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ESTIMATING TIMING OF LIFE-HISTORY EVENTS WITH COARSE DATA

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Populations often are sampled with a coarse scale of measurement. As scale becomes increasingly coarse, the variance estimate can become biased; Sheppard's method has been used to correct that bias. Sheppard's correction, however, also becomes inadequate when the scale of measurement is too coarse. We develop a rule to decide how coarse a scale should be for a particular population variance. In addition, we propose a generalization of Sheppard's correction that allows for divisions of the scale to be unequal. Divisions of a measurement scale (intervals or bins) should be no larger than twice the population *SD* (σ). When the population variance (σ^2) is small, a large amount of variation in interval size can produce inaccurate results. As σ^2 becomes large, more variation in interval size can be allowed without producing large inaccuracies. This new methodology has wide application for estimating timing of life-history events for mammals where dates must be pooled into intervals. We demonstrate this method with computer simulations and with an example for estimating mean and variance of birth date in Dall's sheep (*Ovis dalli*).

Key words: coarse data, Dall's sheep, date of birth, interval censoring, *Ovis dalli*, Sheppard's correction, timing of life-history events, unequal intervals, variance

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Making continuous observations of animals under field conditions is seldom possible, and sampling date of parturition (or other life-history characteristics of populations) often occurs in intervals of >1 day. Thus, mean date of birth is estimated from the number of births in each interval instead of the actual day of birth (Caughley 1977). Caughley and Caughley (1974) proposed using probit analysis for estimating median date of birth. Unless a large sample is collected at each tail of the distribution, however, probit analysis can give inaccurate estimates of variance (Finney 1952) because large samples near the tails seldom occur in the distributions of births (Rachlow and Bowyer 1991) or other life-history phenomena that wane over time. In addition, probit analysis requires the assumption of a normally distributed population (Ashford 1986).

Along with mean date of parturition, synchrony of births is a topic that has received much attention (Berger 1992; Bowyer 1991; Bowyer et al. 1998; Keech et al. 2000; Rachlow and Bowyer 1991). Gochfeld (1980) noted that the standard deviation (*SD*) is the best measure of reproductive synchrony. Estimates of the *SD* provide a description of synchrony that is readily comparable with values in the literature or for making between-year comparisons within a population. With some adjustment, estimates of *SD* also can be calculated from data measured with a coarse scale.

Sampling with a coarse scale has advantages over sampling with a continuous one; using a coarse scale often is more time efficient. More quadrats of herbaceous cover can be sampled within a specified time if cover classes are used instead of the exact measurement for each quadrat (Müeller-Dombois and Ellenberg 1974). Coarse scales can reduce costs of aerial sampling (Nicholson et al. 1997) because of less expense in sampling at intervals >1 day. Finally, field conditions or logistics might preclude sampling continuously. Because rounding of observed values is a special case of interval censoring, the use of coarse scales is near ubiquitous (Heitjan 1989).

Sampling with coarse scales, however, can have disadvantages, because of the loss of knowledge about the exact position of a datum. Parameters estimated from a coarse sample have to be corrected to account for this loss of information (Goldsmith 1968; Lindley 1950; McNeil 1966). The best known of these methods is Sheppard's correction (Sheppard 1898). Sheppard's correction is a general method for correcting the bias of a parameter estimator calculated from course data (see description in the following section). Caughley (1977)

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proposed using Sheppard's correction to adjust variance estimates of data on timing of parturition. Indeed, variance correction is the most common application of Sheppard's correction. Sheppard's correction, however, is only an approximation of the information lost. This method does not create an unbiased estimator, but can provide a good approximation if the scale is not too coarse (Goldsmith 1968).

Sheppard's correction for the sample variance estimator has 1 main drawback that has kept it from becoming applied regularly. The interval of the course sampling scale must be constant. When sampling parturition in wild mammals, it might be impossible to sample births on a fixed schedule. Weather or other unforeseen occurrences might prevent sampling from taking place at regular intervals. Therefore, Sheppard's correction could not be applied. To correct this problem, we develop a generalization to Sheppard's correction for the sample variance estimator that can be used if the intervals of the course sampling scale are allowed to vary. We also examined the properties of the generalized correction through a series of simulation experiments that illustrate that the new correction removes much of the bias of the sample variance when calculated from data collected on a coarse scale of measurement.

Sheppard's correction.—Sheppard (1898) assumed that the density of a distribution was smooth especially at the ends of the support (range) of the density, and could be well approximated by a Taylor series. He further assumed that the proportion in each equal-width interval was known. Sheppard then considered the difference between the true expectation of an arbitrary function (i.e., distribution parameter) and a crude estimate of that expectation derived from data rounded to the midpoints of each interval. He used integration by parts to convert this difference to an interpolation problem with equal spacing. Under those circumstances, an excellent approximation is obtained by using the Euler-Maclaurin method (Kellison 1975). The resulting correction for variance estimates proposed by Sheppard (1898) was given by

$$S^2_{corr} = S^2_{uncorr} - h/12 \tag{1}$$

where S^2_{uncorr} is the usual formula for sample variance and h is the interval width (in units of time) of the coarse sampling scale.

