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Propagator in Random Systems: the Black-Scholes example and beyond

Bailey Hsu* and Jean-François Van Huele†

Department of Physics, Brigham Young University

Provo, UT 84602

*britchap@gmail.com
†vanhuele@byu.edu

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Abstract

We highlight similarities and differences between financial and quantum systems and illustrate how they affect the construction of propagators used to study the evolution of the systems. In particular we discuss the transferability of propagators between the two systems. The comparison provides inspiration for the introduction of new degrees of freedom in the description of the evolution of the systems.

Introduction

Both the microworld and the financial world are infused with randomness and uncertainty. Still, rigorous methods have been developed using mathematical tools such as differential equations to study quantum physics and the market quantitatively. We look at two such equations: the Schrödinger equation and the Black-Scholes equation for option trading. They exhibit interesting similarities. In this paper we use the propagator, a useful tool in the study of the evolution of a system to compare the quantum and the financial domains in the hope of detecting cross-fertilizing elements.

Randomness and Uncertainty

In financial markets, randomness originates in the extremely large number of degrees of freedom, with many buyers and sellers of stocks connected to economic, political, and other factors. Whether or not individual traders act rationally, the system behaves stochastically. In the Black-Scholes financial model for European options [1] the traders of the option can only exercise their right to buy or sell at the time of maturity. The resulting Black-

Scholes equation provides the individual investor with a deterministic tool to hedge financial risk by getting information on portfolio composition. In the quantum world randomness enters in the description of individual particles. The wavefunction of an electron represents its probability amplitude in space. The Schrödinger equation is a fully deterministic process to study the evolution of this probability distribution. Usually the randomness enters in the measurement process [2] independently of the Schrödinger equation, except for the fact that the wavefunction contains information on measurement outcomes. Simultaneous knowledge on complementary variables such as position x and momentum p is limited by the Heisenberg Uncertainty Relation $\Delta x \Delta p \geq \frac{\hbar}{2}$, where $\hbar$ is Planck's constant. Interestingly a similar incompatibility between financial variables, such as stock price x and stock velocity $\dot{x}$ can be derived $\Delta x \Delta \dot{x} \geq \frac{\sigma^2}{2}$, where σ is the volatility which corresponds to the standard deviation of price fluctuations over a specified time horizon [3]. Clearly, the origins of the randomness in microphysics and finance are quite different.

Correspondence between the Equations

The Schrödinger equation for an electron of mass *m* in a potential *V* is,

$$i\hbar\frac{\partial\psi}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial_x^2} + V(x, \frac{\partial}{\partial_x}))\psi \equiv H_S\psi \quad (1)$$

where ψ is the wavefunction and H_S is the Hamiltonian operator. The Black-Scholes equation for European options is,

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial^2 S} = rC \quad (2)$$

where $C(S,t)$ is the portfolio of an investor, $S(t)$ is the stock price, r is the spot (or instantaneous) interest rate. To enhance the similarity between Eq. (1) and Eq. (2) we introduce a new variable $x = lnS$ and obtain

$$\frac{\partial C}{\partial t} = (-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + r)C. \quad (3)$$

By formally associating ψ with C and σ^2 with $\frac{\hbar^2}{m}$, we can define a Black-Scholes Hamiltonian,

$$H_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + r. \tag{4}$$

A cautionary remark is in order. Eq. (1) has an additional factor of i compared to Eq. (3) and H_{BS} is not hermitian. In quantum physics hermiticity is required for measurable variables because it guarantees real outcomes, in particular real energies.

Construction of Propagators

The quantum propagator, $K(x, x_0; t, 0)$ represents the conditional probability of finding a particle at location x and time t if located at position x_0 at time 0. For time-independent Hamiltonians, we use a time-development operator $T(t, 0) = \exp(\frac{Ht}{i\hbar})$ to describe the evolution of the states representing the system. By projecting the $T(t, 0)$ into position eigenstates (states with precise location x), we can follow the dynamics in position space $K(x, x_0; t, 0) =< x|T(t, 0)|x_0 >$. There are several methods to construct propagators [4, 5, 6]. Here we focus on the eigenfunction expansion method [4]. Since eigenstates $|E_n>$ form a complete set $I = \sum_n |E_n><E_n|$ and obey the property $H|E_n>= E_n|E_n>$, we get

$$K(x,x_0;t,0) = \sum_n <x|exp(-\frac{i}{\hbar}Ht)|E_n><E_n|x_0>$$
$$= \sum_n \psi_n{}^*(x_2)\psi_n(x_1)exp(-\frac{i}{\hbar}E_n(t)) \quad (5)$$

