ_1^-
$2m$	M^0	1	1	1	2	$\cdots$	2	2	1
m	M^{+-}		1	0	1	$\cdots$	1	1	1
m	M^{-+}			1	1	$\cdots$	1	1	1
$2\lfloor \frac{m+1}{2} \rfloor - 1$	$M^{\lfloor (m+1)/2 \rfloor}$				1	$\cdots$	1	1	1
$\vdots$	$\vdots$								$\vdots$
5	M^3						1	1	1
3	M^2							1	1
1	M^1								1

TABLE 4. Kostka-Foulkes polynomials for $I_2(m)$

	χ_1^+	χ_i^{+-}	χ_1^{-+}	$\cdots$	$\chi_2^{\lambda-1}$	$\cdots$	χ_2^{μ}	$\cdots$	χ_1^-	$1 \leq \mu \leq \lfloor \frac{m-1}{2} \rfloor$
M^0	1	$t^{m/2}$	$t^{m/2}$	$\cdots$		$\cdots$	$t^{\mu} + t^{m-\mu}$	$\cdots$	t^m	
M^{+-}		1	0	$\cdots$		$\cdots$	$t^{(m/2)-\mu}$	$\cdots$	$t^{m/2}$	
M^{-+}			1	$\cdots$		$\cdots$	$t^{(m/2)-\mu}$	$\cdots$	$t^{m/2}$	
$\vdots$									$\vdots$	
M^{λ}					1	$\cdots$	$t^{\lambda-\mu}$	$\cdots$	$t^{\lambda-1}$	$1 \leq \lambda \leq \lfloor \frac{m+1}{2} \rfloor$
$\vdots$									$\vdots$	
M^2								1	t	
M^1									1	

TABLE 5. Character table of WH_3

	1	s_1	s_{12}	s_{13}	s_{23}	s_{123}	s_{1212}	s_{12123}	$s_{121232123}$	w_0
χ_1^+	1	1	1	1	1	1	1	1	1	1
$\bar{\chi}_3^+$	3	1	$\bar{\tau}$	-1	0	$-\tau$	τ	0	$-\bar{\tau}$	-3
χ_3^+	3	1	τ	-1	0	$-\bar{\tau}$	$\bar{\tau}$	0	$-\tau$	-3
χ_5^+	5	1	0	1	-1	0	0	-1	0	5
χ_4^+	4	0	-1	0	1	1	-1	-1	1	-4
χ_4^-	4	0	-1	0	1	-1	-1	1	-1	4
χ_5^-	5	-1	0	1	-1	0	0	1	0	-5
χ_3^-	3	-1	τ	-1	0	$\bar{\tau}$	$\bar{\tau}$	0	τ	3
$\bar{\chi}_3^-$	3	-1	$\bar{\tau}$	-1	0	τ	τ	0	$\bar{\tau}$	3
χ_1^-	1	-1	1	1	1	-1	1	-1	-1	-1