Among mammals, Sheppard's correction has been used to correct estimates of the mean date of birth for Himalayan tahr (*Hemitragus jemlahicus*—Caughley 1977), mule deer (*Odocoileus hemionus*—Bowyer 1991), Dall's sheep (*Ovis dalli*—Rachlow and Bowyer 1991), Alaskan moose (*Alces alces*—Bowyer et al. 1998), and American bison (*Bison bison*—Berger and Cain 1999). Estimated mean dates of migration of mule deer also have been corrected in this fashion (Nicholson et al. 1997), as have dates of shedding of velvet and casting of antlers in mule deer (Bowyer 1986). In addition, the correction was used to estimate percentage cover for plant species (Müeller-Dombois and Ellenberg 1974; Ver Hoef et al. 1988), an advantage for those studying herbivorous mammals (Barten et al. 2001; Lenhart et al. 2002).

The quality of Sheppard's approximation relies upon the density function having a range that begins and ends at an interval boundary, and the density function having the first few derivatives equal to 0 at those boundaries (this property is known as having a high degree of contact). Sheppard's correction should work best when the density is smooth, close to unimodal, symmetric, and with the curvature not changing too rapidly.

Sheppard's proof has been repeatedly augmented, because the requirement of high contact is difficult to verify and often is unrealistic in applications. Alternate proofs that yield or justify Sheppard's correction were offered by Fisher (1922) and Abernethy (1933). Using a piecewise-quadratic approximation to the density, Lewis (1935) derived a result similar to that of Sheppard. For densities with abrupt boundaries (not smooth), Elderton (1933), Jaiswal (1974), Pairman and Pearson (1919), and Sandon (1924) derived different corrections.

Other authors have investigated corrections derived from maximum-likelihood calculations to try and adjust Sheppard's corrections using distributional assumptions (Heitjan 1989; Kulldorff 1961; Lindley 1950; McNeil 1966). McNeil (1966) provided corrections that were more accurate than Sheppard's when the population was assumed to be normally distributed and group intervals were equal. Those assumptions would seldom be met in distributions of births (Bowyer 1991; Rachlow and Bowyer 1991) or many other life-history events.

A generalized correction for unequal intervals.—If the intervals of sampling (h in equation 1) are not constant, then a method for dealing with variation in interval size is needed. We propose a generalization of Sheppard's correction for a variance estimate calculated with data from unequal group sizes based on the weighted mean of the interval variances. When interval sizes are equal, the general form reverts to the standard form of Sheppard's correction. The variance of a grouped sample can be corrected in the following manner:

$$S^2_{corr} = S^2_{uncorr} - \sum_{i=1}^{k} n_i h_i / 12n \tag{2}$$

where n equals the total sample size, n_i equals the number of observations in interval i, h_i equals the width of interval i, S^2_{uncorr} is the usual sample variance computed with interval midpoints instead of the (unobserved) actual values, and k equals the number of intervals. Because Sheppard's original proof relies heavily upon the assumption of equal interval length, the mathematical justification for the result (Appendix I) follows the approach of Lindley (1950), in which Sheppard's correction is considered the first step in the adjustment of the coarse statistic to the maximum-likelihood estimate using Newton's method (Lange 1999:61).

We discuss how to use Sheppard's correction to adjust for bias in estimates created by using coarse scales. The relationship between accuracy of the corrected parameter estimate and the coarseness of measurement are investigated with computer simulations. We also discuss assumptions and requirements for using Sheppard's correction. In addition, an extension of Sheppard's correction for scales of unequal intervals (i.e., bin sizes) is proposed and demonstrated with computer simulations and from data on parturition in Dall's sheep (*Ovis dalli*) from Rachlow and Bowyer (1991). This methodology relaxes the

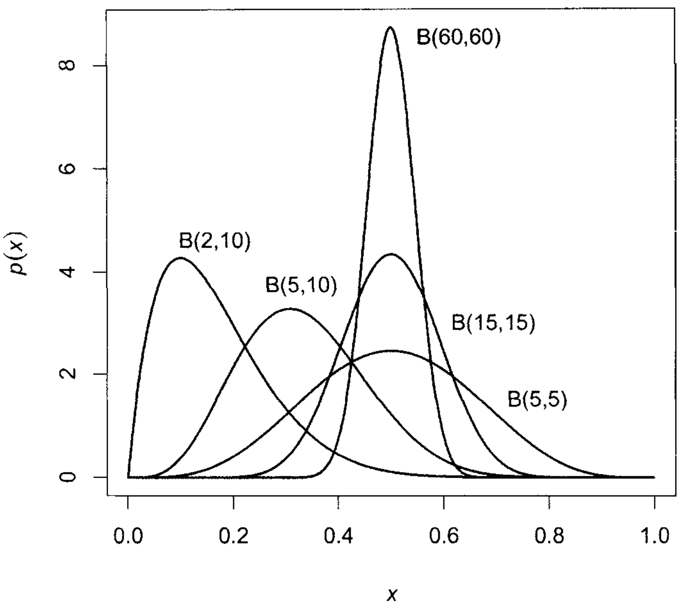

Fig. 1.—Plots of the 5 beta distributions used for each simulation analysis. The 1st parameter is the α parameter; the 2nd value is the β parameter. In the actual simulations, the sampled values were rescaled to range between 0 and 30. This makes no difference in the shape of their distributions, but only rescales the *x* axis.

