# Unitary representations of the icosahedral graded Hecke algebra

AMS Southeastern Section Meeting, LA - March 29, 2008

Cathy Kriloff
Idaho State University
krilcath@isu.edu

Joint work in progress with
Yu Chen
Idaho State University

Partial support provided by a grant from the ISU WeLEAD Project
which is funded by NSF ADVANCE grant SBE-0620073

Slides will be available at www.isu.edu/∼krilcath/pubs.

(1) Icosahedral group and graded Hecke algebra

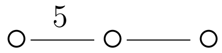
(2) Tempered and unitary representations

(3) Intertwining operators and other ingredients

(4) Statement of results for $\mathbb{H}(H_3)$

## Icosahedral reflection group

$H_3$ = 120 symmetries of the icosahedron or dodecahedron

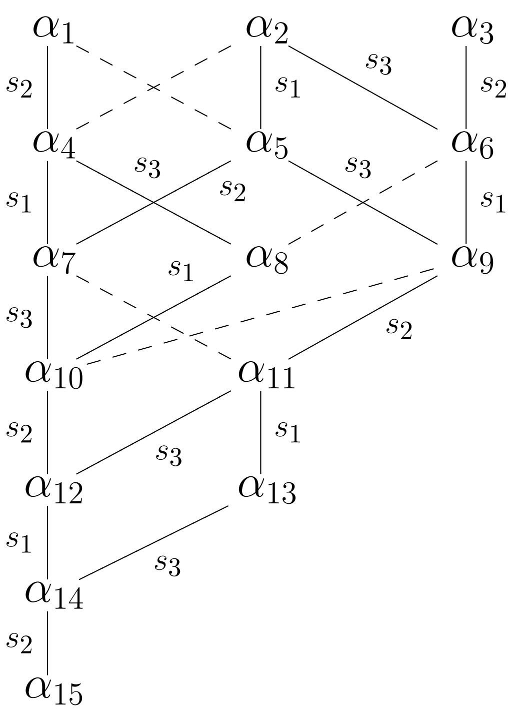
= Coxeter group $\langle s_1, s_2, s_3 \rangle$ with relations given by 



$\mathfrak{h}^*_{\mathbb{R}}$ = reflection representation of $W$ containing root system $R$

Positive roots $\alpha_1, \ldots \alpha_{15}$, and actions of $s_i$ are given below.

FIGURE 1. Positive roots of $H_3$, actions of $s_i$, and the root order





where for $\tau = \frac{1+\sqrt{5}}{2}$,

$$
\begin{aligned}
\alpha_4 &= \alpha_1 + \tau\alpha_2 \\
\alpha_5 &= \tau\alpha_1 + \alpha_2 \\
\alpha_6 &= \alpha_2 + \tau\alpha_3 \\
\alpha_7 &= \tau\alpha_1 + \tau\alpha_2 \\
\alpha_8 &= \alpha_1 + \tau\alpha_2 + \tau\alpha_3 \\
\alpha_9 &= \tau\alpha_1 + \alpha_2 + \alpha_3 \\
\alpha_{10} &= \tau\alpha_1 + \tau\alpha_2 + \tau\alpha_3 \\
\alpha_{11} &= \tau\alpha_1 + (\tau+1)\alpha_2 + \alpha_3 \\
\alpha_{12} &= \tau\alpha_1 + (\tau+1)\alpha_2 + \tau\alpha_3 \\
\alpha_{13} &= (\tau+1)\alpha_1 + (\tau+1)\alpha_2 + \alpha_3 \\
\alpha_{14} &= (\tau+1)\alpha_1 + (\tau+1)\alpha_2 + \tau\alpha_3 \\
\alpha_{15} &= (\tau+1)\alpha_1 + 2\tau\alpha_2 + \tau\alpha_3
\end{aligned}
$$

**Why noncrystallographic root systems?**

- Crystallographic graded Hecke algebras have been well-studied by geometric, algebraic, and combinatorial means. Noncrystallographic algebras have no known geometry, but the representation theory is similar in many ways and intriguingly different in others.