where $\psi(x) =<x|E_n>$ and $\psi^*(x) =<E_n|x>$. We apply this method to the quantum free-particle Hamiltonian $H_{FP} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ and to H_{BS} in Eq. (4). Rewriting the Hamiltonian in terms of p where $p \equiv i\frac{\partial}{\partial x}$, we obtain

$$H_{FP} = \frac{p^2}{2m} \quad (6)$$
$$H_{BS} = \frac{\sigma^2 p^2}{2\hbar^2} + (\frac{1}{2}\sigma^2 - r)\frac{i}{\hbar}p + r$$
$$= \frac{\sigma^2}{2\hbar^2}(p + \frac{i\hbar}{\sigma^2}(\frac{\sigma^2}{2} - r))^2 + r + \frac{\hbar^2}{2\sigma^2}(\frac{\sigma^2}{2} - r)^2 \quad (7)$$

We see that, except for a constant term the H_{BS} looks very much like H_{FP} which has a continuous set of eigenstates and thus

$$K(x,x_0;t,0)_{FP} = \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}(\frac{p^2 t}{2m})} e^{-\frac{i}{\hbar}p(x-x_0)} dp$$

$$
\begin{aligned}
&= \sqrt{\frac{m}{2\pi i\hbar t}} e^{-\frac{m(x-x_0)^2}{2i\hbar t}} \\
K(x, x_0; t, 0)_{BS} &= \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}(\frac{\sigma^2 p^2}{2\hbar^2} - (\frac{\sigma^2}{2} - r)\frac{i}{\hbar}p + r)t} e^{-\frac{i}{\hbar}p(x-x_0)} dp \\
&= e^{-rt} \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-\frac{1}{2t\sigma^2}(x - x_0 + \tau(r - \frac{\sigma^2}{2}))^2} \qquad (8)
\end{aligned}
$$

We have verified that the same result can be derived using Feynman's path-integral method [3]. Thus constructed, the financial propagators allow us to generate fair pricing for call or put options.

Spin Systems

The quantum systems that we considered so far did not include spin. In reality spin plays an important role in many atomic systems and solids. Therefore it becomes important to extend the propagator technique to spin systems [7]. In atomic physics, an electron in the electric field of the nucleus experiences a magnetic field in its own rest frame. This spin-orbit coupling effect has a counterpart in spintronics [8] originating in the asymmetry of the potential (Rashba effect) [9, 10] or the asymmetry of the crystal structure (Dresselhaus effect) [11]. The Rashba and Dresselhaus

Hamiltonians are,

$$H_R = \frac{\mathbf{p}^2}{2m} + \frac{\alpha}{\hbar}(p_x\sigma_y - p_y\sigma_x), \quad H_D = \frac{\mathbf{p}^2}{2m} + \frac{\beta}{\hbar}(p_x\sigma_y - p_y\sigma_x) \tag{9}$$

where α and β correspond to the spin-orbit coupling strength for the Rashba and Dresselhaus couplings and σ_x, σ_y, σ_z are the spin Pauli matrices. By comparing Eq. (9) with to Eq. (7), we make the following observations: **1**. Both systems contain a term drift (linear) in p. Rashba and Dresselhaus propagators can therefore be constructed in analogy with K_{BS} in Eq. (8). **2**. Rashba and Dresselhaus are 2D systems while Black-Scholes is 1D, but 1D quantum wires with spin or 2D extensions of Black-Scholes model can be considered. **3**. Pauli matrices appear in the drift term in Eq. (9) which leads to spin-dependent phenomena while no such matrices appear explicitly in Eq. (7). Instead the coefficient of the linear term is the spot interest rate, r. The raises the question as to whether a degree of freedom similar to spin can be found for Black-Scholes systems and by extension in the physics of financial markets. Mining of financial data bases could be used to substantiate its existence in the way that spectral analysis revealed the existence of electron spin.

Conclusion

We have seen that random systems can be studied using propagators. Stocks and quantum particles obey differential equations that exhibit some striking similarities and are both amenable to the propagator method. The differences between them point to possible extensions such as the existence of a financial spin and argues for its search. Is there an intrinsic discrete two-valued degree of freedom in the financial world and can it be observed in markets? Vice versa can the presence of interest rate terms in the Black-Scholes model inform us about existing "drifting" terms in physical equations? It is hard to see how the interest rate r could play the role of spin since r is not discrete or two valued. Indeed a mathematical similarity need not imply a correspondence between variables in physics and finance. Still, the joint appearance of randomness and common formal methods can be a source of cross-fertilization as illustrated in the study of propagators.

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