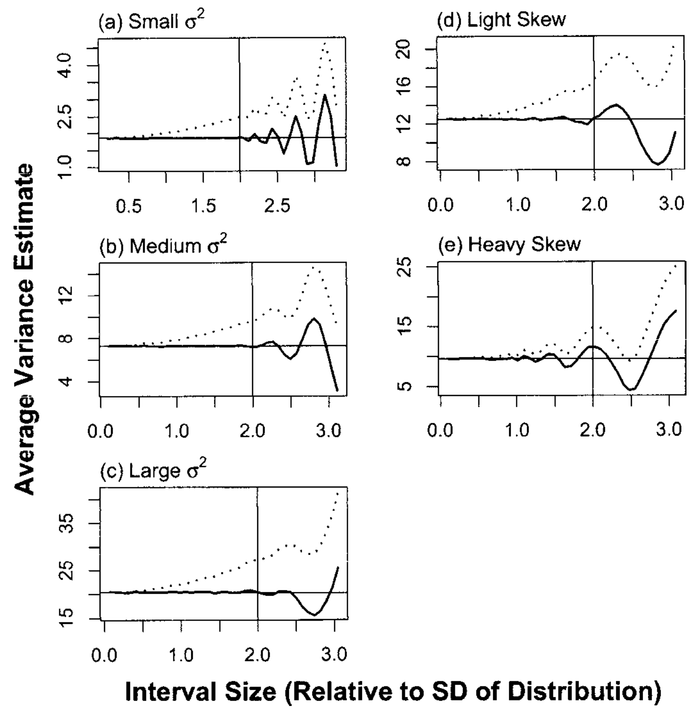

Fig. 2.—Corrected variance estimates with Sheppard's method versus interval size for 5 scaled beta distributions. Symmetrically distributed data were produced using α and β equal to 60, 15, and 5. Skew distributed data were generated using $\alpha = 5$ and $\beta = 10$ (light skew), and $\alpha = 2$, and $\beta = 10$ (heavy skew). Figure 1 illustrates these distributions. The horizontal line is the true population variance, whereas the thick line represents the corrected variance estimate with Sheppard's method. The dotted line represents the uncorrected variance estimate. Vertical lines represent interval size equal to 2σ. For each interval size, 1,000 samples were randomly drawn from each of the respective distributions and grouped according to interval size. Variance was estimated from grouped data and corrected with Sheppard's method.

need to sample in an equal interval, or to adjust all intervals to the size of the bin with the most missing data. Our purpose is to provide a new method to allow calculation of the mean and *SD* of a birthing distribution (or any other life-history characteristic) from data gathered at unequal sampling intervals. Quantifying such phenomena is an essential step in fully understanding and comparing life-history characteristics of mammals and other taxa (Caughley 1977).

Methods

Computer simulations.—Our new correction for unequal interval size was tested with samples from 5 different beta distributions. The beta distribution was selected because of its flexibility and boundedness of its range. A beta distribution is determined by 2 parameters α and β. Appropriate choices for these 2 parameters can yield normal-like symmetric distributions, or distributions with right or left skew. In addition, the beta distribution is defined only for values on the interval between 0 and 1. One of the assumptions of Sheppards' correction is that the beginning and end of the distribution is captured in one of the intervals. This assumption is not violated in reality, but can be difficult to deal with in simulations if the range of the simulated distribution is infinite, such as the normal distribution. In our simulations we used values of α and β equal to 60, 15, and 5 to produce samples from a symmetric distribution with a small to large variance, respectively (Fig. 1). In addition, we used values of $\alpha = 5$ and $\beta = 10$ to simulate a lightly skewed distribution and $\alpha = 2$ and $\beta = 10$ to simulate a heavily skewed distribution (Fig. 1). To make to simulations more realistic, the range was scaled up to values between 0 and 30 instead of 0 to 1.

In the first set of simulations, we held intervals constant to determine an upper limit for interval size of Sheppard's correction. One thousand samples (each with a size of 100) were drawn from each of the 5 distributions mentioned previously. We grouped each sample according to the constant interval size and estimated the population variance using Sheppard's correction for constant interval width. This process was repeated for intervals ranging in size from about 0.1 to 3 standard deviations in width for each of the 5 distributions sampled.