TABLE 6. Tempered representations of $\mathbb{H}H_3$

Label	Tempered Representation	Dimension	Central Character in $\alpha^\vee$ Basis	$Z(\gamma)$ $P(\gamma)$
M_1^+	$\mathrm{Ind}_{S(\mathfrak{h}^*)}^{\mathbb{H}}(0)$	120	$(0,0,0)$	$Z(\gamma)=R^+$ $P(\gamma)=\emptyset$
$\bar{M}_3^+$	$\mathrm{Ind}_{\mathbb{H}_1}^{\mathbb{H}}(St\otimes 0)$	60	$\left(\frac{1}{2}\tau+\frac{1}{2},\tau,\frac{1}{2}\tau\right)$	$Z(\gamma)=\{\alpha_1,\alpha_3\}$ $P(\gamma)=\{\alpha_{15}\}$
M_3^+	$\mathrm{Ind}_{\mathbb{H}_{12}}^{\mathbb{H}}\left(\frac{2}{1+\tau}\rho\otimes 0\right)$	36	$(\tau,2,1)$	$Z(\gamma)=\{\alpha_1,\alpha_3\}$ $P(\gamma)=\{\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14}\}$
M_5^+	$\mathrm{Ind}_{\mathbb{H}_{13}}^{\mathbb{H}}(St\otimes 0)$	30	$\left(\frac{1}{2}\tau+1,\tau+\frac{1}{2},\frac{1}{2}\tau+\frac{1}{2}\right)$	$Z(\gamma)=\{\alpha_2\}$ $P(\gamma)=\{\alpha_8,\alpha_{13}\}$
M_4^+	$\mathrm{Ind}_{\mathbb{H}_{23}}^{\mathbb{H}}(St\otimes 0)$	20	$(\tau+1,2\tau,\tau)$	$Z(\gamma)=\{\alpha_1,\alpha_3\}$ $P(\gamma)=\{\alpha_4,\alpha_7,\alpha_8,\alpha_{10}\}$
M_4^-	cuspidal	13	$\left(\frac{1}{2}\tau+2,2\tau,-\frac{1}{2}\tau+\frac{5}{2}\right)$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\alpha_7,\alpha_8,\alpha_9\}$
M_5^-	$\mathrm{Ind}_{\mathbb{H}_{12}}^{\mathbb{H}}(St\otimes 0)$	12	$(2\tau+1,2\tau+2,\tau+1)$	$Z(\gamma)=\{\alpha_1,\alpha_3\}$ $P(\gamma)=\{\alpha_2,\alpha_5,\alpha_6,\alpha_9\}$
M_3^-	cuspidal	4	$(2\tau+2,2\tau+3,\tau+2))$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\alpha_3,\alpha_4,\alpha_5\}$
$\bar{M}_3^-$	cuspidal	3	$\left(\frac{3}{2}\tau+2,2\tau+2,\frac{3}{2}\tau+\frac{1}{2}\right)$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\alpha_4,\alpha_5,\alpha_6\}$
M_1^-	cuspidal Steinberg	1	$\left(\frac{9}{2}\tau+3,5\tau+4,\frac{5}{2}\tau+\frac{5}{2}\right)$	$Z(\gamma)=\emptyset$ $P(\gamma)=\{\alpha_1,\alpha_2,\alpha_3\}$

Table 7. Decomposition matrix for H_3

dim.		χ_1^+	$\bar{\chi}_3^+$	χ_3^+	χ_5^+	χ_4^+	χ_4^-	χ_5^-	χ_3^-	$\bar{\chi}_3^-$	χ_1^-
120	M_1^+	1	3	3	5	4	4	5	3	3	1
60	$\bar{M}_3^+$		1	1	2	2	2	3	2	2	1
36	M_3^+			1	1	1	1	2	2	1	1
30	M_5^+				1	1	1	2	1	1	1
20	M_4^+					1	1	1	1	1	1
13	M_4^-						1	1	1	0	1
12	M_5^-							1	1	1	1
4	M_3^-								1	0	1
3	$\bar{M}_3^-$									1	0
1	M_1^-										1

Table 8. Kostka-Foulkes polynomials for H_3

	χ_1^+	$\bar{\chi}_3^+$	χ_3^+	χ_5^+	χ_4^+	χ_4^-	χ_5^-	χ_3^-	$\bar{\chi}_3^-$	χ_1^-
M_1^+	1	$t^3+t^5+t^7$	$t^1+t^5+t^9$	$t^2+t^4+t^6+t^8+t^{10}$	$t^3+t^7+t^9+t^{11}$	$t^4+t^6+t^8+t^{12}$	$t^5+t^7+t^9+t^{11}+t^{13}$	$t^6+t^{10}+t^{14}$	$t^8+t^{10}+t^{12}$	t^{15}
$\bar{M}_3^+$		1	t^2	$t+t^3$	t^2+t^4	$t+t^5$	$t^2+t^4+t^6$	t^3+t^7	t^3+t^5	t^8
M_3^+			1	t	t^2	t^3	t^2+t^4	$t+t^5$	t^3	t^6
M_5^+				1	t	t^2	$t+t^3$	t^4	t^2	t^5
M_4^+					1	t	t^2	t^3	t	t^4
M_4^-						1	t	t^2	0	t^3
M_5^-							t	t^2	1	t^3
M_3^-								1	0	t
$\bar{M}_3^-$									1	0
M_1^-										1