- Understanding the representation theory of noncrystallographic graded Hecke algebras may help reveal some as yet unknown geometry.

- Efforts to generalize aspects of Lie theory to the setting of complex reflection groups must include the noncrystallographic cases. Since these are still real reflection groups they may serve as good boundary or test cases. Since they differ in fundamental ways they may help show what differences need to be accounted for.

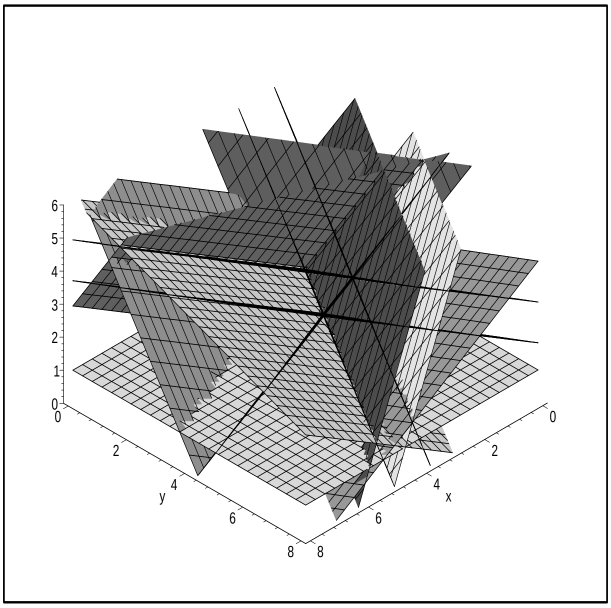
- On a combinatorial level, these or similar algebras may help to relate objects that exhibit common numerology (e.g., noncrossing partitions, nonnesting partitions, vertices in generalized associahedra) but whose connections are not entirely understood.

## Associated geometric and combinatorial objects

Reflecting hyperplanes $H^0_{\alpha_i} = \{x \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle x, \alpha_i^\vee \rangle = 0\}$ for $1 \le i \le 15$

Affine hyperplanes $H^1_{\alpha_i} = \{x \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle x, \alpha_i^\vee \rangle = 1\}$ for $1 \le i \le 15$

$\overline{C}$ = closure of fundamental chamber
is divided into 3-, 2-, 1-, and 0-dimensional cells by $\{H^1_{\alpha_i}\}_{i=1}^{15}$




Characterize the 0-dimensional cells (vertices) using

$Z(x) = \{\alpha_i \mid \langle x, \alpha_i^\vee \rangle = 0\}$ = reflecting hyperplanes containing $x$

$P(x) = \{\alpha_i \mid \langle x, \alpha_i^\vee \rangle = 1\}$ = affine hyperplanes containing $x$.

**Theorem**(Chen-K, 2007). *Ideals in the poset of positive roots under the root order characterize the 3-dimensional cells.*
(See final page.)

## (Icosahedral) graded Hecke algebra $\mathbb{H}$

Additional ingredients:

$W$ acts on complexification $\mathfrak{h}_{\mathbb{C}}^* = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$.

$W$ acts on symmetric algebra $S(\mathfrak{h}_{\mathbb{C}}^*) \cong$ polynomials on $\mathfrak{h}_{\mathbb{C}}$.

$\mathbb{C}W = \left\{ \sum_{w \in W} a_w t_w \mid a_w \in \mathbb{C} \right\}$ with multiplication as in $W$.

Graded Hecke algebra is $\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}_{\mathbb{C}}^*)$
with multiplication as in $S(\mathfrak{h}_{\mathbb{C}}^*)$ and in $\mathbb{C}W$ and

$$\begin{aligned} x t_{s_i} &= t_{s_i} s_i(x) + \langle x, \alpha_i^\vee \rangle \\ &= t_{s_i} s_i(x) + \frac{x - s_i(x)}{\alpha_i} \qquad \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*. \end{aligned}$$

Center is $Z(\mathbb{H}) = S(\mathfrak{h}_{\mathbb{C}}^*)^W = W$-invariant polynomials on $\mathfrak{h}_{\mathbb{C}}$.