To test accuracy of the correction when intervals were allowed to vary, we established 20 random-interval scales with increasing variability in interval size. We accomplished this for each of the 5 distributions by forming a scale from 0 to 30 of constant bin widths that were approximately 0.5σ wide. Therefore, the proportion of the total scale (0–30) taken up by each bin is constant and determined by σ for each distribution. To add some variability to the bin sizes, the proportion of the total scale that each bin encompasses was perturbed from the constant state with a random, logistic-normal error term (created from a vector of independent log-normal variables scaled to add to 1—Aitchison and Shen 1980). Variability of the error term was increased steadily from 0.1 to 0.75 when constructing the 20 random interval scales. This produced random bins with variation in interval widths equal to about 0.1σ to 0.5σ for each of the 5 sampled distributions. In total, for each of the 21 interval treatments within each of the 5 distributions, 1,000 samples of size 100 were drawn, binned according to the interval scale, and the variance estimated with and without the correction.

Estimating mean date of parturition.—We illustrated application of the generalized correction on variance estimates for the birthing

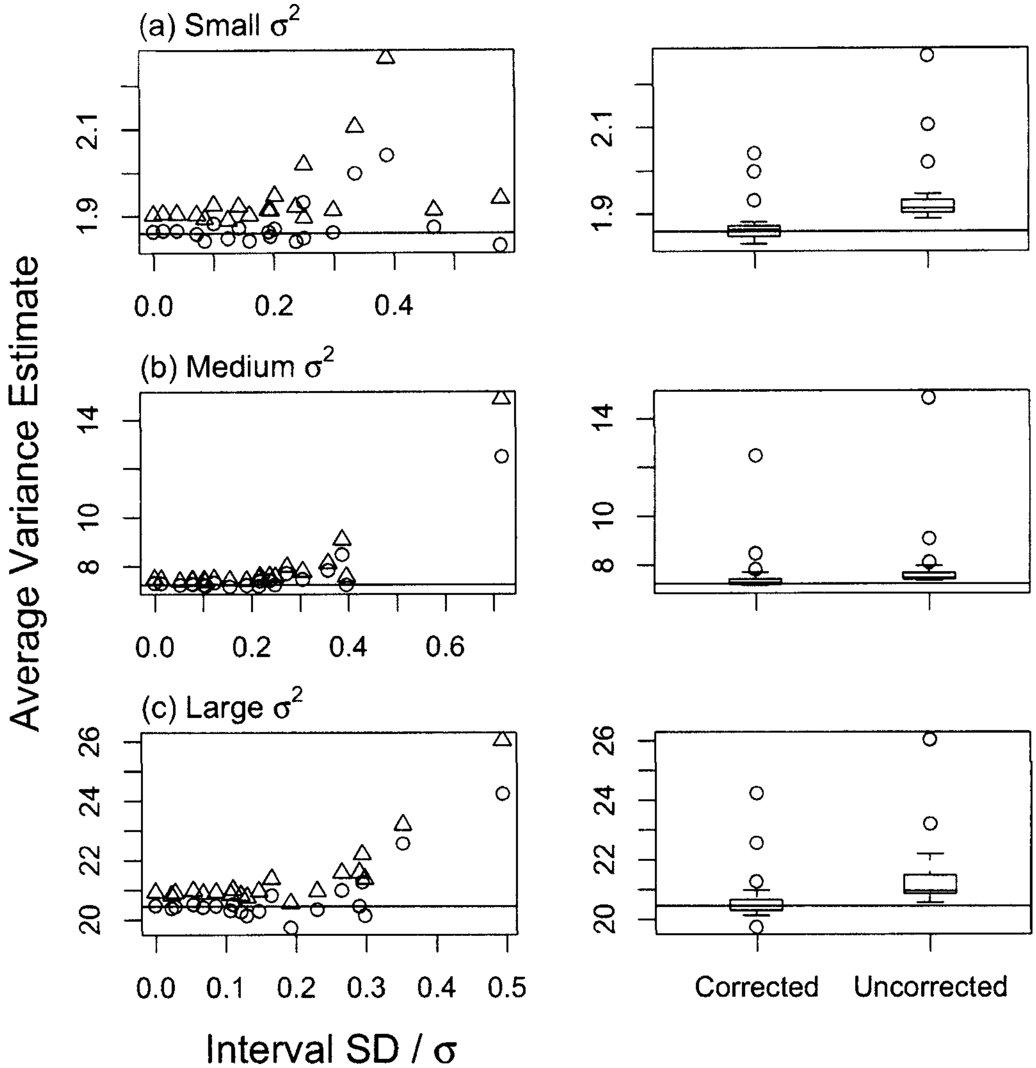

FIG. 3.—Results of the simulation for variable interval width for symmetrical distributions. The left graph illustrates the average variance estimates for each random bin size. For each of the 20 random bin-size allocations plus 1 constant bin allocation (interval size variance = 0), 1,000 samples of size 100 were drawn, binned, and the variance estimated with and without the generalized correction. Plotted circles represent the average corrected variance estimate, whereas triangles represent the average uncorrected estimate. Average estimates were plotted against the ratio of the standard deviation of interval size to the population deviation. Boxplots on the right hand side summarize the average estimates over all 21 interval size schemes.

