Anosov automorphisms of nilmanifolds

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Over one hundred years ago, Henri Poincaré realized that it is possible for simple deterministic dynamical systems to have behaviour that seems chaotic and unpredictable. Since the work of Lorenz, Mandelbrot, Smale and others in the 1960's, these types of systems have been the focus of a great deal of mathematical research, having many interdisciplinary applications, and arousing a great deal of popular interest.

"Chaos" is a loose term applied to dynamical systems sharing certain qualitative properties, such as sensitivity to initial conditions and a complex orbit structure. Although they seem random and unpredictable, there can actually be a lot of large-scale regularity to them.

A C^1 diffeomorphism of a compact Riemannian manifold M is called *Anosov* if, roughly speaking, the derivative map at each point is hyperbolic. Anosov maps have all the features associated with chaos, and they form a tractable class of dynamical systems for the study of chaotic behavior.

One of the simplest examples of an Anosov diffeomorphism is Arnold's cat map: the diffeomorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$ defined by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. An obvious generalization of the cat map is a diffeomorphism of the torus $\mathbb{R}^n/\mathbb{Z}^n$ defined by a matrix in $GL_n(\mathbb{Z})$ with no eigenvalues of modulus one. Even more generally, a hyperbolic automorphism of a simply connected nilpotent Lie group N that fixes a lattice Γ in N gives an Anosov diffeomorphism of the compact homogeneous space N/Γ.

And, up to dynamical equivalence, these examples are essentially all the known examples of Anosov diffeomorphisms! It is conjectured that every Anosov diffeomorphism is topologically conjugate to an example of one of these types, or a finite quotient of one of these examples. Thus, in order to understand Anosov diffeomorphisms in general, it is fundamental to understand examples arising from automorphisms of nilpotent Lie groups. Until recently, very few nontoral examples were known.

In this talk we will describe the basic properties of Anosov maps. We will then discuss lattices and rational structures for Lie groups, and how to find automorphisms preserving lattices. That done, we will show some general methods of constructing Anosov automorphisms of nontoral manifolds. The constructions use units in algebraic number fields, so algebraic number theory is a key tool.