$Z(\mathbb{H})$ acts on simple $\mathbb{H}$-module $V$ by central character $\gamma \in \mathfrak{h}_{\mathbb{C}}$ $(W\gamma)$.

We will often restrict to real central character $\gamma \in \mathfrak{h}_{\mathbb{R}}$.

**Goal:** Describe the real support of spherical unitary representations of $\mathbb{H}(H_3)$ ("SPUD").

## Tempered representations of $\mathbb{H}$

Any simple $\mathbb{H}$-module $V$ has generalized weight space decomposition

$$V = \bigoplus_{\gamma \in \mathfrak{h}_{\mathbb{C}}} V_\gamma^{\mathrm{gen}},$$

where $\gamma$ called a weight of $V$ if $V_\gamma^{\mathrm{gen}} \neq 0$.

$\mathbb{H}$-module $V$ is tempered (discrete series) if for all weights $\lambda \in \mathfrak{h}_{\mathbb{C}}$ of $V$ and all fundamental weights $\omega_i \in \mathfrak{h}_{\mathbb{C}}^*$, $\omega_i(\lambda) \leq 0$ ($\omega_i(\lambda) < 0$).

## Unitary representations of $\mathbb{H}$

$*$-operation defined on $\mathbb{H}$ by

$$t_w^* = t_{w^{-1}}, \qquad x^* = -t_{w_0}(w_0 x)t_{w_0} = -\bar{x} + \sum_{\alpha \in R^+} \bar{x}(\alpha^\vee) t_{s_\alpha},$$

for $w \in W$ and $x \in \mathfrak{h}^*$.

$\mathbb{H}$-module $V$ is Hermitian if there is nondegenerate form such that

$$\langle h v_1, v_2 \rangle = \langle v_1, h^* v_2 \rangle, \qquad \text{for all } h \in \mathbb{H},\ v_1, v_2 \in V,$$

and $V$ is unitary if $\langle\ ,\ \rangle$ is also positive definite.

An invariant Hermitian form on $V$ is equivalent to an $\mathbb{H}$-module isomorphism from $V$ to its Hermitian dual.

**Langlands classification for $\mathbb{H}$**

Fix subset $\Pi_M$ of simple roots and root system $R_M$ they generate.

$\mathbb{H}$ has subalgebra $\mathbb{H}_M = \mathbb{H}_{M_0} \otimes S(\mathfrak{t}^*) \leftrightarrow (\mathfrak{h}^*_{\mathbb{C}}, R_M)$ where
$\mathbb{H}_{M_0} \leftrightarrow (\mathbb{C}\langle \Pi_M \rangle, R_M)$, and define

$$\mathfrak{t} = \{\nu \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(\nu) = 0 \text{ for all } \alpha \in \Pi_M\}$$
$$\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^*_{\mathbb{C}} \mid \lambda(\alpha^\vee) = 0 \text{ for all } \alpha \in \Pi_M\}$$
$$\mathfrak{t}^+ = \{\nu \in \mathfrak{t} \mid Re(\alpha(\nu)) > 0 \text{ for all } \alpha \in \Pi \setminus \Pi_M\}.$$

For an $\mathbb{H}_M$-module $U$ let $I(M, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

**Theorem**(Langlands Classification - Evens, 1996).

(1) *Every simple $\mathbb{H}$-module is a quotient of a <u>standard module</u> $X(M, \sigma, \nu) = I(M, \sigma \otimes \mathbb{C}_\nu)$, $\sigma$ a tempered $\mathbb{H}_{M_0}$-module, $\nu \in \mathfrak{t}^+$.*
(2) *$X(M, \sigma, \nu)$ has unique irreducible (Langlands) quotient $L(M, \sigma, \nu)$.*
(3) *$L(M, \sigma, \nu)$ is unique up to conjugacy of the triple $(M, \sigma, \nu)$.*

The $W$-module structure of $X(M, \sigma, \nu)$ is constant on cells.