distributions of Dall's sheep. Percentage of cumulative births for Dall's sheep in Denali National Park and Preserve, Alaska, was determined from ratios of young to adult females in 1988 and 1989 (Rachlow and Bowyer 1991). Those data were collected at variable intervals ranging from 1 to 6 days. We used the cumulative birthing curve provided by Rachlow and Bowyer (1991) from polynomial regressions so that we could illustrate our method effectively, while maintaining the properties of a real birthing distribution. Number of confirmed births was used as the sample size for each year (1988, $n = 12$; 1989, $n = 10$). Data from 1989 were adjusted to the onset of lambing in 1988 so they could be compared directly with data from 1988 (Rachlow and Bowyer 1991).

RESULTS

Figure 2 illustrates the need for some type of correction to the standard sample variance estimator. The dotted line represents the average variance estimate for the uncorrected estimator. Clearly, as interval size increases the uncorrected estimator becomes increasingly biased. The solid line represents the corrected estimator, which remains nearly unbiased for a large range of interval sizes.

In addition, Figure 2 illustrates that Sheppard's correction performed well until interval size became about twice the *SD* of the population for each of the symmetric distributions and the lightly skewed distribution. When <2 full intervals encompassed nearly the entire distribution, too much information was lost for Sheppard's correction to adequately adjust the estimate, and the estimate became unstable. For the heavily skewed distribution, the estimate became unstable at about 1.5σ, but oscillated around the correct value and did not become greatly unstable until the 2σ, similar to the other distributions.

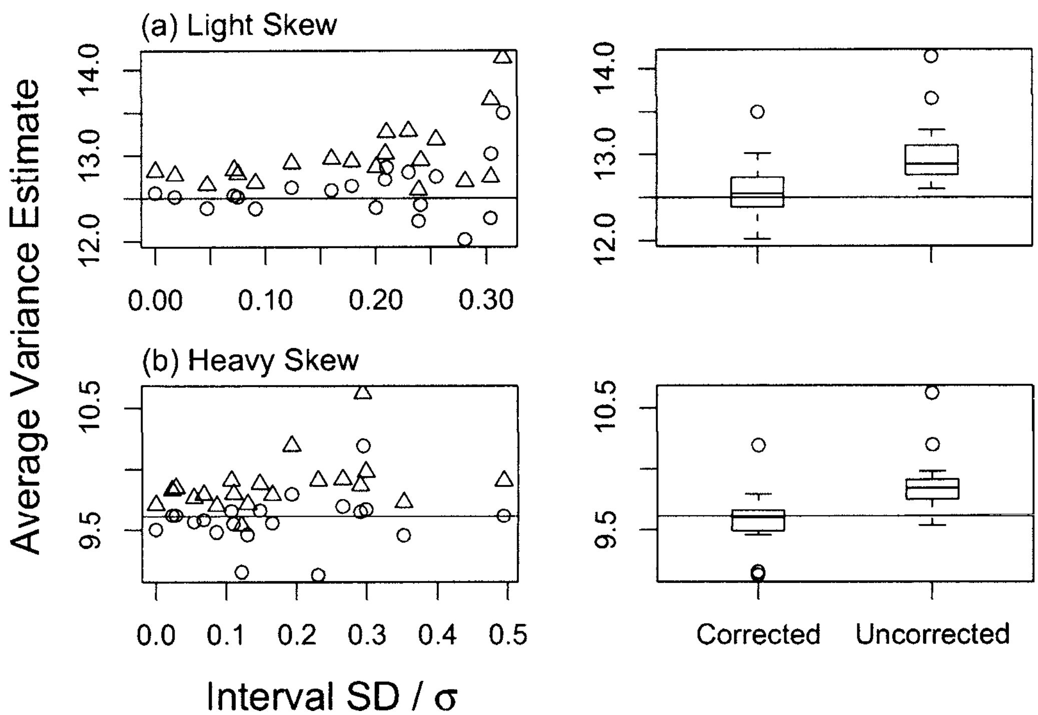

FIG. 4.—Results of the variable interval-width simulation for skewed distributions. The left graph illustrates the average variance estimates for each random bin size allocation. For each of the 20 random bin-size allocations, plus 1 constant bin allocation (interval size variance = 0), 1,000 samples of size 100 were drawn, binned, and the variance estimated with and without the generalized correction. Plotted circles represent the average corrected variance estimate, whereas triangles represent the average uncorrected estimate. Average estimates were plotted against the ratio of the standard deviation of interval size to the population standard deviation. The boxplots on the right hand side summarize the average estimates over all 21 interval size schemes.

When interval sizes were allowed to vary, generally, the correction performed well (Figs. 3 and 4). However, the correction did not perform as well as when intervals were kept equidistant; the correction produced an estimator with much less bias. Figs. 3 and 4 illustrate that the arrangement of bins plays a large role in determining the bias of the uncorrected estimator as well as the amount of improvement the correction will produce. If the interval containing the true mean (μ) was large compared with the intervals that contained the tails of the distribution, the correction was overly large. The reverse situation also occurred. The left-hand sides of Figs. 3 and 4 show that as bin variability increases relative to the *SD* of the distribution, the corrected and uncorrected estimators become increasing unstable, most likely because of the inclusion of some overly large intervals (those greater that 2σ). For each of the 5 distributions, however, the correction worked well as long as the *SD* of interval sizes remained less than 0.25σ. The right hand sides of Figs. 3 and 4 summarize the average variance estimate over all of the 21 bin treatments. In general, the new correction removes nearly all of the bias of the uncorrected sample variance when examined over all of the different random-interval scales.

Variances of the birthing distributions of Dall's sheep were calculated according to the generalized Sheppard's correction. Mean ($\pm$ *SD*) dates of parturition for Dall's sheep in 1988 and 1989 were May 20 ($\pm$ 10.3 days) and June 3 ($\pm$ 12.2 days), respectively. Methods for calculating the mean and *SD* for 1988 were simple compared to the computationally intensive maximum likelihood estimators. Computation of the generalized correction estimate for the 1988 data is detailed in Table 1. Parameter estimates for 1989 were calculated in an identical fashion.

DISCUSSION

Grouping temporal data into intervals is a useful technique if sampling on a continuous scale is impossible or impractical, which often occurs for life-history characteristics such as birthing distributions. Caughley (1977) reported that Sheppard's correction can be used only if sampling intervals are contiguous; if both the beginning and the end of the sampling distribution are contained in a sampled interval; and if divisions of the group boundaries are equidistant. Maximum-likelihood corrections also require conditions 1 and 2, but not 3 (Kulldorff 1961). Maximum-likelihood corrections, however, also require knowledge or assumptions about the shape of the distribution for the population and are computer intensive. The new generalized form of Sheppard's correction requires only conditions 1 and 2. Although our generalized form of Sheppard's correction requires the smallest number of conditions, this method also can be inaccurate. Our results indicate the need for a hierarchy of considerations in selecting the best method for determining the mean date of birth, or timing of other life-history events.

Table 1.—Calculations for sample mean and corrected variance of date of birth in Dall's sheep (*Ovis dalli*) in Alaska during spring 1988 (data modified from Rachlow and Bowyer 1991).

Day	Interval midpoint y_i	h_i	Proportion of births n_i/n	$y_i\ (n_i/n)$	Weighted squared error	$h_i^2/12$	$n_i h_i^2/12n$
0			0.0000	0.0000	0.0000		
1	0.5	1	0.0896	0.0488	17.2309	0.0833	0.0075
4	2.5	3	0.2119	0.5298	29.8465	0.75	0.1589
8	6	4	0.1745	1.0469	12.2163	1.3333	0.2327
10	9	2	0.0528	0.4752	1.5212	0.3333	0.0176
12	11	2	0.0369	0.4057	0.4182	0.3333	0.0123
18	15	6	0.0679	1.0188	0.0272	3.0	0.2037
20	19	2	0.0260	0.494	0.5579	0.3333	0.0087
24	22	4	0.0887	1.9518	5.1683	1.3333	0.1183
27	25.5	3	0.1195	3.0462	14.8048	0.75	0.0896
30	28.5	3	0.1322	3.7683	26.4078	0.75	0.0992
			$\sum =$	12.7855	108.1993		0.941

Mean date of birth $= \sum_{i=1}^{k} y_i(n_i/n) = 12.7885 = 20$ May.
Weighted squared error $= (n_i/n)(y_i - \bar{y})^2$.
Variance of the grouped sample $= (n-1)^{-1}\sum_{i=1}^{k} n_i(y_i - \bar{y})^2 = 108.1993$ days.
Variance correction $= -\sum_{i=1}^{k} n_i h_i^2/12n = -0.941$.
Corrected variance $= (n-1)^{-1}\sum_{i=1}^{k} n_i(y_i - \bar{y})^2 - \sum_{i=1}^{k}(n_i h_i^2/12n) = 107.2583$ days.
Corrected *SD* = 10.3 days.

First, recording data on a continuous scale is better than grouping. Retaining information on individual position is superior to losing it and attempting to correct for its absence. Corrections we mentioned previously do not create unbiased estimators, but can be very accurate. Consequently, consideration should be given to the level of difficulty in collecting data and cost per sample unit. If significantly more samples can be obtained, or obtained at a substantially lower cost, by sampling with a coarse scale, then it may be advisable to do so.