**Intertwining operator**

Restrict to spherical representations $V$ such that $\mathrm{Hom}_W(V,1) \neq 0$.

Writing $w_0 = s_{i_1} \cdots s_{i_\ell} = (s_1 s_2 s_1 s_2 s_3)^3$ induces the ordering

$$\beta_1 = s_\ell \cdots s_2 \alpha_1, \ldots, \quad \beta_{\ell-1} = s_\ell \alpha_{\ell-1}, \quad \beta_\ell = \alpha_\ell.$$

$$\{\alpha_1, \alpha_5, \alpha_7, \alpha_4, \alpha_{13}, \alpha_2, \alpha_{11}, \alpha_{15}, \alpha_{14}, \alpha_{12}, \alpha_9, \alpha_{10}, \alpha_8, \alpha_6, \alpha_3\}.$$

For each $\nu \in V$, define the following element of $\mathbb{R}[W]$:

$$\begin{aligned} A(\nu) &= (1 + \langle \nu, \beta_1^\vee \rangle s_{i_1})(1 + \langle \nu, \beta_2^\vee \rangle s_{i_2}) \cdots \big(1 + \langle \nu, \beta_\ell^\vee \rangle s_{i_\ell}\big) \\ &= (1 + \langle \nu, \alpha_1^\vee \rangle s_1)(1 + \langle \nu, \alpha_5^\vee \rangle s_2) \cdots (1 + \langle \nu, \alpha_3^\vee \rangle s_3). \end{aligned}$$

- $A(\nu)$ is independent of the choice of reduced expression for $w_0$.
- $A(\nu)$ is invertible if and only if $\langle \nu, \alpha_i^\vee \rangle \neq \pm 1$ for all $i$.

Define the principal series module $X(\nu) = \mathbb{H} \otimes_{S(\mathfrak{h}_\mathbb{C}^*)} \mathbb{C}_\nu$ for $\nu \in \mathfrak{h}_\mathbb{R}$.

If $\nu \in C$ then $X(\nu)$ has a unique simple quotient $L(\nu)$, which is spherical, and any spherical $\mathbb{H}$-module appears in this way.