Second, if sampling with a coarse scale is chosen, intervals should not be larger than twice the *SD* for the population. This outcome agrees well with the computations of Goldsmith (1968), who noted that Sheppard's correction begins to fail when the *SD* is 0.6 interval widths. The smallest interval possible is best. There is less likelihood of those approximations being inaccurate if the intervals are very small. For example, if sampling occurs in intervals of 3 days, the raw grouped estimate will only need to be corrected by three-fourths. Unfortunately, the population *SD* usually is unknown. When constructing sampling intervals, a crude estimate of the *SD* is the probable range of data divided by 4. That estimate makes use of the knowledge that 95% of a normal probability density falls within 2 *SD* of the mean. The assumption does not have to be made that data collected will be normally distributed; this estimate is only to determine a relative magnitude. If we assumed that the interval from the first to the last birth was approximately 1 month (31 days) for designing sampling intervals for Dall's sheep, then the crude estimate of *SD* would be 7.75. Therefore, no intervals should be planned longer than 15 days in length.

Finally, if sampling with a coarse scale is chosen, sampling with constant interval size is preferable to sampling with variable interval size, unless σ^2 is large in relation to the largest interval size. We propose that the *SD* of interval sizes should not exceed 0.25σ. Inaccuracy in the correction could result if interval size varies too greatly. If large intervals are used near the mean and smaller intervals are used near the tails of the true distribution, the generalized correction will overadjust the variance estimate. This result occurs because data farther away from the sample mean are not shifted greatly in the grouping process, and there are many data in the large, central interval (Appendix I). Because positions of data near the edges of the distribution are not greatly altered in the grouping process, the variance of the grouped data might not be altered as much as the correction predicts. The opposite condition can occur with small intervals near the mean and large intervals near the tails of the distribution. In that instance, data in the tails are shifted more than the interior data. A variance larger than predicted by the correction is produced because of many data in the small interval. This problem is alleviated as the intervals become more constant in relation to the population variance.

There are additional complexities that could be considered in the generalization of Sheppard's correction to more closely approximate the shape of a particular naturally occurring distribution. Simpson's rule and the trapezoid rule have been used for many years to approximate values of definite integrals with divisions of a scale (Ellis and Gullick 1994). In addition, maximum-likelihood techniques can be used when the form of the distribution is known. This might be especially important if the distribution is J-shaped (i.e., exponential distribution) or otherwise has an abrupt lower or upper bound. We believe that the greater flexibility offered by our approach will be useful to those interested in quantifying the timing of life-history events we discussed, as well as other applications including entry and emergence of animals from hibernation, plant green-up and senescence, and mating activities of mammals. So long as the guidelines we offer are followed, researchers can now relax the need to sample in a continuous or equal-interval scale to accurately estimate timing of life-history events.

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Appendix I

This derivation of the Sheppard's correction for the case of unequal bin widths closely follows the derivation by Lindley (1950) for equal bin widths. In his approach, the uncorrected variance (with coarse data) is considered an initial guess at the true maximum likelihood estimate of the variance, and 1 iteration of Newton's method is used to correct that value. The outcome is Sheppard's correction.

Suppose the true values $\{x_1,\ldots,x_n\}$ (unobserved) are realizations of the independent, identically distributed random variables $\{X_1,\ldots, X_n\}$. Suppose the range of X is partitioned into intervals, and x_i is contained in an interval of width h_i, centered on y_i. Then the observed data are the rounded values $y_1,\ldots,y_n$. If the density of the true values, X, is $f(x;\theta)$ (the value depends on an unknown parameter θ), then the maximum likelihood estimate is the solution of

$$\sum_{i=1}^{i=n}\frac{\partial}{\partial\theta}\log\int_{y_i-h_i/2}^{y_i+h_i/2}f(x;\theta)dx=0 \qquad (3)$$

because the probability of observing y_i is the integral of $f(x;\theta)$ over the interval containing y_i.

If, instead, coarse data are used in the maximum-likelihood equation that does not correct for rounding, the estimate $\hat{\theta}_0$ is obtained, which is the uncorrected estimator that solves

$$\sum_{i=1}^{i=n}\frac{\partial}{\partial\theta}\log f(y_i;\theta)=0. \qquad (4)$$

This solution is not the true maximum-likelihood estimator of θ, because the coarse values are used, and the likelihood in (4) assumes non-rounded values.

Because the integral in (3) is, in general, intractable, we can approximate it with the Taylor expansion of $f(x;\theta)$:

$$\begin{aligned}\int_{y-h/2}^{y+h/2}f(x;\theta)dx &= \int_{y-h/2}^{y+h/2}\left\{f(y;\theta)+f'(y;\theta)(x-y)+\frac{1}{2}f''(y;\theta)(x-y)^2\right.\\ &\qquad\left.+\frac{1}{6}f'''(y;\theta)(x-y)^3+R(x-y)^4\right\}dx\\ &= hf(y;\theta)+\frac{h^3}{12}f''(y;\theta)+O(h^4)\\ &= hf(y;\theta)\left\{1+\frac{h_i^2 f''(y;\theta)}{12\,f(y;\theta)}+O(h_i^3)\right\}. \qquad (5)\end{aligned}$$

The order h_i^4 is an asymptotic result as h_i becomes small. Further,

$$\log \int_{y-h/2}^{y+h/2} f(x;\theta)dx = \log hf(y;\theta) + \log\left\{1 + \frac{h^2}{12}\frac{f''(y;\theta)}{f(y;\theta)} + O(h_i^3)\right\}. \tag{6}$$

Therefore, the likelihood equation we should be solving when we use coarse values is

$$\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\log \int_{y_i-h_i/2}^{y_i+h_i/2} f(x;\theta)dx = \sum_{i=1}^{n} \frac{\partial}{\partial\theta}\left[\log h_i f(y_i;\theta) + \log\left\{1 + \frac{h_i^2}{24}\frac{f''(y_i;\theta)}{f(y_i;\theta)} + O(h_i^3)\right\}\right] = 0. \tag{7}$$

If we have a rough guess $\hat{\theta}_0$ obtained with the rounded values and maximum likelihood equation for non-rounded values (4), how do we get closer to the correct maximum-likelihood estimate shown in the last equation? One way is to use our rough guess as a first step using Newton's method to find the solution to equation (3). Recall that Newton's method for finding the roots of an equation $g(\theta) = 0$ is to start with $\hat{\theta}_0$, and then move to the corrected value $\hat{\theta}_1 = \hat{\theta}_0 - g(\hat{\theta}_0)/g'(\hat{\theta}_0)$. Usually the process is repeated, but as the likelihood equation is quite complicated, we will use just the first iteration of the method. What is the correction $g(\hat{\theta}_0)/g'(\hat{\theta}_0)$, when g is the right-hand side of equation (7)?

$$\frac{g(\hat{\theta}_0)}{g'(\hat{\theta}_0)} = \frac{\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\big|_{\theta=\hat{\theta}_0}\left[\log h_i f(y_i;\theta) + \log\left\{1 + \frac{h_i^2}{24}\frac{f''(y_i;\theta)}{f(y_i;\theta)} + O(h_i^3)\right\}\right]}{\frac{\partial^2}{\partial\theta^2}\big|_{\theta=\hat{\theta}_0}\sum_{i=1}^{n}\left[\log h_i f(y_i;\theta) + \log\left\{1 + \frac{h_i^2}{24}\frac{f''(y_i;\theta)}{f(y_i;\theta)} + O(h_i^3)\right\}\right]}. \tag{8}$$

Because $\hat{\theta}_0$ solves equation (4), $\sum_{i=1}^{n} \partial/\partial\theta|_{\theta=\hat{\theta}_0} \log f(y_i;\theta) = 0$; so equation (8) simplifies to

$$\frac{g(\hat{\theta}_0)}{g'(\hat{\theta}_0)} = \frac{\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\big|_{\theta=\hat{\theta}_0} \log\left\{1 + \frac{h_i^2}{24}\frac{f''(y_i;\theta)}{f(y_i;\theta)} + O(h_i^3)\right\}}{\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2}\big|_{\theta=\hat{\theta}_0} \log h_i f(y_i;\theta)}. \tag{9}$$

Finally, taking the derivatives yields the correction

$$\frac{g(\hat{\theta}_0)}{g'(\hat{\theta}_0)} = \frac{\sum_{i=1}^{n} \frac{h_i^2}{24}\frac{\partial}{\partial\theta}\big|_{\theta=\hat{\theta}_0}\frac{f''(y_i;\theta)}{f(y_i;\theta)} + O(h_i^3)}{\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2}\big|_{\theta=\hat{\theta}_0} \log f(y_i;\theta)}. \tag{10}$$

The preceding derivation generalizes Lindley's result, which was derived under the assumption of equal bin width. Lindley then showed that the normal distribution yields exactly Sheppard's correction. Therefore, we examined our generalized correction under the assumption of normality.

Example: For $f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}}\exp\{-x^2/2\theta\}$,

$$\frac{\partial}{\partial\theta}\frac{f''(x;\theta)}{f(x;\theta)} = \frac{1}{\theta^2} - \frac{2x^2}{\theta^3}.$$

Also,

$$\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$

Thus, the correction becomes

$$-\frac{\sum_{i=1}^{n}\left(\frac{1}{2\theta^2} - \frac{x_i^2}{\theta^3}\right)\left(\frac{h_i^2}{12}\right)}{\sum_{i=1}^{n}\left(\frac{1}{2\theta^2} - \frac{x_i^2}{\theta^3}\right)}.$$

If h_i^2 is approximately uncorrelated with x_i^2, the correction can be approximated with

$$-\frac{n^{-1}\left[\sum_{i=1}^{n}\left(\frac{1}{2\theta^2} - \frac{x_i^2}{\theta^3}\right)\right]\left[\sum_{i=1}^{n}\frac{h_i^2}{12}\right]}{\sum_{i=1}^{n}\frac{1}{2\theta^2} - \frac{x_i^2}{\theta^3}} = -\frac{1}{n}\sum_{i=1}^{n}\left(\frac{h_i^2}{12}\right).$$

For distributions that resemble the shape of the normal distribution, the correction should be about the same. Among the properties of good distributions are rough unimodality and smoothness. J-shaped distributions or uniform distributions have sharp drops at the edges; for those distributions, Sheppard's correction might not perform as well.