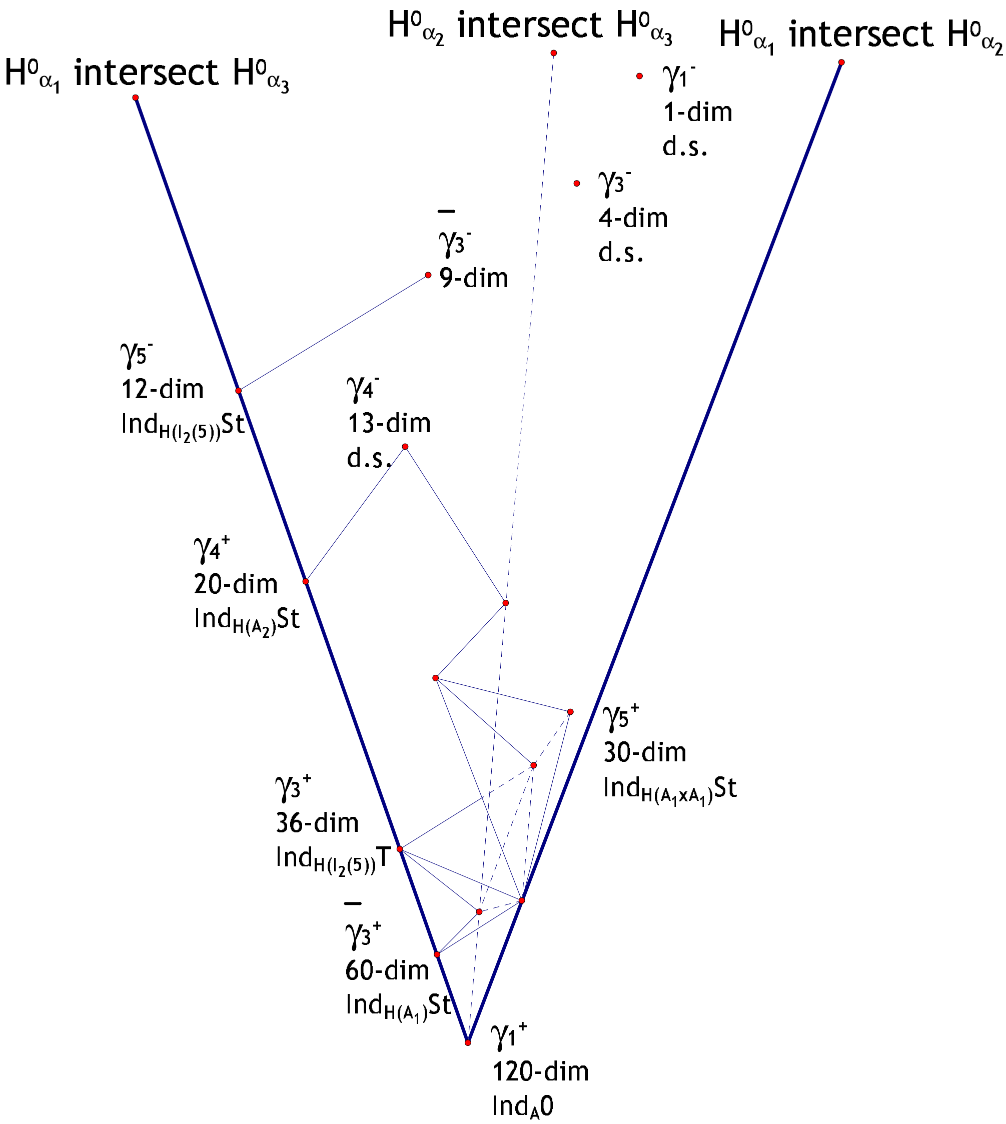
$L(\nu)$ is Hermitian if and only if $w_0 \nu = -\nu$.

- $A(\nu)$ is an intertwining operator from $X(\nu)$ to $X(-\nu)$ with image $L(\nu)$ if $w_0 \nu = -\nu$.
- $L(\nu)$ is unitary if and only if $\rho(A(\nu))$ is positive semi-definite for all $\rho \in \widehat{W}$.
- The signature of $\rho(A(\nu))$ is constant on cells.

## Results for $\mathbb{H}(H_3)$

The spherical unitary dual of $\mathbb{H}(H_3)$ consists of

- three connected closed 3-dimensional pieces connected to
- three 1-dimensional pieces,
- a single isolated 1-dimensional piece, and
- two isolated points.



## Techniques

(1) Obtain vertices $\nu$ of 3-d polyhedral regions

  (a) Convert half-space description from exact inequalities over $\mathbb{Q}[\sqrt{5}]$ to approximate inequalities.

  (b) Use Polymake to convert approximate half-space description to vertex description. (Is there an algorithm over $\mathbb{Q}[\sqrt{5}]$?)

  (c) Compute inner products in Maple to "guess" $P(\nu)$, $Z(\nu)$ for each vertex $\nu$.

  (d) Solve systems of equations exactly in Maple to obtain an exact form for each vertex over $\mathbb{Q}[\sqrt{5}]$.

(2) Analyze signs of eigenvalues for each $\rho(A(\nu))$ where
$\rho$ is an irreducible representation of $H_3$ (from $W$-graphs),
$\nu$ is a vertex or average of vertices (chosen efficiently)

  (a) Compute matrix $\rho(A(\nu))$ and characteristic polynomial $\chi_{\rho,\nu}$ over $\mathbb{Q}[\sqrt{5}]$ .

  (b) Analyze signs of coefficients in $\chi_{\rho,\nu}$ by inspection if possible and by computing an approximation if not.

  (c) Use Descartes' Rule of Signs to detect or rule out negative eigenvalues.

Figure 2. Ideals in the positive root poset or 3-dim cells for $H_3$